Balancing non-Wieferich primes in arithmetic progression and *abc* conjecture

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Abstract: In this note, we shall define the balancing Wieferich prime which is an analogue of the famous Wieferich primes. We prove that, under the *abc* conjecture for the number field $\mathbf{Q}(\sqrt{2})$, there are infinitely many balancing non-Wieferich primes. In particular, under the assumption of the *abc* conjecture for the number field $\mathbf{Q}(\sqrt{2})$ there are at least $O(\log x/\log\log x)$ such primes $p \equiv 1(\bmod k)$ for any fixed integer $k > 2$.

Key words: Balancing number; Wieferich prime; arithmetic progression; *abc* conjecture.

1. Introduction. It is well known that if p is a prime and a is any integer not divisible by p, then

$$a^{p-1} \equiv 1(\bmod p). \tag{1.1}$$

The quotient $q_p(a) = \frac{a^{p-1}-1}{p}$ is called the Fermat quotient for p with base a. These quotients have been studied by several authors. However, while doing the first case of Fermat's last theorem, Wieferich used the Fermat's quotient with base 2 and proved the following theorem.

Theorem 1.1. *Let p be an odd prime, and x, y, z be integers, not divisible by p, satisfying the equation*

$$x^p + y^p = z^p. \tag{1.2}$$

Then

$$2^{p-1} \equiv 1(\bmod p^2). \tag{1.3}$$

Primes satisfying (1.3) are called Wieferich primes with base 2. Till today two Wieferich primes are known for the base 2, that are 1093 and 3511, found respectively by Meissner in 1913 and by Beegner in 1922.

Before going to the analogue theory of Wieferich primes for the sequence of balancing numbers, our foremost task is to discuss the concept of balancing numbers. A balancing number is a positive integer n which is a solution of the Diophantine equation

$$\begin{aligned} 1 + 2 + \cdots + (n-1) \\ = (n+1) + (n+2) + \cdots + m \end{aligned} \tag{1.4}$$

for some natural number m [1,3]. Equivalently, the solutions (x, y) of the Pell's equation $8x^2 + 1 = y^2$ are called balancing and Lucas balancing numbers. Let us denote the n-th balancing and Lucas balancing numbers by B_n and C_n respectively. Balancing numbers can also be obtained from the recurrence relation $B_{n+1} = 6B_n - B_{n-1}$ for $n \geq 1$ with initial values $B_0 = 0$, $B_1 = 1$ [3] and the Binet formula for balancing number is

$$B_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{1.5}$$

where $\alpha = 3 + 2\sqrt{2}$ and $\beta = 1/\alpha$. Panda and Rout [4] studied the periodicity of balancing number sequences and proved that $(B_n \bmod m)_{m=0}^{\infty}$ is periodic. It is also known that for any odd prime p, $B_{p-(\frac{8}{p})} \equiv 0 \ (\bmod p)$, where $(\frac{8}{p})$ denotes the Jacobi symbol [4]. A prime p is called a *balancing Wieferich prime* if

$$B_{p-(\frac{8}{p})} \equiv 0 \ (\bmod p^2). \tag{1.6}$$

Though Panda and Rout [4] did not discuss the balancing Wieferich primes but they formulated a conjecture that there are three primes 13, 31, and 1546463 such that periods of balancing sequence modulo these three primes are equal to the periods modulo its square. Hence, these are the three balancing Wieferich primes. Sun and Sun [7] proved that if the first case of Fermat's last theorem fails for an odd prime p then $F_{p-(\frac{5}{p})} \equiv 0 \ (\bmod p^2)$, where

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F_n is the n-th Fibonacci number. However, the finiteness or infinitude of the balancing Wieferich primes are still unknown. Also we do not know whether there are infinitely many balancing non-Wieferich primes or not.

The objective of this paper is to show that, for any arbitrary integer $k > 2$, there are infinitely many balancing non-Wieferich primes p with $p \equiv 1 \pmod{k}$ under the assumption of the *abc* conjecture. Our proof closely follows the paper of Graves and Ram Murty [2].

2. Preliminaries. Let us start with definition of radical of a number and cyclotomic polynomial.

Definition 2.1. The radical of a positive integer n is defined as the product of the distinct prime numbers dividing n, i.e.,

$$\operatorname{rad}(n) = \prod_{\substack{p|n \\ p \text{ prime}}} p.$$

Definition 2.2. Given an integer x, $x = \prod_p p^\alpha$, where the product is over the distinct primes $p \mid x$, we define its *powerful part* as the product of prime powers such that $p^\alpha \| x$ with $\alpha \geq 2$.

Definition 2.3. For any integer $m \geq 1$, the m-th cyclotomic polynomial can be defined as

$$\Phi_m(X) = \prod_{\substack{(d,m)=1 \\ 0<d<m}} (X - \zeta_m^d), \tag{2.1}$$

where ζ_m is the primitive m-th root of unity.

We now state the following generating formula for cyclotomic polynomials

$$\Phi_m(X) = \frac{X^m - 1}{\underbrace{\prod}_{\substack{d|m \\ 0<d<m}} \Phi_d(X)}. \tag{2.2}$$

We need the following estimate of Thangadurai and Vatwani [8] which relates the cyclotomic polynomial $\Phi_n(x)$ with the Euler totient function ϕ.

Proposition 2.4. *For all integers $n \geq 2$ and $b \geq 2$,*

$$\Phi_n(b) \geq \frac{1}{2} b^{\phi(n)}.$$

The proof of the following result is available in [5].

Proposition 2.5. *If $p \mid \Phi_n(b)$, then either $p \mid n$ or $p \equiv 1 \pmod{n}$.*

We need the following important inequality due to Rosser [6].

Proposition 2.6. *The n-th prime is strictly greater than $n \log n$.*

In 1980, Masser and Oesterlé formulated the following *abc* conjecture.

Conjecture 2.7. Let a, b, c be mutually coprime integers satisfying $a + b = c$ and let $\epsilon > 0$ be given. Then there is a constant $\kappa(\epsilon)$ such that

$$\max(|a|, |b|, |c|) \leq \kappa(\epsilon)(\operatorname{rad}(abc))^{1+\epsilon}. \tag{2.3}$$

From the Binet formula of B_n in (1.5), one can observe that, the ring $\mathbf{Z}$ is not sufficient for our purpose. Instead, we will work in a larger ring $\mathbf{Z}[\sqrt{2}]$ as $\mathbf{Z}[\sqrt{2}]$ is the ring of integers of the number field $\mathbf{Q}(\sqrt{2})$.

2.1. The *abc* conjecture in number fields (See [9]). Let K be an algebraic number field and $\mathcal{O}_K$ be its ring of integers. Let V_K be the set of primes on K, that is $v \in V_K$ is an equivalence class of non-trivial norms on K (finite or infinite). Let $\|x\|_v := N_{K/\mathbf{Q}}(\mathfrak{p})^{-v_\mathfrak{p}(x)}$ if v is the prime defined by a prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$ and $v_\mathfrak{p}$ is the corresponding valuation. Let $\|x\|_v := |\sigma(x)|^e$ for all non-conjugate embeddings $\sigma : K \to \mathbf{C}$ with $e = 1$ if $\sigma(K) \subset \mathbf{R}$ and $e = 2$ otherwise. Then height of any triple $(a, b, c) \in (K^*)^3$ is

$$H_K(a, b, c) := \prod_{v \in V_K} \max(\|a\|_v, \|b\|_v, \|c\|_v). \tag{2.4}$$

Suppose $I_K(a, b, c)$ is the set of all prime ideals $\mathfrak{p}$ of $\mathcal{O}_K$ for which $\|a\|_v, \|b\|_v, \|c\|_v$ are not equal. Then the radical of the triple (a, b, c) is given by

$$\operatorname{rad}_K(a, b, c) := \prod_{\mathfrak{p} \in I_K(a,b,c)} N_{K/\mathbf{Q}}(\mathfrak{p}). \tag{2.5}$$

Conjecture 2.8. For any $\epsilon > 0$, there exists a positive constant $C_{K,\epsilon}$ such that for all $a, b, c \in K^*$ satisfying $a + b + c = 0$, we have

$$H_K(a, b, c) \leq C_{K,\epsilon}(\operatorname{rad}_K(a, b, c))^{1+\epsilon}. \tag{2.6}$$

We would like to give the definition of rank of apparition of a number in an integer sequence which will play a vital role in proving some lemmas. The rank of apparition of k in an integer sequence $\{U_n\}$ is the least index t such that $U_t \neq 0$ and $k \mid U_t$. We denote the rank of apparition of p for balancing sequence as $\alpha(p)$ if it exists. An important property of balancing number is $B_n \equiv 0 \pmod{p}$ if and only if $n \equiv 0 \pmod{\alpha(p)}$.

Lemma 2.9. *For $n \geq 2$ and $\alpha = 3 + 2\sqrt{2}$, the n-th balancing number satisfies the following* *inequality*

$$\alpha^{n-1} < B_n < \alpha^n.$$

Proof. This inequality can be easily obtained by using induction on n. □

A similar type of inequality also holds if we replace the integer b with real number b in Proposition 2.4 and hence the following lemma.

Lemma 2.10. *For any real number b with $|b| > 1$, there exists $C > 0$ such that*

$$|\Phi_n(b)| \geq C \cdot |b|^{\phi(n)}.$$

Proof. From the proof of Proposition 2.4, we will find that

$$S = \frac{\Phi_n(b)}{b^{\phi(n)}} = \prod_{d|n}\left(1 - \frac{1}{b^d}\right)^{\mu(n/d)}.$$

Now, $|\prod_{d|n}(1 - \frac{1}{b^d})^{\mu(n/d)}| > \prod_{d=1}^{\infty}(1 - \frac{1}{|b|^d})$. Then taking logarithm we have

$$\log|S| > \log\prod_{d=1}^{\infty}\left(1 - \frac{1}{|b|^d}\right) = \sum_{d=1}^{\infty}\log\left(1 - \frac{1}{|b|^d}\right).$$

Now using the following expansion for $x \in [0,1)$

$$\log(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right) \geq -(x + x^2 + x^3 + \cdots),$$

we get

$$\log|S| > -\sum_{d=1}^{\infty}\frac{1}{|b|^d} = \frac{-1}{|b|-1}.$$

Taking exponentiation of the above inequality we conclude $|S| > e^{-1/(|b|-1)}$ and this completes the proof. □

Lemma 2.11. *Let p be an odd prime. Suppose $B_n \equiv 0 \pmod{p}$ and $B_n \not\equiv 0 \pmod{p^2}$. Then $B_{\alpha(p)} \not\equiv 0 \pmod{p^2}$.*

Proof. Since $B_n \equiv 0 \pmod{p}$ we must have $\alpha(p) \mid n$. Thus $n = \alpha(p)e$ for some $e \geq 1$. Assume on the contrary that $B_{\alpha(p)} \equiv 0 \pmod{p^2}$. Thus $B_{\alpha(p)} = pl$ where $p \mid l$. Using the De-Moivres theorem for balancing numbers [3], i.e.,

$$C_{\alpha(p)e} + \sqrt{8}B_{\alpha(p)e} = (C_{\alpha(p)} + \sqrt{8}B_{\alpha(p)})^e,$$

we have

$$\begin{aligned} B_n = B_{\alpha(p)e} &= \binom{e}{1}C_{\alpha(p)}^{e-1}B_{\alpha(p)} \\ &+ \binom{e}{3}C_{\alpha(p)}^{e-3}B_{\alpha(p)}^3 + \cdots + 8^{\frac{p-1}{2}}B_{\alpha(p)}^e \\ &= pl\binom{e}{1}C_{\alpha(p)}^{e-1} + p^3l^3\binom{e}{3}C_{\alpha(p)}^{e-3} + \cdots + 8^{\frac{p-1}{2}}p^el^e \\ &\equiv pleC_{\alpha(p)}^{e-1} \pmod{p^2}. \end{aligned}$$

Since $p \mid l$, $B_n \equiv 0 \pmod{p^2}$, which is a contradiction to our hypothesis. □

Lemma 2.12. *Suppose B_n is factored into X_nY_n, where X_n is the squarefree part of B_n and Y_n is the powerful part of B_n. If $p \mid X_n$, then*

$$B_{p-(\frac{8}{p})} \not\equiv 0 \pmod{p^2}.$$

Proof. Since $\gcd(X_n, Y_n) = 1$ and $p \mid X_n$, we have $p \| B_n$. Thus, by Lemma 2.11, $p^2 \nmid B_{\alpha(p)}$. Since $B_{p-(\frac{8}{p})} \equiv 0 \pmod{p}$ we have $\alpha(p) \mid (p - (\frac{8}{p}))$. Write $(p - (\frac{8}{p})) = \alpha(p)f$. Also $B_{\alpha(p)} = pl$ where $p \nmid l$. Hence

$$\begin{aligned} B_{p-(\frac{8}{p})} &= B_{\alpha(p)f} \\ &\equiv plfC_{\alpha(p)}^{f-1} \pmod{p^2}. \end{aligned}$$

As $p \nmid l$ and $\gcd(f, p) = 1$ as $f \mid p-1$ or $f \mid p+1$ implies $p^2 \nmid plf$. Using $B_{\alpha(p)} \equiv 0 \pmod{p}$ in $C_{\alpha(p)}^2 = 8B_{\alpha(p)}^2 + 1$ we conclude that $C_{\alpha(p)} \equiv \pm 1 \pmod{p}$. Therefore, $B_{p-(\frac{8}{p})} \not\equiv 0 \pmod{p^2}$. □

Lemma 2.12 says that if p divides X_n (i.e., squarefree part of B_n) then p is a *balancing non-Wieferich prime.*

3. Main results.

Theorem 3.1. *If abc conjecture for the number field $\mathbf{Q}(\sqrt{2})$ is true and $k \geq 2$ is an integer then there are infinitely many primes p such that*

$$B_{p-(\frac{8}{p})} \not\equiv 0 \pmod{p^2} \quad \text{and} \quad p \equiv 1 \pmod{k}.$$

Proof. Let us denote the n-th prime number by p_n which is relatively prime to k. Also, write $B_{p_nk} = X_{p_nk}Y_{p_nk}$ where Y_{p_nk} is the powerful part of B_{p_nk}. Using the Binet formula for balancing numbers in (1.5) we have,

$$4\sqrt{2}B_{p_nk} - \alpha^{p_nk} + \beta^{p_nk} = 0.$$

Thus, the *abc* conjecture in (2.6) with $K = \mathbf{Q}(\sqrt{2})$, implies that, for any $\epsilon > 0$, there exists a constant C_ϵ such that

$$\begin{aligned} (3.1)\qquad & H(4\sqrt{2}B_{p_nk}, -\alpha^{p_nk}, \beta^{p_nk}) \\ &\leq C_\epsilon(\mathrm{rad}(4\sqrt{2}B_{p_nk}, -\alpha^{p_nk}, \beta^{p_nk}))^{1+\epsilon}. \end{aligned}$$

Now from the definition of the height in (2.4) we have,

$$\begin{aligned} & H(4\sqrt{2}B_{p_nk}, -\alpha^{p_nk}, \beta^{p_nk}) \\ &= \max\{|4\sqrt{2}B_{p_nk}|, |-\alpha^{p_nk}|, |\beta^{p_nk}|\} \end{aligned}$$

$$\begin{aligned} &\cdot \max\{|-4\sqrt{2}B_{p_nk}|, |-\beta^{p_nk}|, |\alpha^{p_nk}|\} \\ &\geq |4\sqrt{2}B_{p_nk}| \cdot |-4\sqrt{2}B_{p_nk}| \\ &= 32B_{p_nk}^2 = 32X_{p_nk}^2Y_{p_nk}^2. \end{aligned}$$

Since α and β are units, all prime factors are coming from $4\sqrt{2}B_{p_nk}$. Therefore, we have

$$\begin{aligned} \operatorname{rad}(4\sqrt{2}B_{p_nk}, -\alpha^{p_nk}, \beta^{p_nk}) &= \prod_{\mathfrak{p}|4\sqrt{2}B_{p_nk}} N(\mathfrak{p}) \\ &\leq 8X_{p_nk}^2Y_{p_nk}. \end{aligned}$$

Thus from (3.1),

$$32X_{p_nk}^2Y_{p_nk}^2 \leq C_\epsilon(8X_{p_nk}^2Y_{p_nk})^{1+\epsilon}$$

which implies that

$$4Y_{p_nk} \leq C_\epsilon(2X_{p_nk}^2Y_{p_nk})^\epsilon$$

and hence we conclude

$$Y_{p_nk} \ll_\epsilon B_{p_nk}^{2\epsilon}. \tag{3.2}$$

Let us take

$$X'_{p_nk} = \gcd(X_{p_nk}, \Phi_{p_nk}(\alpha/\beta))$$

and

$$Y'_{p_nk} = \gcd(Y_{p_nk}, \Phi_{p_nk}(\alpha/\beta)).$$

From (2.2), we have

$$\Phi_{p_nk}(\alpha/\beta) \mid \left(\left(\frac{\alpha}{\beta}\right)^{p_nk} - 1\right)\beta^{p_nk-1}.$$

Therefore,

$$\Phi_{p_nk}(\alpha/\beta) \mid \left(\left(\frac{\alpha}{\beta}\right) - 1\right)B_{p_nk} = \Phi_1(\alpha/\beta)X_{p_nk}Y_{p_nk}.$$

As $\gcd(\Phi_{p_nk}, \Phi_1) = 1$ and $\gcd(X_{p_nk}, Y_{p_nk}) = 1$, we conclude that

$$\Phi_{p_nk}(\alpha/\beta) \mid X_{p_nk} \text{ or } \Phi_{p_nk}(\alpha/\beta) \mid Y_{p_nk}.$$

Thus, $X'_{p_nk} = \Phi_{p_nk}(\alpha/\beta)$ and $Y'_{p_nk} = 1$ in the former case and $X'_{p_nk} = 1$ and $Y'_{p_nk} = \Phi_{p_nk}(\alpha/\beta)$ in the latter case. Therefore in any case we have

$$X'_{p_nk}Y'_{p_nk} = \Phi_{p_nk}(\alpha/\beta). \tag{3.3}$$

Using $\beta = 1/\alpha$ in (3.3) and then from Lemma 2.10,

$$|X'_{p_nk}Y'_{p_nk}| = |\Phi_{p_nk}(\alpha^2)| \gg C \cdot |\alpha|^{2\phi(p_nk)}.$$

Since $\{B_{p_nk}\}$ is a positive integer sequence and using Lemma 2.9,

$$X'_{p_nk}Y'_{p_nk} \gg C \cdot \alpha^{2\phi(p_nk)} \gg C \cdot B_{\phi(k)}^{2\phi(p_n)}. \tag{3.4}$$

Thus, from (3.2) and (3.4),

$$X'_{p_nk}B_{p_nk}^{2\epsilon} \gg X'_{p_nk}Y'_{p_nk} \gg C \cdot B_{\phi(k)}^{2\phi(p_n)},$$

which will be further simplified as

$$X'_{p_nk} \gg B_{\phi(k)}^{2(\phi(p_n)-\epsilon)}.$$

Choosing $\epsilon < \frac{1}{2}$ and using Proposition 2.6

$$X'_{p_nk} \gg B_{\phi(k)}^{2\phi(p_n)-1} \gg B_{\phi(k)}^{n\log n}. \tag{3.5}$$

But since we know that X'_{p_nk} is a product of distinct primes from (3.5) we get

$$\lim_{n\to\infty} \#\{\text{primes } p : p \mid X'_{p_ik}, i \leq n\} = \infty.$$

From (3.3) we know that if the prime p divides X'_{p_nk} then it also divides $\Phi_{p_nk}(\alpha/\beta)$ and hence by Proposition 2.5, p is congruent to 1 modulo p_nk. Also from Lemma 2.12, if p divides X_{p_nk} then $B_{p-(\frac{8}{p})} \not\equiv 0 \pmod{p^2}$. Thus there are infinitely many primes p such that $B_{p-(\frac{8}{p})} \not\equiv 0 \pmod{p^2}$ and $p \equiv 1 \pmod{k}$. □

Theorem 3.2. *Let $k > 2$ and $n > 1$ be the positive integers and also assume the abc conjecture for the number field $\mathbf{Q}(\sqrt{2})$. Then*

$$\begin{aligned} \#\{\text{primes } p \leq x : p &\equiv 1 \pmod{k}, \\ B_{p-(\frac{8}{p})} &\not\equiv 0 \pmod{p^2}\} \\ &\gg \frac{\log x}{\log\log x}. \end{aligned}$$

Proof. From the proof of Theorem 3.1, we get

$$X'_{p_nk} \leq B_{p_nk}.$$

We also know that if $p \mid X'_{p_nk}$ then $B_{p-(\frac{8}{p})} \not\equiv 0 \pmod{p^2}$. Now our aim is to count the number of primes p such that p divides $X'_{p_nk} \leq x$. To achieve this, we first show that for all large n, there exists a prime p such that $p \mid X'_{p_nk}$ but $p \nmid X'_{p_ik}$ for $i < n$.

Let us assume on the contrary, i.e., $p \mid X'_{p_nk}$ and $p \mid X'_{p_ik}$ for $i < n$. Then

$$X'_{p_nk} \leq \prod_{i=1}^{n-1} \gcd(X'_{p_ik}, X'_{p_nk}). \tag{3.6}$$

Also $X'_{p_ik} \mid B_{p_ik}$ and $X'_{p_nk} \mid B_{p_nk}$, thus

$$\gcd(X'_{p_ik}, X'_{p_nk}) \mid \gcd(B_{p_ik}, B_{p_nk}).$$

As the sequence of balancing numbers is a strongly divisible sequence, i.e., $\gcd(B_m, B_n) = B_{\gcd(m,n)}$ [3], we have $\gcd(X'_{p_ik}, X'_{p_nk}) \mid B_{\gcd(p_ik, p_nk)} = B_k$. Now from (3.6) and Lemma 2.9, we conclude that

$$X'_{p_nk} \le B_k^{n-1} < \alpha^{k(n-1)}. \tag{3.7}$$

Again from (3.5) and (3.7), we have

$$\alpha^{k(n-1)} > B_{\phi(k)}^{n\log n}.$$

Then using the lower bound of $B_{\phi(k)}$ given in Lemma 2.9,

$$\alpha^{k(n-1)} > \alpha^{(\phi(k)-1)n\log n}. \tag{3.8}$$

Taking logarithm on both sides of (3.8), we get $kn > k(n-1) > (\phi(k)-1)n\log n$, i.e.,

$$\frac{k}{\phi(k)-1} > \log n$$

which is not true for large n. Therefore, if $p \mid X'_{p_nk}$ then $p \nmid X'_{p_ik}$ for $i < n$.

As $p \mid X'_{p_nk}$ and $X'_{p_nk} < B_{p_nk} \le x$,

$$\begin{aligned}\#\{\text{primes } p \le X'_{p_nk} : p \equiv 1 \pmod{k},\\ B_{p-(\frac{8}{p})} \not\equiv 0 \pmod{p^2}\}\\ \gg n.\end{aligned}$$

Also it is easy to realize that,

$$\max\{n \ge 0 : \alpha^n \le x\} \ge \frac{\log x}{\log(\alpha)}.$$

Since $B_k < \alpha^k$, therefore the largest n such that $B_{p_nk} \le x$ is $\ll \frac{\log x}{\log\log x}$. Thus,

$$\begin{aligned}\#\{\text{primes } p \le x : p \equiv 1 \pmod{k},\\ B_{p-(\frac{8}{p})} \not\equiv 0 \pmod{p^2}\}\\ \gg \frac{\log x}{\log\log x}.\end{aligned} \qquad \square$$

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