Fano manifolds with nef tangent bundle and large Picard number

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Abstract: We study Fano manifolds with nef tangent bundle and large Picard number.

Key words: Fano manifold; nef tangent bundle; homogeneous manifold; large Picard number.

1. Introduction. The classical Gauss-Bonnet Theorem implies that the only compact Riemann surface with positive curvature is the Riemann sphere. In the higher dimensional case, the Frankel conjecture claims that a compact Kähler manifold with positive bisectional curvature is the projective space. This conjecture was solved by Mori [12] and Siu-Yau [20], independently. Mori's proof is purely algebraic and he obtained a more general result. In fact, he solved the Hartshorne conjecture, which says that the projective space is the only projective manifold with ample tangent bundle [12]. After that, in complex geometry, Mok proved the generalized Frankel conjecture on compact Kähler manifolds with semipositive bisectional curvature [11]. As a generalization of their works, complex projective manifolds with nef tangent bundle have been studied by many authors (for instance, see [14]). By the result of Demailly, Peternell and Schneider [4], the study can be reduced to the case of Fano manifolds. Let us recall the following conjecture posed by Campana and Peternell.

Conjecture 1.1 ([2]). *Any Fano manifold with nef tangent bundle is rational homogeneous.*

This conjecture holds if the dimension is at most four [6], and this is also true for five-folds whose Picard number greater than one [21]. Recently Kanemitsu [9] proved the above conjecture for five-folds of Picard number one. In this paper, we will generalize a result of [21] to the higher dimensional case. Our main result is

Theorem 1.2. *Let X be a Fano manifold with nef tangent bundle. Let m be the dimension, n the Picard number and i_X the pseudoindex of X. Then we have*

(GM): $n(i_X - 1) \leq m$.

Furthermore, X is rational homogeneous if one of the following holds:

(1) $m \leq n(i_X - 1) + 1$.

(2) $m \leq n + 3$.

In general, it is expected that the above inequality (GM) holds for any Fano manifold, which is the so-called Generalized Mukai conjecture [1]. It is easy to prove this inequality for Fano manifolds with nef tangent bundle. So the main part of this paper is to prove the homogeneity under the above assumptions (1) and (2).

While preparing this note, Akihiro Kanemitsu informed the author that he also proved the rational homogeneity for the case of (2) in Theorem 1.2 independently and was preparing a manuscript for publication. By using his result [9], he also proved the rational homogeneity for the case of $m = n + 4$ very recently [23].

2. Preliminaries. Throughout this paper, we work over the field of complex numbers. In this section, we set up our notation and recall some results on Fano manifolds.

2.1. Results on Fano manifolds. A *projective manifold* means a smooth projective variety. A *Fano manifold* is a projective manifold whose anticanonical divisor $-K_X$ is ample. Given a Fano manifold X, we will denote by $N_1(X)$ the vector space of 1-cycles in X with real coefficients, modulo numerical equivalence. The dimension of this vector space, that we denote by ρ_X, is called the *Picard number* of X. The *Kleiman-Mori cone* of X is defined as the closure $\overline{\mathrm{NE}}(X)$ of the convex cone

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generated by effective 1-cycles. On the other hand, a surjective morphism with connected fibers $f: X \to Y$ to a normal projective variety Y is called a *contraction* of X. A contraction f is said to be of *fiber type* if $\dim(X) > \dim(Y)$. By the Contraction Theorem, given an extremal face F of $\overline{\mathrm{NE}}(X)$, there exists a contraction $\varphi_F : X \to Y$ satisfying that, for every irreducible curve $C \subset X$, the numerical class of C is in $F \subset \overline{\mathrm{NE}}(X)$ if and only if $\varphi_F(C)$ is a point. A contraction φ_F is called *elementary* if the corresponding face F is one dimensional, i.e. if F is an extremal ray.

For a Fano manifold X, the *pseudoindex* i_X is defined as

$$i_X := \min\{-K_X \cdot C \mid C \subset X \text{ rational curve}\}.$$

The pseudoindex is upper bounded for any Fano manifold and the extremal cases are classified:

Theorem 2.1 ([3], [10]). *Let X be a Fano manifold of dimension $m \geq 2$. Then $i_X \leq m+1$. Furthermore,*

(1) *if $i_X = m+1$, then X is a projective space $\mathbf{P}^m$*;
(2) *if $i_X = m$, then X is a smooth quadric hypersurface Q^m.*

The following theorem is also used in this paper.

Theorem 2.2 ([8, Main Theorem]). *Let $f: X \to Y$ be a surjective morphism from a rational homogeneous manifold of $\rho = 1$ to a projective manifold of positive dimension. Then Y is isomorphic to $\mathbf{P}^m$ or X.*

2.2. Fano manifolds with nef tangent bundle. Here we overview some results on Fano manifolds with nef tangent bundle. We begin by defining the notion of CP-manifolds and FT-manifolds.

Definition 2.3 (See [13, Definition 1]). A Fano manifold is a *CP-manifold* if it has a nef tangent bundle. Especially, a CP-manifold is said to be an *FT-manifold* if any elementary contraction is a $\mathbf{P}^1$-fibration.

Let us recall some classification results of CP-manifolds and FT-manifolds.

Theorem 2.4. *Let X be a CP-manifold of Picard number 1. Then the following holds*:

(1) *if $i_X = 2$, then X is $\mathbf{P}^1$*;
(2) *if $i_X = 3$, then X is $\mathbf{P}^2, Q^3$ or $K(G_2)$, where $K(G_2)$ is the 5-dimensional contact homogeneous manifold of type G_2 (see* [14, Example 2.17]*).*

Proof. The first statement is well known. For instance, see [14, Proposition 2.10 (5)]. The second statement was proved in [7, pp. 623–624]. One can also find an alternative proof in [22, Theorem 1.3]. □

Theorem 2.5 (See [21, Theorem 3.1, Theorem 4.1]). *Let X be a CP-manifold of dimension $m \leq 5$ and Picard number n. If $m \leq 4$, or $m = 5$ and $n > 1$, then X is rational homogeneous. In particular, if $n = 2$, then X is one of the following $(\mathbf{P}^2)^2$, $\mathbf{P}^2 \times \mathbf{P}^3$, $\mathbf{P}^2 \times Q^3$, $\mathbf{P}(T_{\mathbf{P}^3})$, $\mathbf{P}(\mathcal{S}_i)$ or $Y \times H$, where $\mathcal{S}_i$ $(i = 1, 2)$ are the two spinor bundles on Q^4 and H is an FT-manifold.*

Remark 2.6. For the details of spinor bundles $\mathcal{S}_i$, we refer the reader to [21, Example 3.7]. We remark that $\mathbf{P}(\mathcal{S}_1)$ and $\mathbf{P}(\mathcal{S}_2)$ are isomorphic to the flag manifold $F(1, 2, \mathbf{P}^3)$ parametrizing pairs (l, P), where l is a line in a plane $P \subset \mathbf{P}^3$. Hence $\mathbf{P}(\mathcal{S}_1) = \mathbf{P}(\mathcal{S}_2)$ admits not only a $\mathbf{P}^1$-bundle structure over Q^4 but also a $\mathbf{P}^2$-bundle structure over $\mathbf{P}^3$.

Theorem 2.7 ([17, Theorem 1.2]). *Let X be a Fano manifold whose elementary contractions are smooth $\mathbf{P}^1$-fibrations. Then X is isomorphic to a complete flag manifold G/B, where G is a semisimple algebraic group and B a Borel subgroup. In particular, any FT-manifold is rational homogeneous.*

Proposition 2.8 ([13, Proposition 5]). *Let X be a CP-manifold admitting a contraction $f: X \to Y$ onto an FT-manifold Y. Then there exists a projective manifold Z such that $X \cong Y \times Z$.*

The next result tells us that the nefness of the tangent bundle imposes strong restrictions on Fano manifolds of Picard number greater than one.

Proposition 2.9 ([13, Proposition 4]). *Let X be a CP-manifold. Then the following properties hold*:

(1) *Every contraction $\pi: X \to Y$ is smooth and, moreover, its image Y and every fiber $\pi^{-1}(y)$ are CP-manifolds.*
(2) *For every contraction $\pi: X \to Y$, the Picard number of a fiber $\pi^{-1}(y)$ equals $\rho_X - \rho_Y$. Moreover, being $j: \pi^{-1}(y) \to X$ the inclusion and $j_*: \mathrm{N}_1(\pi^{-1}(y)) \to \mathrm{N}_1(X)$ the induced linear map, we have $j_*(\mathrm{NE}(\pi^{-1}(y))) = \mathrm{NE}(X) \cap j_*(\mathrm{N}_1(\pi^{-1}(y)))$.*
(3) *The Kleiman-Mori cone $\mathrm{NE}(X)$ is simplicial.*

Notation 2.10. Along the rest of this paper, we always assume that X is a CP-manifold of dimension m and Picard number n. We will denote by R_i, $i = 1, \dots, n$ its extremal rays, and by Γ_i a rational curve of minimal degree such that $[\Gamma_i] \in R_i$. If I is any subset of $D := \{1, \ldots, n\}$ we will denote by R_I the extremal face spanned by the rays R_i such that $i \in I$, by $\pi_I : X \to X_I$ the corresponding extremal contraction. We will also denote by $\pi^I : X \to X^I$ the contraction of the face R^I spanned by the rays R_i such that $i \in D \setminus I$. For $I \subset J \subset D$ we will denote the contraction of the extremal face $\pi_{I*}(R_J) \subset \mathrm{N}_1(X_I)$ by $\pi_{I,J} : X_I \to X_J$ or by $\pi^{D\setminus I, D\setminus J} : X^{D\setminus I} \to X^{D\setminus J}$. The fiber of $\pi_{I,J}$ is denoted by $F_{I,J}$ or $F^{D\setminus I, D\setminus J}$. If I is empty, the fiber of $\pi_{I,J}$ will also be denoted by F_J or $F^{D\setminus J}$.

3. Proof of Theorem 1.2. Here we introduce two invariants for CP-manifolds.

Definition 3.1. Given a CP-manifold as in 2.10, we define two invariants as follows:

$$f(X) := \sum_{i=1}^{n} \dim F_i, \;\; s(X) := \sum_{i=1}^{n} (-K_X \cdot \Gamma_i - 1).$$

Lemma 3.2. *Let X be a CP-manifold as in 2.10. For a partition $D = I_1 \sqcup \cdots \sqcup I_l$, we set $J_k := I_1 \cup \cdots \cup I_k$. Then we have the following*

(1) *The restriction of $\pi_{J_{k-1}}$ defines a finite morphism $F_{I_k} \to F_{J_{k-1},J_k}$.*

(2) *We have inequalities*

- $m = \sum_{k=1}^{l} \dim F_{J_{k-1},J_k} \geq \sum_{k=1}^{l} \dim F_{I_k}$, and
- $m \geq f(X) \geq s(X) \geq n(i_X - 1)$.

Proof. Let us consider the commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & X_{I_k} \\ \downarrow & & \downarrow \\ X_{J_{k-1}} & \longrightarrow & X_{J_k}. \end{array}$$

Then, by Proposition 2.9 (2), we have a finite morphism $F_{I_k} \to F_{J_{k-1},J_k}$. In particular, $\dim F_{I_k} \leq \dim F_{J_{k-1},J_k}$. On the other hand, by Proposition 2.9 (1), we have

$$m = \sum_{k=1}^{l} \dim F_{J_{k-1},J_k}.$$

Hence we obtain the first inequality as desired. In particular, we have $m \geq f(X)$. The remaining part follows from Theorem 2.1. □

Proposition 3.3. *Let X be a CP-manifold as in 2.10. Then*

$$n(i_X - 1) \leq m$$

with equality if and only if X is isomorphic to $(\mathbf{P}^{i_X-1})^n$.

Proof. The inequality follows from Lemma 3.2. To prove the latter part, we assume that $m = n(i_X - 1)$. Then Lemma 3.2 tells us that $\dim F_i + 1 = -K_X \cdot \Gamma_i = i_X$ for any i. Let V^i be a family of rational curves on X containing Γ_i, which is unsplit and covering by the minimality of $-K_X \cdot \Gamma_i$ and the nefness of the tangent bundle of X. By Proposition 2.9 (3), the numerical classes of $V^1, \cdots, V^n$ are linearly independent in $N_1(X)$. Applying [16, Theorem 1.1], we see that X is isomorphic to $(\mathbf{P}^{i_X-1})^{\rho_X}$. □

Remark 3.4. The inequality $n(i_X - 1) \leq m$ also follows from Proposition 2.9 (3) and [1, Corollaire 5.3].

By Lemma 3.2, we have the following

Lemma 3.5. *Let X be a CP-manifold as in 2.10. Assume that $m = n(i_X - 1) + 1$. Then we have*

$$(f(X), s(X)) = (m, m), (m, m-1) \text{ or } (m-1, m-1).$$

Proposition 3.6. *Let X be a CP-manifold as in 2.10. Assume that $m = f(X) = s(X) = n(i_X - 1) + 1$. Then X is isomorphic to $(\mathbf{P}^{i_X-1})^{n-1} \times \mathbf{P}^{i_X}$.*

Proof. By Lemma 3.2, there exists an integer $s \in D$ such that

$$-K_X \cdot \Gamma_i = \begin{cases} i_X & (i \neq s) \\ i_X + 1 & (i = s) \end{cases}.$$

Applying the same argument as in the proof of Proposition 3.3, X is isomorphic to $(\mathbf{P}^{i_X-1})^{n-1} \times \mathbf{P}^{i_X}$. □

Proposition 3.7. *Let X be a CP-manifold as in 2.10. Assume that $m = f(X) = s(X) + 1 = n(i_X - 1) + 1$. Then X is isomorphic to $(\mathbf{P}^{i_X-1})^{n-1} \times Q^{i_X}$.*

Proof. We proceed by induction on n. If $n = 1$, then it follows from Theorem 2.1 that X is isomorphic to Q^{i_X}. So suppose our assertion for $n - 1$.

By Lemma 3.2, we have $-K_X \cdot \Gamma_i = i_X$ for any i, and

$$\dim F_i = \begin{cases} i_X - 1 & (i \neq s) \\ i_X & (i = s) \end{cases}$$

for some $s \in D$. Without loss of generality, we may assume that $s = n$. By Theorem 2.1, $F_i \cong \mathbf{P}^{i_X-1} (i \neq n)$ and $F_n \cong Q^{i_X}$. Set $I_k := \{1, 2, \cdots, k\} \subset D$. By Lemma 3.2, we have $\dim F_k = \dim F_{I_{k-1},I_k}$. Since $\pi_{I_{k-1}}|_{F_k} : F_k \to F_{I_{k-1},I_k}$ is a finite morphism, it is surjective. Applying Theorem 2.2, $X^n = F_{I_{n-1},I_n}$ is isomorphic to $\mathbf{P}^{i_X}$ or Q^{i_X}, and F_{I_{k-1},I_k} is isomorphic to $\mathbf{P}^{i_X-1}$ for any $k \neq n$. In a similar way, we see that X^1 is isomorphic to $\mathbf{P}^{i_X-1}$. Since X^n is rational, its Brauer group is trivial. Hence $\pi_{I_{n-2},I_{n-1}} : X_{I_{n-2}} \to X_{I_{n-1}} = X^n$ is a projective bundle (see, for instance, [21, Proposition 2.5]). This implies that $X_{I_{n-2}}$ is also rational. By applying this argument repeatedly to $\pi_{I_{k-1},I_k} : X_{I_{k-1}} \to X_{I_k}$, we see that $\pi_1 : X \to X_1$ is a $\mathbf{P}^{i_X-1}$-bundle. Since we have seen that X^1 is isomorphic to $\mathbf{P}^{i_X-1}$, [15, Lemma 4.1] concludes that $X \cong \mathbf{P}^{i_X-1} \times F^1$. From the induction hypothesis, we see that X is isomorphic to $(\mathbf{P}^{i_X-1})^{n-1} \times Q^{i_X}$. □

Proposition 3.8. *Let X be a CP-manifold as in* 2.10. *Assume that $m = f(X) + 1 = s(X) + 1 = n(i_X - 1) + 1$. Then X is isomorphic to $\mathbf{P}(T_{\mathbf{P}^{i_X}}) \times (\mathbf{P}^{i_X-1})^{n-2}$.*

Proof. We proceed by induction on n. Under our assumption, Theorem 2.1 tells us that $n > 1$. If $n = 2$, then our assertion follows from [18, Theorem 2] directly. We assume that $n > 2$. Then we have

$$-K_X \cdot \Gamma_i = i_X \text{ and } \dim F_i = -K_X \cdot \Gamma_i - 1 \text{ for any } i.$$

Hence, by Theorem 2.1, F_i is isomorphic to $\mathbf{P}^{i_X-1}$. Set $I_k := \{i_1, i_2, \cdots, i_k\} \subset D$ ($i_s \neq i_t$ if $s \neq t$). Then there exists $s \in D$ such that

$$\dim F_{I_k,I_{k+1}} = \begin{cases} i_X - 1 & (k \neq s) \\ i_X & (k = s). \end{cases}$$

Since $F_{I_{n-1},I_n} = X^{i_n}$, we see that $\dim X^i = i_X$ or $i_X - 1$ for any i. From the commutative diagram

$$\begin{array}{ccc} X_{I_s} & \longrightarrow & X_{I_{s+1}} \\ \downarrow & & \downarrow \\ X^{s+1} & \longrightarrow & \{*\}\,, \end{array}$$

we obtain a finite morphism $F_{I_s,I_{s+1}} \to X^{s+1}$. Since $\dim F_{I_s,I_{s+1}} = i_X$, we see that $\dim X^{s+1} = i_X$. By reordering, we may assume $\dim X^1 = i_X$. We claim that there exits $i \neq 1$ such that $\dim X^i = i_X$. To prove this, we set $i_j := j$ for any j. Then we may find $s' \in D$ such that $\dim F_{I_{s'},I_{s'+1}} = i_X$, and this implies that $\dim X^{s'+1} = i_X$. Consequently, by reordering again, we may assume $\dim X^1 = \dim X^2 = i_X$. Let us prove that X is isomorphic to $\mathbf{P}(T_{\mathbf{P}^{i_X}}) \times (\mathbf{P}^{i_X-1})^{n-2}$.

It follows from [18, Theorem 2] that $X^{1,2}$ is isomorphic to $\mathbf{P}(T_{\mathbf{P}^{i_X}})$. Then $X^{1,2,3} \to X^{1,2}$ is a smooth $\mathbf{P}^{i_X-1}$-fibration. Since the Brauer group of $\mathbf{P}(T_{\mathbf{P}^{i_X}})$ is trivial, there exists a vector bundle $\mathcal{E}$ of rank i_X on $X^{1,2}$ such that $X^{1,2,3} \to X^{1,2}$ is a $\mathbf{P}^{i_X-1}$-bundle $\mathbf{P}(\mathcal{E}) \to X^{1,2}$. On the other hand, $\mathbf{P}(\mathcal{E}|_{F^{\{1,2\},j}})$ is a CP-manifold which satisfies the assumption as in Proposition 3.3 for $j = 1, 2$. So it is isomorphic to $(\mathbf{P}^{i_X-1})^2$. Hence we may assume $\mathcal{E}|_{F^{\{1,2\},j}} \cong \mathcal{O}_{\mathbf{P}^{i_X-1}}^{i_X}$ for $j = 1, 2$. By applying Grauert's theorem [5, III. Corollary 12.9], we see that $(\pi^{\{1,2\},1})_*(\mathcal{E})$ is a rank i_X vector bundle on $X^1 \cong \mathbf{P}^{i_X}$, and for any point $x \in F^{\{1,2\},1}$ we have $(\pi^{\{1,2\},1})^*((\pi^{\{1,2\},1})_*(\mathcal{E})) \otimes k(x) \cong H^0(F^{\{1,2\},1}, \mathcal{E}|_{F^{\{1,2\},1}})$. Then the natural map

$$(\pi^{\{1,2\},1})^*((\pi^{\{1,2\},1})_*(\mathcal{E})) \to \mathcal{E}$$

is surjective. As a consequence, we see that

$$(\pi^{\{1,2\},1})^*((\pi^{\{1,2\},1})_*(\mathcal{E})) \cong \mathcal{E}.$$

By restricting this isomorphism to $F^{\{1,2\},2}$, we have

$$((\pi^{\{1,2\},1})_*(\mathcal{E}))|_{\pi^{\{1,2\},1}(F^{\{1,2\},2})} \cong \mathcal{E}|_{F^{\{1,2\},2}}.$$

Since $\pi^{\{1,2\},1}(F^{\{1,2\},2})$ is a hyperplane in $X^1 \cong \mathbf{P}^{i_X}$, the Horrocks's splitting criterion and Van de Ven's Theorem (for example see [19, Chap. 1. Theorem 2.3.2, Chap. 2. Theorem 2.2.2]) conclude that $(\pi^{\{1,2\},1})_*(\mathcal{E})$ is trivial, and thus so is $\mathcal{E}$. This implies $X^{1,2,3} \cong \mathbf{P}(T_{\mathbf{P}^{i_X}}) \times \mathbf{P}^{i_X-1}$. In particular, we see that X^3 is isomorphic to $\mathbf{P}^{i_X-1}$. By applying [15, Lemma 4.1], we see that $X \cong \mathbf{P}^{i_X-1} \times Z$ for some CP-manifold Z. By the induction hypothesis, our assertion holds. □

Summing up, we have proved the rational homogeneity for the case of (1) in Theorem 1.2. Finally we complete the proof of Theorem 1.2. In fact, we shall show the following

Theorem 3.9. *Let X be a CP-manifold as in* 2.10. *Then X is a rational homogeneous manifold provided that $m \leq n + 3$.*

Proof. We proceed by induction on n. By Theorem 2.5, we may assume that $m \geq 6$. Then it turns out that $n \geq 3$, because we have $n + 3 \geq m$. From now on, we assume that our assertion holds when the Picard number is less than n. Set $m = n + r$ $(0 \leq r \leq 3)$.

If X dominates an FT-manifold Y, then it follows from Proposition 2.8 that X is isomorphic to $Y \times Z$ for some CP-manifold Z. Since we have $\dim Z \leq \rho_Z + r$ and $\rho_Z < n$, we see that Z is rational homogeneous by the induction hypothesis. Hence we may assume that X does not dominate any FT-manifold.

By Lemma 3.2, for any j, k, l, we have

$$m = n + r \geq \dim X^{j,k,l} + (n-3)$$
$$\geq \dim X^{k,l} + (n-2).$$

This implies that

$$\dim X^{k,l} \leq r + 2 \leq 5 \text{ and } \dim X^{j,k,l} \leq r + 3 \leq 6.$$

Since X does not dominate any FT-manifold, it follows from Theorem 2.5 that $X^{k,l}$ is one of the following

$$(\mathbf{P}^2)^2, \mathbf{P}^2 \times \mathbf{P}^3, \mathbf{P}^2 \times Q^3, \mathbf{P}(T_{\mathbf{P}^3}) \text{ or } \mathbf{P}(\mathcal{S}_i).$$

If $\dim X^{k,l} = 5$ for any two distinct integers $k, l \in \{1, 2, 3\}$, we see that $X^{1,2,3}$ is an FT-manifold. This is a contradiction. Hence there exists $k, l \in \{1, 2, 3\}$ such that $X^{k,l}$ is isomorphic to $(\mathbf{P}^2)^2$. Without loss of generality, we may assume that $X^{1,2} \cong (\mathbf{P}^2)^2$. Since X^1 and X^2 are $\mathbf{P}^2$, $X^{1,3}$ and $X^{2,3}$ are isomorphic to $\mathbf{P}^2 \times V$, where V is $\mathbf{P}^2$, $\mathbf{P}^3$ or Q^3. We claim that V is $\mathbf{P}^2$. Assume by contradiction that V is $\mathbf{P}^3$ or Q^3. Then we see that $F^{\{1,2,3\},\{1,2\}} \cong \mathbf{P}^2$ and $F^{\{1,2,3\},\{i,3\}} \cong \mathbf{P}^1$ for $i = 1, 2$. Then a CP-manifold $F^{\{1,2,3\},\{1\}}$ admits a smooth $\mathbf{P}^1$-fibration structure over V and a smooth $\mathbf{P}^2$-fibration structure over $\mathbf{P}^2$. However this contradicts Theorem 2.5. Hence V is $\mathbf{P}^2$.

Now $X^{1,2,3}$ has three smooth $\mathbf{P}^2$-fibration structures over $(\mathbf{P}^2)^2$. By [21, Proposition 2.5], these fibrations are nothing but projective bundles. Applying [15, Lemma 4.1], we see that $X^{1,2,3}$ is isomorphic to $(\mathbf{P}^2)^3$. If $n > 3$, the same argument implies that $X^{j,k,l} \cong (\mathbf{P}^2)^3$ for any $j, k, l \in \{1, 2, 3\}$. Then $X^{1,2,3,4}$ is an FT-manifold. This is a contradiction. Hence X is isomorphic to $(\mathbf{P}^2)^3$. □

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