Admissible representations, multiplicity-free representations and visible actions on non-tube type Hermitian symmetric spaces

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Abstract: This paper presents new characterization for a non-compact Hermitian symmetric space G/K to be of tube type (or non-tube type) by multiplicities in some branching laws and visible actions. The study in this paper gives an example of a kind of the Cartan decomposition for non-symmetric homogeneous spaces.

Key words: Multiplicity-free representation; admissible representation; visible action; (real) spherical variety; Hermitian symmetric space; tube type domain.

1. Introduction. Let us begin this paper with two (apparently, quite different) facts on the relationship between multiplicities of representations and geometry.

The first fact is proved by T. Kobayashi and T. Oshima in [16]. Let $\widehat{G}_{\mathrm{ad}}$ be the set of equivalence classes of irreducible admissible representations of a real reductive Lie group G. Here, a representation π of G is admissible if $\dim \mathrm{Hom}_K(\mu, \pi|_K) < \infty$ for any irreducible representation μ of a maximal compact subgroup K of G. For a pair $G \supset H$ of algebraic reductive groups, the homogeneous space G/H is real spherical if the dimension of intertwiners for any irreducible admissible representation $\pi \in \widehat{G}_{\mathrm{ad}}$ into the space $C^\infty(G/H)$ of continuous functions on G/H is finite, namely, $\dim \mathrm{Hom}_G(\pi, C^\infty(G/H)) < \infty$, and vice versa ([16, Theorem A]). Here, G/H is *real spherical* if there exists an open P-orbit in G/H where P is a minimal parabolic subgroup of G ([9]). Moreover, the complexification $G_{\mathbf{C}}/H_{\mathbf{C}}$ of G/H is *spherical*, namely, $G_{\mathbf{C}}/H_{\mathbf{C}}$ has an open Borel orbit, if and only if the multiplicity is uniformly bounded in the sense of $\sup_{\pi \in \widehat{G}_{\mathrm{ad}}} \dim \mathrm{Hom}(\pi, C^\infty(G/H)) < \infty$ ([16, Theorem B]).

The second fact is concerned with the complex geometry. Let H be a Lie group. The space $\mathcal{O}(D, \mathcal{V})$ of holomorphic sections of an H-equivariant Hermitian holomorphic vector bundle $\mathcal{V} \to D$ over a complex manifold D defines a continuous representation of H with respect to the Fréchet topology. Let $\mathcal{H}$ be a unitary representation of H which is realized in $\mathcal{O}(D, \mathcal{V})$, namely, there exists a continuous and injective H-homomorphism from the Hilbert space $\mathcal{H}$ into $\mathcal{O}(D, \mathcal{V})$. Now, we consider a general setting where the H-action on D is not transitive, and also a basic question when $\mathcal{H}$ is multiplicity-free. In general, the property of multiplicity-freeness of $\mathcal{H}$ is not fulfilled even though each fiber $\mathcal{V}_x$ $(x \in D)$ is multiplicity-free as a representation of the isotropy subgroup H_x. However, this does hold if H acts on the base space D in a strongly visible fashion in the sense of [11]. We say that this theory is *propagation theory of multiplicity-freeness* which is established by T. Kobayashi (see [12,17]). A part of the idea of proof goes back to Gelfand–Kazhdan, S. Kobayashi [7], and Faraut–Thomas [4].

Among irreducible bounded symmetric domain, there are two types: Hermitian symmetric spaces of tube type; Hermitian symmetric spaces of non-tube type. The Hermitian symmetric spaces $G/K = SU(n,n)/S(U(n) \times U(n))$, $SO^*(4n)/U(2n)$, $SO_0(n,2)/(SO(n) \times SO(2))$, $Sp(n,\mathbf{R})/U(n)$, and $E_{7(-25)}/(E_6 \cdot \mathbf{T})$ are of tube type, whereas $G/K = SU(p,q)/S(U(p) \times U(q))$ with $p \neq q$, $SO^*(4n+2)/U(2n+1)$, and $E_{6(-14)}/(Spin(10) \cdot \mathbf{T})$ are of non-tube type. We shall see in Theorem 1 below that aforementioned two theories applied to the associated Stein manifolds $G_{\mathbf{C}}/[K_{\mathbf{C}}, K_{\mathbf{C}}]$ will reveal sharp differences between tube and non-tube types, giving new characterization of tube type domains from the

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viewpoint of multiplicities in some branching laws and also from the viewpoint of visibility of holomorphic actions.

2. Visible actions on complex manifolds. Let us review from [11] (see also [12]) the notion of strongly visible actions on complex manifolds. A holomorphic action of a Lie group H on a connected complex manifold D is called *strongly visible* if there exist a real submanifold S in D and an anti-holomorphic diffeomorphism σ on D satisfying the following conditions:

(V.1) S meets every H-orbit in D,

(S.1) $\sigma|_S = \mathrm{id}_S$,

(S.2) σ preserves each H-orbit in D.

We say that the submanifold S is a *slice*. The slice S is automatically a totally real submanifold ([12, Remark 3.3.2]).

We allow that S meets every H-orbit twice and more than twice, namely, S is not necessary a complete representative of H-orbits in D.

In Kobayashi's original definition [12, Definition 3.3.1], the concept of strongly visible actions is slightly wider, namely, he calls that this action is strongly visible if a complex manifold D contains an open set satisfying the conditions (V.1)–(S.2). For an application to multiplicity-free representations, this wider definition is sufficient. However, for simplicity, we adopt the narrower one throughout this paper.

3. Multiplicity-freeness and visible action. Taking a pair $G_u \supset K$ of compact Lie groups as an example, the theory of visible actions gives a geometric explanation for multiplicity-free representations as follows: We want to understand which irreducible representation μ of K the multiplicity-freeness holds in the sense that $\dim \mathrm{Hom}_K(\mu, \lambda|_K) \leq 1$ for any irreducible representation λ of G_u. By the Frobenius reciprocity, this dimension is nothing but the one of intertwiners from λ to the space $\mathcal{O}(D, \mathcal{V})$ of holomorphic sections for the $G_{\mathbf{C}}$-equivariant Hermitian holomorphic vector bundle $\mathcal{V} = G_{\mathbf{C}} \times_{K_{\mathbf{C}}} \mu$ on $D = G_{\mathbf{C}}/K_{\mathbf{C}}$. Then, the multiplicity-freeness holds if the G_u-action on D is strongly visible and μ is multiplicity-free as a representation of M, where M is the stabilizer of a generic element of a slice for the strongly visible G_u-action on D. If (G_u, K) is a symmetric pair, then a slice can be taken as the A-orbit under the Cartan decomposition $G_{\mathbf{C}} = G_u A K_{\mathbf{C}}$ for symmetric $G_{\mathbf{C}}/K_{\mathbf{C}}$. Thus, M is the centralizer of A in K.

Not only for finite-dimensional representations of a compact Lie group but also for infinite-dimensional representations of a non-compact real form, we give an explanation of the multiplicity-freeness by the complex geometric viewpoint. In fact, by switching a compact real form G_u by a non-compact one $G_{\mathbf{R}}$ in the above example, we can show that the Hilbert space $L^2(G_{\mathbf{R}}/K, \mu)$ of square integrable sections on the non-compact $G_{\mathbf{R}}/K$ is multiplicity-free as a representation of $G_{\mathbf{R}}$.

4. Characterization of tube type Hermitian symmetric spaces. We are ready to state our main results of this paper.

Let G/K be a non-compact irreducible Hermitian symmetric space. Then, K has a one-dimensional center, and hence the commutator subgroup $K^s := [K, K]$ is of codimension one in K. Therefore, the homogeneous space G/K^s is not a symmetric space. We note that the complexified $G_{\mathbf{C}}/K^s_{\mathbf{C}}$ of G/K^s is a Stein manifold by Matsushima's theorem.

Our main result characterizes tube type (or non-tube type) among Hermitian symmetric spaces by visible actions, and also by multiplicities in branching laws, and is stated as follows:

Theorem 1. *The following six conditions are equivalent for a non-compact irreducible Hermitian symmetric space G/K:*

(i) *G/K is of non-tube type.*

(ii) *$G_{\mathbf{C}}/K^s_{\mathbf{C}}$ is spherical.*

(iii) *The action of a compact real form G_u of $G_{\mathbf{C}}$ on $G_{\mathbf{C}}/K^s_{\mathbf{C}}$ is strongly visible.*

(iv) *The K^s-action on the Hermitian symmetric space G/K is strongly visible.*

(v) *The restriction $\pi|_{K^s}$ is K^s-admissible for a (equivalently, for any) holomorphic discrete series representation π of G.*

(vi) *For a (equivalently, for any) holomorphic discrete series representation π of G of scalar type, the restriction $\pi|_{K^s}$ is multiplicity-free.*

Concerning to (v) of Theorem 1, it follows from the corollary of [10, Theorem 2.4.5] that the restriction $\pi|_{K^s}$ is K^s-admissible, namely, the irreducible decomposition of $\pi|_{K^s}$ contains only discrete spectra and $\dim \mathrm{Hom}_{K^s}(\mu, \pi|_{K^s}) < \infty$ holds for any $\mu \in \widehat{K^s}$.

Our strategy of the proof of Theorem 1 is as follows: Krämer's classification of spherical affine irreducible complex homogeneous spaces [18] shows the equivalence (i) $\Leftrightarrow$ (ii). The equivalence between

(i) and (iii) is proved in [19]. We discuss the equivalence (i) $\Leftrightarrow$ (v) in Section 4.1; the implication (vi) $\Rightarrow$ (i) in Section 4.2; (i) $\Rightarrow$ (iv) in Section 4.3; and (iv) $\Rightarrow$ (vi) in Section 4.4. We summarize the strategy as follows:

$$\begin{tikzcd}
 & \text{(iii)} \arrow[d, Leftrightarrow, "{[19]}"'] & \\
\text{(ii)} \arrow[r, Leftrightarrow, "{[18]}"] \arrow[dr, Rightarrow, "\S4.5"'] & \text{(i)} \arrow[r, Rightarrow, "\S4.3"] \arrow[d, Leftrightarrow, "\S4.1"'] & \text{(iv)} \arrow[d, Rightarrow, "\S4.4"] \\
 & \text{(v)} & \text{(vi)} \arrow[ul, Rightarrow, "\S4.2"]
\end{tikzcd}$$

4.1. Proof of (i) $\Leftrightarrow$ (v). Our proof of the equivalence (i) $\Leftrightarrow$ (v) is based on Kobayashi's criterion for admissible restrictions of representations [10,13], namely, criterion for discretely decomposability of the restriction with finite-multiplicities. We remark that the relation between (i) and (v) was announced by Duflo–Vargas [1].

First, we summarize his criterion briefly, see [13, Section 6.2]. Let $\mathfrak{k}_0$ be the Lie algebra of a compact Lie group K. We fix a maximal torus $\mathfrak{t}_0$ of $\mathfrak{k}_0$ and a positive system $\Delta^+(\mathfrak{k}_0, \mathfrak{t}_0)$. We write $C_+ \subset \sqrt{-1}\mathfrak{t}_0^*$ for the corresponding closed Weyl chamber. We regard the unitary dual $\widehat{T}$ as a lattice of $\sqrt{-1}\mathfrak{t}_0^*$ and put $\Lambda_+ := \widehat{T} \cap C_+$. For a representation ϖ of G, we define the K-support of ϖ by

$$\mathrm{Supp}_K(\varpi) := \{\lambda \in \Lambda_+ : \mathrm{Hom}_K(\tau_\lambda, \varpi|_K) \neq 0\},$$

and the asymptotic K-support of ϖ introduced by Kashiwara–Vergne [6], see also [10],

$$\mathrm{AS}_K(\varpi) := \mathrm{Supp}_K(\varpi)\infty \tag{1}$$

which is a closed cone in C_+. Here, the asymptotic cone $S\infty$ for a subset S in a vector space $\mathbf{R}^N$ is defined by

$$\begin{aligned} S\infty := \{y \in \mathbf{R}^N :\ & \text{there exists a sequence} \\ & \{(y_n, \varepsilon_n)\} \subset S \times \mathbf{R}_+ \text{ such that} \\ & \lim_{n\to\infty} y_n\varepsilon_n = y, \lim_{n\to\infty} \varepsilon_n = 0\}. \end{aligned}$$

Let L be a closed subgroup of K and $\mathfrak{l}_0$ the Lie algebra of L. The inclusion $\mathfrak{l}_0 \hookrightarrow \mathfrak{k}_0$ defines the natural projection $\mathrm{pr} : \mathfrak{k}_0^* \to \mathfrak{l}_0^*$. We set $\mathfrak{l}_0^\perp := \ker \mathrm{pr}$ and define a closed cone $C_K(L)$ in $\sqrt{-1}\mathfrak{t}_0^*$ by

$$C_K(L) := C_+ \cap \sqrt{-1}\,\mathrm{Ad}^*(K)\mathfrak{l}_0^\perp. \tag{2}$$

The criterion for admissible restrictions of representations is written by two closed cones (1), (2) in C_+ as follows:

Lemma 2 ([13, Theorem 6.3.3]). *The restriction $\varpi|_L$ is L-admissible if and only if* $\mathrm{AS}_K(\varpi) \cap C_K(L) = \{0\}$.

Next, we return to our setting of (v). Let $\mathfrak{k}_0^s$ be the Lie algebra of K^s, which coincides with the derived ideal $[\mathfrak{k}_0, \mathfrak{k}_0]$ of $\mathfrak{k}_0$. In view of the natural projection $\mathrm{pr} : \mathfrak{k}_0^* \to (\mathfrak{k}_0^s)^*$, the kernel $(\mathfrak{k}_0^s)^\perp$ is isomorphic to the dual $\mathfrak{c}(\mathfrak{k}_0)^*$ of the center $\mathfrak{c}(\mathfrak{k}_0)$ in $\mathfrak{k}_0$. Then, $\sqrt{-1}\,\mathrm{Ad}^*(K)(\mathfrak{k}_0^s)^\perp = \sqrt{-1}\mathfrak{c}(\mathfrak{k}_0)^*$, from which we obtain

$$C_K(K^s) = C_+ \cap \sqrt{-1}\mathfrak{c}(\mathfrak{k}_0)^*. \tag{3}$$

An explicit formula of the asymptotic K-support $\mathrm{AS}_K(\pi)$ is given for a holomorphic discrete series representation π of G of scalar type as follows: Let $Z \in \mathfrak{c}(\mathfrak{k}_0)$ be the characteristic element such that $\mathfrak{g} := \mathfrak{g}_0 \otimes_{\mathbf{R}} \mathbf{C} = \mathfrak{k} + \mathfrak{p}_+ + \mathfrak{p}_-$ is the eigenspace decomposition of $\mathrm{ad}(Z)$ with eigenvalues 0, $\sqrt{-1}$, $-\sqrt{-1}$, respectively. Let $\nu_1, \dots, \nu_r$ be strongly orthogonal roots in $\Delta(\mathfrak{p}_+)$ such that ν_1 is the highest root among $\Delta(\mathfrak{p}_+)$ and that ν_{j+1} is the highest root in $\Delta(\mathfrak{p}_+)$ strongly orthogonal to $\nu_1, \dots, \nu_j$ where $r = \mathrm{rank}\, G/K$. By using the K-type formula [20] of π and the stability of $\mathrm{AS}_K(\pi)$ [10, Lemma 3.1] under the tensor product, we have:

Lemma 3. *$\mathrm{AS}_K(\pi)$ is expressed by*

$$\mathrm{AS}_K(\pi) = \left\{ \sum_{i=1}^r a_i\nu_i : a_1 \geq a_2 \geq \cdots \geq a_r \geq 0 \right\}.$$

Combining [2,3] with Lemma 3, we have:

Lemma 4. *$\{\sum a_i\nu_i : a_1 \geq a_2 \geq \cdots \geq a_r \geq 0\} \cap \sqrt{-1}\mathfrak{c}(\mathfrak{k})^* = \{0\}$ if and only if G/K is of non-tube type.*

Now, we are ready to give a proof of the equivalence (i) $\Leftrightarrow$ (v).

Proof of (i) $\Leftrightarrow$ (v). Since the K^s-admissibility is presented by taking the tensor product with finite-dimensional representations [8, Corollary 1.3], it is sufficient for the proof to deal with the case where π is of scalar type. By Lemma 2, the restriction $\pi|_{K^s}$ is K^s-admissible if and only if $\mathrm{AS}_K(\pi) \cap C_K(K^s) = \{0\}$. It follows from the equality (3) and Lemma 3 that $\mathrm{AS}_K(\pi) \cap C_K(K^s) = \{\sum a_i\nu_i : a_1 \geq a_2 \geq \cdots \geq a_r \geq 0\} \cap \sqrt{-1}\mathfrak{c}(\mathfrak{k})^* \cap C_+$. Applying Lemma 4 to the right-hand side, we conclude that $\mathrm{AS}_K(\pi) \cap C_K(K^s) = \{0\}$ if and only if G/K is of non-tube type. Therefore, the equivalence (i) $\Leftrightarrow$ (v) has been proved. $\square$

4.2. Proof of (vi) $\Rightarrow$ (i). The equivalence (i) $\Leftrightarrow$ (v) brings us to the implication (vi) $\Rightarrow$ (i) as follows:

Proof of (vi) $\Rightarrow$ (i). Suppose that G/K is of tube type. By the equivalence (i) $\Leftrightarrow$ (v), the restriction $\pi|_{K^s}$ is not K^s-admissible, in particular, not multiplicity-free for any holomorphic discrete series representation π of G. This is the contraposition of the implication (vi) $\Rightarrow$ (i). □

4.3. Proof of (i) $\Rightarrow$ (iv). The key of the proof for the implication (i) $\Rightarrow$ (iv) is to construct a slice for the K^s-action on G/K explicitly.

Let $\mathfrak{g}_0$, $\mathfrak{k}_0$, and $\mathfrak{k}_0^s$ be the Lie algebras of G, K, and K^s, respectively. We write $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ for the corresponding Cartan decomposition. Let $\mathfrak{a}_0$ be a maximal abelian subspace in $\mathfrak{p}_0$ and $A := \exp \mathfrak{a}_0$. Then, we have the Cartan decomposition

$$G = KAK. \tag{4}$$

Let $\mathfrak{m}_0$ be the centralizer of $\mathfrak{a}_0$ in $\mathfrak{k}_0$. We recall:

Lemma 5 (cf. [5, Lemma 3.1]). *G/K is of tube type if and only if $\mathfrak{m}_0 \subset \mathfrak{k}_0^s$.*

Proof of (i) $\Rightarrow$ (iv). Suppose that G/K is of non-tube type. By Lemma 5, $\mathfrak{m}_0$ is not contained in $\mathfrak{k}_0^s$. We take $X \in \mathfrak{m}_0$ such that $X \notin \mathfrak{k}_0^s$. Then,

$$\mathfrak{k}_0 = \mathfrak{k}_0^s + \mathbf{R}X, \tag{5}$$

because $\mathfrak{k}_0^s$ is of codimension one in $\mathfrak{k}_0$. Thus, we obtain

$$K = K^s(\exp \mathbf{R}X) = (\exp \mathbf{R}X)K^s. \tag{6}$$

Combining (4) and (6), we get

$$\begin{aligned} G &= KAK \\ &= K^s(\exp \mathbf{R}X)AK \\ &= K^sA(\exp \mathbf{R}X)K \quad (\because\ X \in \mathfrak{m}_0) \\ &= K^sAK \quad (\because\ \exp \mathbf{R}X \subset K). \end{aligned}$$

This implies that the real submanifold $S := AK/K$ meets every K^s-orbit in G/K.

The existence of an anti-holomorphic diffeomorphism σ on G/K satisfying (S.1) and (S.2) for the K^s-action on G/K with S follows from [15, Lemmas 2.2 and 2.4].

Hence, the K^s-action on G/K is strongly visible. In particular, one can take a slice S for this action to be $\dim S = \operatorname{rank} G/K$. □

As a corollary of our proof, we get a new decomposition for the non-symmetric pair (G, K^s) as follows:

Theorem 6. *For a non-tube type Hermitian symmetric space G/K, one can find an abelian subgroup A of G with $\dim A = \operatorname{rank} G/K$ such that the following group decomposition holds*:

$$G = K^sAK.$$

4.4. Proof of (iv) $\Rightarrow$ (vi). The idea of the proof of the implication (iv) $\Rightarrow$ (vi) is based on that of [15, Corollary 6.3].

Let $(\pi, \mathcal{H})$ be a holomorphic discrete series representation of G. It is known that there is a natural injective continuous G-homomorphism from the Hilbert space $\mathcal{H}$ to the Fréchet space $\mathcal{O}(G/K, \mathcal{V})$ consisting of holomorphic sections over a holomorphic line bundle $\mathcal{V} = G \times_K \mu$ for some $\mu \in \widehat{K}$. In order to prove the multiplicity-freeness property of the restriction $\pi|_{K^s}$, it is sufficient to show that $\mathcal{O}(G/K, \mathcal{V})$ is multiplicity-free as a representation of K^s.

Proof of (iv) $\Rightarrow$ (vi). Let π be of scalar type. Then, each fiber $\mathcal{V}_x$ is one-dimensional. In particular, the representation of the isotropy subgroup K_x^s on the fiber $\mathcal{V}_x$ is obviously multiplicity-free. If the K^s-action on G/K is strongly visible, then the assumption of propagation theory of multiplicity-freeness property [17] is satisfied, from which we conclude that $\mathcal{O}(G/K, \mathcal{V})$ is multiplicity-free as a representation of K^s.

Therefore, the implication (iv) $\Rightarrow$ (vi) has been proved. □

As a conclusion, the proof of Theorem 1 has been completed.

4.5. Remark. Here is a direct proof of (ii) $\Rightarrow$ (v).

Suppose that $G_{\mathbf{C}}/K_{\mathbf{C}}^s$ is spherical. It follows from the theory of spherical manifolds [16, Theorem B] that there exists a constant $C > 0$ such that

$$\dim \operatorname{Hom}_G(\pi, C^\infty(G/K^s, G \times_K \mu)) \leq C$$

for any $\pi \in \widehat{G}_{\mathrm{ad}}$ and $\mu \in \widehat{K^s}$. Since the left-hand side of this inequality is given by $\dim \operatorname{Hom}_{K^s}(\mu, \pi|_{K^s})$ by the Frobenius reciprocity, it follows that the restriction $\pi|_{K^s}$ is admissible.

5. Generalization of Cartan decomposition to non-symmetric $G_{\mathbf{C}}/K_{\mathbf{C}}^s$. Let us explain our construction of a slice S satisfying (V.1) for non-tube type G/K from the group theoretic viewpoint, which is an essential part of the proof for the implication (i) $\Rightarrow$ (iii) of Theorem 1. More precisely, we find a subset B giving a decomposition

$$G_{\mathbf{C}} = G_u B K_{\mathbf{C}}^s, \tag{7}$$

which implies that $S := BK_{\mathbf{C}}^s/K_{\mathbf{C}}^s$ meets every G_u-orbit in $G_{\mathbf{C}}/K_{\mathbf{C}}^s$ (see (11) below).

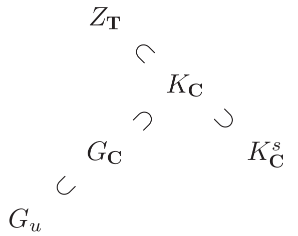

Fig. 1. Herringbone stitch method

The key ingredients are as follows: One is that we have a Cartan decomposition for any (not necessary Riemannian) symmetric space G/H of a reductive Lie group G, namely, there exists an abelian subgroup A of G such that $G = KAH$ owing to Flensted–Jensen [5]. In the case where H coincides with a maximal compact subgroup K of G, the decomposition $G = KAK$ is nothing but the classical Cartan decomposition (see (4) in Section 4.3). The other is to apply the herringbone stitch method [14] for our setting via the symmetric subgroup $K_{\mathbf{C}}$ (see Fig. 1).

Let us retain the notation as in Section 4.3. According to our strategy, we first focus on the symmetric space $G_{\mathbf{C}}/K_{\mathbf{C}}$. Then, the Cartan decomposition for symmetric pairs [5] gives the following decomposition

$$G_{\mathbf{C}} = G_u A K_{\mathbf{C}}, \tag{8}$$

where $A := \exp \mathfrak{a}_0$ and $\mathfrak{a}_0$ is a maximal abelian subspace in $\mathfrak{p}_0$.

Next, we treat the one-dimensional complex manifold $K_{\mathbf{C}}/K^s_{\mathbf{C}}$. As G/K is of non-tube type, we have the decomposition (5) for some $X \in \mathfrak{m}_0$ satisfying $X \notin \mathfrak{k}^s_0$ (see Lemma 5). Then, the complexification $\mathfrak{k} = \mathfrak{k}_0 \otimes_{\mathbf{R}} \mathbf{C}$ is decomposed as follows:

$$\mathfrak{k} = \mathfrak{k}^s + \mathbf{R}X + \sqrt{-1}\mathbf{R}X \tag{9}$$

where $\mathfrak{k}^s = \mathfrak{k}^s_0 \otimes_{\mathbf{R}} \mathbf{C}$. We set $Z_{\mathbf{T}} = \exp \mathbf{R}X$ and $Z_{\mathbf{R}} = \exp \sqrt{-1}\mathbf{R}X$. Then, the decomposition (9) gives rise to a global decomposition as follows:

$$K_{\mathbf{C}} = K^s_{\mathbf{C}} Z_{\mathbf{T}} Z_{\mathbf{R}} = Z_{\mathbf{T}} Z_{\mathbf{R}} K^s_{\mathbf{C}}. \tag{10}$$

We are ready to apply the herringbone stitch method to our setting. As $X \in \mathfrak{m}_0$, three Lie groups $Z_{\mathbf{T}}, Z_{\mathbf{R}}, A$ commute with one another. Since $Z_{\mathbf{T}}$ is a subgroup of G_u, we get

$$\begin{aligned} G_{\mathbf{C}} &= G_u A K_{\mathbf{C}} \quad (\because\ (8)) \\ &= G_u A (Z_{\mathbf{T}} Z_{\mathbf{R}} K^s_{\mathbf{C}}) \quad (\because\ (10)) \\ &= G_u Z_{\mathbf{T}} (A Z_{\mathbf{R}}) K^s_{\mathbf{C}} \quad (\because\ A Z_{\mathbf{T}} = Z_{\mathbf{T}} A) \\ &= G_u (A Z_{\mathbf{R}}) K^s_{\mathbf{C}} \quad (\because\ Z_{\mathbf{T}} \subset G_u). \end{aligned}$$

Therefore, (7) holds if we set

$$B := A Z_{\mathbf{R}} = Z_{\mathbf{R}} A. \tag{11}$$

The point is that our B is still abelian even though $G_{\mathbf{C}}/K^s_{\mathbf{C}}$ is not symmetric, from which S is a submanifold in D. In this sense, (7) is a generalization of the Cartan decomposition known for semisimple symmetric spaces [5] to some non-symmetric spherical homogeneous spaces. Consequently, we have proved:

Theorem 7. *For a non-tube type Hermitian symmetric space G/K, one can find an abelian subgroup B of $G_{\mathbf{C}}$ with $\dim B = \operatorname{rank} G/K + 1$ such that the following group decomposition holds*:

$$G_{\mathbf{C}} = G_u B K^s_{\mathbf{C}}.$$

In particular, B is given by (11).

6. Conclusion. Finally, we summarize the relationship between the multiplicity-freeness of representation and the complex geometry in our setting. If $G_{\mathbf{C}}/K^s_{\mathbf{C}}$ is spherical, or equivalently, G/K is of non-tube type, then we have a generalized Cartan decomposition $G_{\mathbf{C}} = G_u B K^s_{\mathbf{C}}$. Then, our slice for the G_u-action on $G_{\mathbf{C}}/K^s_{\mathbf{C}}$ is given by $S := BK^s_{\mathbf{C}}/K^s_{\mathbf{C}}$. Applying the propagation theory of multiplicity-freeness, we obtain another proof for $\mathcal{O}(G_{\mathbf{C}}/K^s_{\mathbf{C}})$ to be multiplicity-free as a representation of $G_{\mathbf{C}}$.

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