Counterexamples to C^∞ well posedness for some hyperbolic operators with triple characteristics

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Abstract: In this paper we prove a well posed and an ill posed result in the Gevrey category for a simple model hyperbolic operator with triple characteristics, when the principal symbol cannot be smoothly factorized, and whose propagation cone is not transversal to the triple characteristic manifold, thus confirming the conjecture that the Ivrii-Petkov condition is not sufficient for the C^∞ well posedness unless the propagation cone is transversal to the characteristic manifold, albeit for a limited class of operators. Moreover we are able not only to disprove C^∞ well posedness, but we can actually estimate the precise Gevrey threshold where well posedness will cease to hold.

Key words: Cauchy problem; well-posedness; Gevrey class; triple characteristics.

1. Introduction. Hyperbolic operators with double characteristics have been thoroughly investigated in the past years, and at least in the case when there is no transition between different types on the set where the principal symbol vanishes of order 2, essentially everything is known, see e.g. [8] and [1] for a general survey and [5] and [3] for classical introductions. When C, the propagation cone, see [8] for a definition, is transversal to the manifold of multiple points, we are again, in a way, effectively hyperbolic. When this happens and the lower order terms satisfy a generic Ivrii-Petkov vanishing condition, it is known that we have well posedness in C^∞. See [6] for a very complete analysis of this situation. Here we prove a well posedness result in the Gevrey category for a simple model hyperbolic operator with triple characteristics and whose propagation cone is not transversal to the triple manifold. Also we are able not only to disprove C^∞ well posedness, but we can actually estimate the precise Gevrey threshold, by exhibiting a special class of solutions, through which we can violate weak necessary solvability conditions. This threshold appears at $s = 2$, thus beyond the canonical value of $s = 3/2$ dictated by the classical result of Bronšteĭn [2]. The choice of the lower order terms will be the easiest possible, i.e. zero. We consider the operator

$$P(x, D) = D_0^3 - (D_1^2 + x_1^2 D_n^2) D_0 - b_0 x_1^3 D_n^3. \tag{1.1}$$

Here $x = (x_0, x') \in \mathbf{R}^{n+1}$ with $x' = (x_1, x'', x_n)$ and the local estimates below will be proven in a neighborhood of $x = 0$. Clearly hyperbolicity is equivalent to $b_0^2 \leq 4/27$. We will also assume that the principal symbol vanishes exactly of order 3 on the triple manifold Σ_3, thus we will require $b_0^2 < 4/27$, i.e. outside Σ_3, P is strictly hyperbolic. We assume familiarity with the definition of $\gamma^{(s)}(\mathbf{R}^n)$, the Gevrey s class and with the notion of locally solvable in $\gamma^{(s)}$ Cauchy problem (see [1]). In this note we say that the Cauchy problem for P is well posed in the Gevrey s class if for any $\phi_j(x') \in \gamma^{(s)}(\mathbf{R}^n)$, $j = 0, 1, 2$ one can find a neighborhood ω of the origin such that there is a $u \in C^3(\omega)$ verifying $Pu = 0$ in ω and $D_0^j u(0, x') = \phi_j(x')$ in $\omega \cap \{x_0 = 0\}$ for $j = 0, 1, 2$.

The main results in this paper are then precisely stated:

Theorem 1.1. *Assume that $b_0^2 < 4/27$. Then the Cauchy problem for P is well posed in the Gevrey 2 class.*

That this is actually the best one can hope for is proven in

Theorem 1.2. *If $s > 2$, it is possible to choose $b_0 \in]0, \frac{2}{3\sqrt{3}}[$ such that the Cauchy problem for P is not locally solvable at the origin in the Gevrey s class.*

2. Estimates in Gevrey classes. Since the coefficients of P are independent of x_n, we first

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make the Fourier transform with respect to x_n and regard ξ_n, the dual variable of x_n, as a parameter. We define

$$\langle u,v\rangle = \int_{\mathbf{R}^{n-1}} \hat{u}(x_0,x_1,x'',\xi_n)\bar{\hat{v}}(x_0,x_1,x'',\xi_n)dx_1dx''$$

with $\hat{u}$ denoting the partial Fourier transform with respect to x_n. In a similar way we have for the L^2 norm $[[u]]^2 = \int_{\mathbf{R}^n} |\hat{u}(x_0,x_1,x'',\xi_n)|^2 dx_1dx''$. Before dealing with the operator (1.1) itself, we need a preliminary result on the multiplier operator M. Let

$$E_j(u) = [[D_0^j u]]^2 + [[D_1^j u]]^2 + [[(x_1\xi_n)^j u]]^2$$

where $E_0(u) = [[u]]^2$. Let $0 < \theta \le 1/2$ and we start by proving the following

Lemma 2.1. *Let $M = D_0^2 - \theta\Omega$ with $\Omega = D_1^2 + x_1^2\xi_n^2$. Then for any $s \ge 1$, $s \in \mathbf{R}$ and any τ large we have for any $u \in C_0^\infty(\mathbf{R}^n)$*

$$\begin{aligned}(2.2)\quad \theta^{-1}\int_0^\infty W[[Mu]]^2dx_0 &\ge W(0)\sum_{j=0}^{1}\tau^{3-2j}\langle\xi_n\rangle^{\frac{3-2j}{s}}E_j(u(0,\cdot))\\ &\quad + \int_0^\infty W\sum_{j=0}^{1}\tau^{4-2j}\langle\xi_n\rangle^{\frac{4-2j}{s}}E_j(u)dx_0,\end{aligned}$$

where $W = \exp(2\tau\langle\xi_n\rangle^{\frac{1}{s}}(x_0-a))$ with $a > 0$ and $\langle\xi_n\rangle = \sqrt{1+\xi_n^2}$.

Proof. We compute

$$-2\mathsf{Im}\langle Mu, D_0u\rangle = -2\mathsf{Im}\langle D_0^2u, D_0u\rangle$$
$$+2\theta\mathsf{Im}\langle\Omega u, D_0u\rangle = \partial_{x_0}\{[[D_0u]]^2 + \theta\langle\Omega u,u\rangle\}.$$

Since $\langle\Omega u,u\rangle = [[D_1u]]^2 + [[x_1\xi_nu]]^2$ and $-W\partial_{x_0} = -\partial_{x_0}W + 2\tau\langle\xi_n\rangle^{\frac{1}{s}}W$ using Cauchy-Schwarz inequality we see

$$\begin{aligned}(2.3)\quad \int_0^\infty W[[Mu]]^2dx_0 &\ge \theta\tau\langle\xi_n\rangle^{\frac{1}{s}}W(0)E_1(u(0,\cdot))\\ &\quad + \tau^2\int_0^\infty W\langle\xi_n\rangle^{\frac{2}{s}}([[D_0u]]^2 + \theta[[D_1u]]^2\\ &\quad + \theta[[x_1\xi_nu]]^2)dx_0.\end{aligned}$$

Repeating similar arguments we have

$$\begin{aligned}\int_0^\infty W[[D_0u]]^2dx_0 &\ge \tau\langle\xi_n\rangle^{\frac{1}{s}}W(0)[[u(0,\cdot)]]^2\\ &\quad + \tau^2\int_0^\infty W\langle\xi_n\rangle^{\frac{2}{s}}[[u]]^2dx_0\end{aligned}$$

and replacing $(1-\theta)\int_0^\infty W[[D_0u]]^2dx_0$ in (2.3) by the above estimate the right-hand side of (2.3) is bounded from below by

$$\begin{aligned}&\theta W(0)\sum_{j=0}^{1}\tau^{3-2j}\langle\xi_n\rangle^{\frac{3-2j}{s}}E_j(u(0,\cdot))\\ &\quad + \theta\tau^2\int_0^\infty W\langle\xi_n\rangle^{\frac{2}{s}}\sum_{j=0}^{1}\tau^{2-2j}\langle\xi_n\rangle^{\frac{2-2j}{s}}E_j(u)dx_0.\end{aligned}$$

It is now easy to see that (2.2) holds. $\square$

We now move to the proof of Theorem 1.1. If $b_0 = 0$ we do have C^∞ well posedness which is an easy consequence of the double characteristics theory. So we will assume $b_0 \ne 0$ in the following.

Proof. We will make use of standard energy estimates. We choose $\theta = 1/3$ and with $M(x,D) = D_0^2 - \Omega/3$ compute $2\mathsf{Im}\langle Pu, Mu\rangle$ which is, with $B = b_0x_1^3\xi_n^3$,

$$\begin{aligned}(2.4)\ 2\mathsf{Im}\Big\langle\Big(D_0M - \frac{2}{3}\Omega D_0 - B\Big)u, Mu\Big\rangle &= -\frac{\partial}{\partial x_0}[[Mu]]^2 + 2\mathsf{Im}\Big\langle -\frac{2}{3}\Omega D_0u, D_0^2u\Big\rangle\\ &\quad + 2\mathsf{Im}\Big\langle\frac{2}{3}\Omega D_0u, \frac{\Omega}{3}u\Big\rangle + 2\mathsf{Im}\langle -b_0x_1^3\xi_n^3u, D_0^2u\rangle\\ &\quad + 2\mathsf{Im}\Big\langle -b_0x_1^3\xi_n^3u, -\frac{\Omega}{3}u\Big\rangle.\end{aligned}$$

From (2.4) we get

$$2\mathsf{Im}\langle Pu, Mu\rangle = -\frac{\partial}{\partial x_0}\mathcal{E}(u) + \mathcal{R}(u),$$

where $\mathcal{R}(u) = b_0\mathsf{Im}\langle[D_1^2, x_1^3]\xi_n^3u, u\rangle/3$ and

$$\begin{aligned}(2.5)\quad \mathcal{E}(u) &= [[Mu]]^2 + \frac{2}{3}\langle\Omega D_0u, D_0u\rangle\\ &\quad + \frac{2}{9}[[\Omega u]]^2 + 2b_0\mathsf{Re}\langle x_1^3\xi_n^3u, D_0u\rangle.\end{aligned}$$

From (2.5) we have

$$\begin{aligned}(2.6)\ \mathcal{E}(u) &= [[Mu]]^2 + 2b_0\mathsf{Re}\langle x_1^2\xi_n^2u, x_1\xi_nD_0u\rangle\\ &\quad + \frac{2}{3}([[D_1D_0u]]^2 + [[x_1\xi_nD_0u]]^2)\\ &\quad + \frac{2}{9}([[D_1^2u]]^2 + [[x_1^2\xi_n^2u]]^2 + 2\mathsf{Re}\langle D_1^2u, x_1^2\xi_n^2u\rangle).\end{aligned}$$

We write (2.6) like this:

$$\begin{aligned}(2.7)\ \mathcal{E}(u) &= [[Mu]]^2 + \frac{2}{3}[[D_1D_0u]]^2\\ &\quad + \left[\left[\sqrt{\frac{2}{3}}x_1\xi_nD_0u + b_0\sqrt{\frac{3}{2}}x_1^2\xi_n^2u\right]\right]^2 + \frac{2}{9}[[D_1^2u]]^2\\ &\quad + \frac{2}{9}\Big(1 - \frac{27}{4}b_0^2\Big)[[x_1^2\xi_n^2u]]^2 + \frac{4}{9}\mathsf{Re}\langle D_1^2u, x_1^2\xi_n^2u\rangle.\end{aligned}$$

Noticing that $\mathrm{Re}\langle D_1^2u, x_1^2u\rangle = [[x_1D_1u]]^2 - [[u]]^2$ we get from (2.7) that

$$\begin{aligned}\mathcal{E}(u) &= [[Mu]]^2 + \frac{2}{3}[[D_1D_0u]]^2 \\ &+ \left[\left[\sqrt{\frac{2}{3}}x_1\xi_nD_0u + b_0\sqrt{\frac{3}{2}}x_1^2\xi_n^2u\right]\right]^2 + \frac{2}{9}[[D_1^2u]]^2 \\ &+ \frac{2}{9}\left(1-\frac{27}{4}b_0^2\right)[[x_1^2\xi_n^2u]]^2 + \frac{4}{9}[[x_1\xi_nD_1u]]^2 \\ &- \frac{4}{9}\xi_n^2[[u]]^2.\end{aligned}$$

Multiplying by W and integrating from 0 to ∞ we have

$$\begin{aligned}(2.8)\qquad \int_0^\infty 2W\mathrm{Im}\langle Pu, Mu\rangle dx_0 &= W(0)\mathcal{E}(u(0,\cdot)) \\ &+ 2\tau\langle\xi_n\rangle^{\frac{1}{s}}\int_0^\infty W\Big\{[[Mu]]^2 + \frac{2}{3}[[D_1D_0u]]^2 \\ &+ \left[\left[\sqrt{\frac{2}{3}}x_1\xi_nD_0u + b_0\sqrt{\frac{3}{2}}x_1^2\xi_n^2u\right]\right]^2 \\ &+ \frac{2}{9}[[D_1^2u]]^2 + \frac{2}{9}\left(1-\frac{27}{4}b_0^2\right)[[x_1^2\xi_n^2u]]^2 \\ &+ \frac{4}{9}[[x_1\xi_nD_1u]]^2 - \frac{4}{9}\xi_n^2[[u]]^2\Big\}dx_0 \\ &- 2b_0\xi_n^3\int_0^\infty W\mathrm{Re}\langle x_1^2u, D_1u\rangle dx_0.\end{aligned}$$

Recalling (2.2) from Lemma 2.1 now with $1\le s\le 2$ we can dispose of the negative contribution in (2.8) $-4\xi_n^2[[u]]^2/9$, choosing τ large because $\langle\xi_n\rangle^{\frac{4}{s}} \ge \xi_n^2$. Let us now deal with the remainder term

$$\int_0^\infty W\mathcal{R}(u)dx_0 = 2b_0\xi_n^3\int_0^\infty W\mathrm{Re}\langle x_1^2u, D_1u\rangle dx_0.$$

Applying Cauchy-Schwarz inequality we get

$$\begin{aligned}&|2\mathrm{Re}\langle x_1^2\xi_n^2u, \xi_nD_1u\rangle| \\ &\le \langle\xi_n\rangle^{\frac{1}{s}}([[x_1^2\xi_n^2u]]^2 + \langle\xi_n\rangle^{2-\frac{2}{s}}[[D_1u]]^2) \\ &\le \langle\xi_n\rangle^{\frac{1}{s}}([[x_1^2\xi_n^2u]]^2 + \langle\xi_n\rangle^{4-\frac{4}{s}}[[u]]^2 + [[D_1^2u]]^2).\end{aligned}$$

It is clear that $[[D_0^2u]] \le 4([[Mu]]^2 + [[x_1^2\xi_n^2u]]^2 + [[D_1^2u]]^2)$. Using (2.8) and $1+2/s \ge 2 \ge 4-4/s$ we obtain for any $u \in C_0^\infty(\mathbf{R}^n)$

$$\begin{aligned}\int_0^\infty W[[Pu]]^2dx_0 &\ge CW(0)\sum_{j=0}^2\tau^{4-3j/2} \\ &\times \langle\xi_n\rangle^{\frac{5-2j}{s}}E_j(u(0,\cdot)) + C\sum_{j=0}^2\tau^{6-2j} \\ &\times \int_0^\infty W\langle\xi_n\rangle^{\frac{6-2j}{s}}E_j(u(x_0))dx_0\end{aligned}$$

if τ is large enough and $1\le s\le 2$. Let $s=2$ and

$$\begin{aligned}\tilde{E}_j(u) &= \int E_j(u)d\xi_n \\ &= \int_{\mathbf{R}^n}(|D_0^ju|^2 + |D_1^ju|^2 + |(x_1D_n)^ju|^2)dx'.\end{aligned}$$

Then for any $u \in C_0^\infty(\mathbf{R}^{n+1})$ vanishing in $x_0 \ge a$ we integrate (2.2) with respect to ξ_n we get

$$\begin{aligned}&\int_0^a \|e^{\tau\langle D_n\rangle^{\frac{1}{2}}(x_0-a)}Pu\|^2dx_0 \\ &\ge C\sum_{j=0}^2\tilde{E}_j(e^{-\tau a\langle D_n\rangle^{\frac{1}{2}}}\langle D_n\rangle^{\frac{5-2j}{4}}u(0,\cdot)) \\ &\quad + C\int_0^a\sum_{j=0}^2\tilde{E}_j(e^{\tau\langle D_n\rangle^{\frac{1}{2}}(x_0-a)}\langle D_n\rangle^{\frac{6-2j}{4}}u(x_0)).\end{aligned}$$

Let us denote $\langle\xi\rangle = \sqrt{1+\sum_{j=1}^n\xi_j^2}$ then it is easy to check

Lemma 2.2. *For any $\ell \in \mathbf{R}$ there exist $C = C_\ell > 0$, $\tau = \tau_\ell > 0$ such that for any $u \in C_0^\infty(\mathbf{R}^{n+1})$ vanishing in $x_0 \ge a$ we have*

$$\begin{aligned}&C\int_0^a \|e^{\tau\langle D_n\rangle^{\frac{1}{2}}(x_0-a)}\langle D\rangle^\ell Pu\|^2dx_0 \\ &\ge \sum_{j=0}^2\tilde{E}_j(e^{-\tau a\langle D_n\rangle^{\frac{1}{2}}}\langle D_n\rangle^{\frac{5-2j}{4}}\langle D\rangle^\ell u(0,\cdot)) \\ &\quad + \int_0^a\sum_{j=0}^2\tilde{E}_j(e^{\tau\langle D_n\rangle^{\frac{1}{2}}(x_0-a)}\langle D_n\rangle^{\frac{6-2j}{4}}\langle D\rangle^\ell u(x_0))dx_0.\end{aligned}$$

Let $\ell > 0$ be large and we assume that

$$\begin{aligned}(2.9)\qquad &e^{\tau a\langle D_n\rangle^{\frac{1}{2}}}\langle D_n\rangle^{\frac{-(5-2j)}{4}}\langle D\rangle^\ell\phi_{2-j} \in L^2(\mathbf{R}^n), \\ &e^{-\tau\langle D_n\rangle^{\frac{1}{2}}(x_0-a)}\langle D_n\rangle^{-\frac{3}{2}}\langle D\rangle^\ell f \in L^2(I\times\mathbf{R}^n)\end{aligned}$$

with $I=(0,a)$. Then from Lemma 2.2 and a standard argument of functional analysis we see that there exists u such that

$$\int_0^a \|e^{-\tau\langle D_n\rangle^{\frac{1}{2}}(x_0-a)}\langle D\rangle^\ell u\|^2dx_0 < +\infty$$

satisfying $Pu = f$ in $I\times\mathbf{R}^n$ and $D_0^ju(0) = \phi_j$, $j = 0,1,2$. Since $e^{-\tau\langle D_n\rangle^{\frac{1}{2}}(x_0-a)}\langle D\rangle^\ell u \in L^2(I\times\mathbf{R}^n)$ it is clear that $u \in L^2(I; H^\ell(\mathbf{R}^n))$. Since we have $Pu \in L^2(I; H^{\ell-3/2}(\mathbf{R}^n))$ from the assumption then from Theorem B.2.9 ([4]) it follows that

$$D_0^j u \in L^2(I; H^{\ell-3/2-j}(\mathbf{R}^n))$$

for $j = 0, 1, 2, 3$. Thus we get a smooth solution in $I \times \mathbf{R}^n$ provided (2.9) is verified and choosing ℓ large. □

3. Optimality of the Gevrey index.

3.1. Sibuya's results. The differential equation

$$w''(y) = (y^3 + \zeta y)w(y) \tag{3.10}$$

will play a very important role in the construction of the family of solutions leading to the optimality of the Gevrey index $s = 2$. Therefore we recap briefly, in this special setting, the general theory of subdominant solutions and Stokes coefficients for the equation (3.10), following the presentation found, for example, in the book of Sibuya [9]. Theorem 6.1 in [9] states that the differential equation (3.10) has a unique solution

$$w(y; \zeta) = \mathcal{Y}(y; \zeta)$$

such that

(i) $\mathcal{Y}(y; \zeta)$ is an entire function of (y, ζ).

(ii) $\mathcal{Y}(y; \zeta)$ and its derivative $\mathcal{Y}'(y; \zeta)$ admit asymptotic representations

$$y^{-3/4}\left[1 + \sum_{N=1}^{\infty} B_N y^{-N/2}\right] e^{-E(y;\zeta)} = Y(y, \zeta),$$

$$y^{3/4}\left[-1 + \sum_{N=1}^{\infty} C_N y^{-N/2}\right] e^{-E(y;\zeta)},$$

uniformly on each compact set in the ζ space as y goes to infinity in any closed subsector of the open sector $|\arg y| < 3\pi/5$ moreover

$$E(y; \zeta) = \frac{2}{5} y^{5/2} + \zeta y^{1/2}$$

and B_N, C_N are polynomials in ζ.

We note that if we set $\omega = \exp(2\pi i/5)$ and

$$\mathcal{Y}_k(y; \zeta) = \mathcal{Y}(\omega^{-k} y; \omega^{-2k}\zeta)$$

where $k = 0, 1, 2, 3, 4$ then $\mathcal{Y}_k(y; \zeta)$ solve (3.10). In particular $\mathcal{Y}_0(y; \zeta) = \mathcal{Y}(y; \zeta)$. Then we have

(i) $\mathcal{Y}_k(y; \zeta)$ is an entire function of (y, ζ).

(ii) $\mathcal{Y}_k(y; \zeta) \sim Y(\omega^{-k} y; \omega^{-2k}\zeta)$ uniformly on each compact set in the ζ space as y goes to infinity in any closed subsector of the open sector

$$|\arg y - 2k\pi/5| < 3\pi/5.$$

Let S_k denote the open sector defined by $|\arg y - 2k\pi/5| < \pi/5$. Since

$$\mathsf{Re}[y^{5/2}] > 0 \quad \text{for} \quad y \in S_0 \tag{3.11}$$

and $\mathsf{Re}[y^{5/2}] < 0$ for $y \in S_{-1} = S_4$ and for S_1 the solution $\mathcal{Y}_0(y; \zeta)$ is subdominant in S_0 and dominant in S_4 and S_1. Similarly $\mathcal{Y}_k(y; \zeta)$ is subdominant in S_k and dominant in S_{k-1} and S_{k+1}. It is clear that $\mathcal{Y}_{k+1}$ and $\mathcal{Y}_{k+2}$ are linearly independent. Therefore $\mathcal{Y}_k$ is a linear combination of those two:

$$\mathcal{Y}_k(y; \zeta) = C_k(\zeta)\mathcal{Y}_{k+1}(y; \zeta) + \tilde{C}_k(\zeta)\mathcal{Y}_{k+2}(y; \zeta)$$

where C_k, $\tilde{C}_k$ are called the Stokes coefficients for $\mathcal{Y}_k(y; \zeta)$. We summarize in the following statements some of the known and useful facts about the Stokes coefficients for our particular equation (3.10). Proofs can be found in Chapter 5 of [9].

Proposition 3.1. *The following results hold.*

(i) $\tilde{C}_k(\zeta) = -\omega$ *for any* k *and* ζ,

(ii) $C_k(\zeta) = C_0(\omega^{-2k}\zeta)$ *for any* k, ζ *and* $C_0(\zeta)$ *is an entire function of* ζ,

(iii) $C_k(0) = 1 + \omega$ *for any* k,

(iv) $\partial_\zeta C_0(\zeta)|_{\zeta=0} \neq 0$.

Proposition 3.2. *We have*

$$C_0(\zeta) + \omega^2 C_0(\omega\zeta)C_0(\omega^4\zeta) - \omega^3 = 0$$

for any $\zeta \in \mathbf{C}$.

3.2. Localization of zeros. We now state a key lemma which is proved in [1].

Lemma 3.1. *The Stokes coefficient* $C_0(\zeta)$ *vanishes in at least one* (*non zero*) ζ_0.

Now that we know that $C_0(\zeta)$ vanishes somewhere, we would like to find out where exactly this happens. We begin with a symmetry result, recalling Proposition 3 in [10]:

Lemma 3.2. *The Stokes coefficient* $C_0(\zeta)$ *verifies the equivalence*

$$C_0(\zeta) = 0 \Longleftrightarrow C_0(\overline{\omega\zeta}) = 0.$$

The following is a very important step in the construction of the null solutions, and is a sharp result on the location of the zeros of the entire function $C_0(\zeta)$.

Lemma 3.3. *There exists* $\zeta_0 \in W = \{z \in \mathbf{C} | \pi < \arg z \leq 19\pi/15\}$ *where* $C_0(\zeta_0) = 0$.

Proof. We recall from Proposition 3.1 in [7] that $C_0(\zeta) = 0$ implies either $\zeta \in W_1 = \{\pi \leq \arg\zeta \leq 19\pi/15\}$ or $\zeta \in W_2 = \{\pi/3 \leq \arg\zeta \leq 3\pi/5\}$. But W_1 and W_2 are symmetric under the mapping $\zeta \to \overline{\omega\zeta}$. We just have to show the $\arg\zeta \neq \pi$. Proposition 3.2 and Lemma 3.2 above together imply that $C_0(\zeta) \neq 0$ if ζ is real. $\square$

3.3. Proof of Theorem 1.2. Consider again the operator:

$$P(x,D) = D_0^3 - (D_1^2 + x_1^2 D_n^2)D_0 - b_0 x_1^3 D_n^3.$$

Let $\lambda > 0$ be a positive large parameter, $R > 0$, $\theta \in]0,\pi[$ to be chosen later and consider:

$$U(x,\lambda,R,\theta) = E(x_0,x_n,\lambda)w(Ax_1+B), \tag{3.12}$$

with $E(x_0,x_n,\lambda) = e^{ix_0\lambda^{\frac{1}{2}Re^{i\theta}+ix_n\lambda}}$ and A, B to be chosen together with w. Sometimes the x'' components of x will be omitted to enhance readability. It is easy to see

$$PU = \Big(\lambda^{3/2}R^3e^{i3\theta} - \lambda^{5/2}Re^{i\theta}x_1^2 - b_0\lambda^3x_1^3 + \lambda^{1/2}Re^{i\theta}A^2\frac{w''}{w}(Ax_1+B)\Big)U. \tag{3.13}$$

Thus setting $y = Ax_1 + B$ we have from (3.13) and the request that $PU = 0$,

$$\begin{aligned} w''(y) = \lambda^{-1/2}R^{-1}e^{-i\theta}A^{-2}\Big[&\frac{b_0\lambda^3}{A^3}y^3 + \Big(-3\frac{b_0\lambda^3B}{A^3} + \frac{\lambda^{5/2}Re^{i\theta}}{A^2}\Big)y^2 \\ &+ \Big(\frac{3b_0\lambda^3B^2}{A^3} - \frac{2\lambda^{5/2}Re^{i\theta}B}{A^2}\Big)y - \frac{b_0\lambda^3B^3}{A^3} \\ &+ \frac{\lambda^{5/2}Re^{i\theta}B^2}{A^2} - \lambda^{3/2}e^{i3\theta}R^3\Big]w(y). \end{aligned} \tag{3.14}$$

The following choices are then made:

$$\lambda^{-1/2}R^{-1}e^{-i\theta}\frac{b_0\lambda^3}{A^5} = 1, \qquad -\frac{3b_0\lambda^3B}{A^3} + \frac{\lambda^{5/2}Re^{i\theta}}{A^2} = 0. \tag{3.15}$$

Then (3.15) yields

$$A = \lambda^{1/2}b_0^{1/5}R^{-1/5}e^{-i\theta/5}, \qquad B = \frac{R^{4/5}b_0^{-4/5}e^{i4\theta/5}}{3}. \tag{3.16}$$

Using these values we have from (3.14)

$$w''(y) = (y^3 + \zeta y + \mu)w(y), \tag{3.17}$$

with

$$\zeta = -\frac{b_0^{-8/5}e^{i8\theta/5}R^{8/5}}{3}, \ \mu = \lambda R^2e^{2i\theta}A^{-2}\Big(\frac{2}{27b_0^2} - 1\Big).$$

It is now clear that choosing $b_0 = \frac{\sqrt{2}}{3\sqrt{3}} \in]0, \frac{2}{3\sqrt{3}}[$ will give us equation (3.10). We now choose $w(y;\zeta) = \mathcal{Y}_0(y;\zeta_0)$ with ζ_0 found in Lemma 3.3 and from (3.16) we take $y = b_0^{\frac{1}{5}}R^{-\frac{1}{5}}\lambda^{\frac{1}{2}}e^{-i\frac{\theta}{5}}x_1 + \frac{1}{3}b_0^{-\frac{4}{5}}R^{\frac{4}{5}}e^{\frac{4i\theta}{5}}$. We have that $b_0^{-\frac{8}{5}}R^{\frac{8}{5}}e^{i\frac{8\theta}{5}+i\pi} = 3|\zeta_0|e^{i\arg\zeta_0}$ and $\pi < \arg\zeta_0 \le 19\pi/15$. This clearly leaves us with $0 < \theta_0 = \theta(\arg\zeta_0) \le \pi/6$, while the number R, still at our disposal, is chosen to fix the absolute values, thus $R = R_0 > 0$, depending on b_0 and $|\zeta_0|$. Recall that $\mathcal{Y}_k(y;\zeta) = \mathcal{Y}(\omega^{-k}y;\omega^{-2k}\zeta)$ and that

$$\begin{aligned} \mathcal{Y}_0(y;\zeta_0) &= -\omega\mathcal{Y}_2(y;\zeta_0) \\ &= -\omega\mathcal{Y}_0(\omega^{-2}y;\omega^{-4}\zeta_0), \end{aligned} \tag{3.18}$$

since $C_0(\zeta_0) = 0$. Thus we notice that when $x_1 > 0$ and λ is large $\arg(y) \in [-\pi/30, 0[$ clearly well inside the subdominant sector S_0. On the other hand if $x_1 < 0$ and λ is large, using (3.18), we have that $\arg(y) \in [\pi/6, \pi/5[$, again within the subdominant sector S_0. This proves in particular that $w(y;\zeta_0)$, $y = Ax_1 + B$ with A, B given by (3.16) with $\theta = \theta_0$, $R = R_0$ is, for every $\lambda > 0$ in the Schwartz space $\mathcal{S}(\mathbf{R})$ and moreover $w(y;\zeta_0)$ is bounded on $\mathbf{R}$ uniformly in λ. Let

$$U_\lambda = e^{i(T-x_0)\lambda^{1/2}R_0e^{i\theta_0} - ix_n\lambda}w(y;\zeta_0)$$

then $PU_\lambda = 0$ because $P(x_1,-D_0,D_1,-D_n) = -P(x_1,D_0,D_1,D_n)$. Suppose now that there exist a neighborhood ω of the origin and $u \in C^3(\omega)$ satisfying

$$\begin{cases} Pu = 0 \text{ in } \omega \\ u(0,x') = 0, D_0u(0,x') = 0 \text{ on } \omega' \\ D_0^2u(0,x') = \bar\phi_1(x_1)\bar\phi_2(x'')\bar\psi(x_n) \text{ on } \omega' \end{cases} \tag{3.19}$$

where $\omega' = \omega \cap \{x_0 = 0\}$ and $\phi_1(x_1) \in \gamma_0^{(s)}(\mathbf{R})$, $\phi_2(x'') \in \gamma_0^{(s)}(\mathbf{R}^{n-2})$, $\psi(x_n) \in \gamma_0^{(s)}(\mathbf{R})$. From the Holmgren uniqueness theorem we can assume that $u(x) = 0$ if $0 \le x_0 \le T$, $|x'| \ge r$ for small $T > 0$ and $r > 0$. Then from $0 = \int_0^T (PU_\lambda, u)dx_0 - \int_0^T (U_\lambda, Pu)dx_0$ we have

$$\begin{aligned} (U_\lambda(0), D_0^2u(0)) = \sum_{j=0}^{2}&(D_0^{2-j}U_\lambda(T), D_0^ju(T)) \\ &- ((D_1^2 + x_1^2D_n^2)U_\lambda(T), u(T)). \end{aligned}$$

The right-hand side is $O(\lambda^2)$ because $w(y;\zeta_0)$, $\lambda^{-1/2}D_1w(y;\zeta_0)$ are bounded uniformly in λ. On the other hand the left-hand side is

$$\hat\psi(\lambda)e^{iT\lambda^{1/2}R_0e^{i\theta_0}}\int w(y;\zeta_0)\phi_1(x_1)\phi_2(x'')dx_1dx''$$

$$= \hat{\psi}(\lambda)e^{iT\lambda^{1/2}R_0e^{i\theta_0}}\left(\int \phi_2(x'')dx''\right) \times \int w(y;\zeta_0)\phi_1(x_1)dx_1.$$

We choose ϕ_2 so that $\int \phi_2(x'')dx'' \neq 0$. Recall that $\psi \in \gamma_0^{(2)}(\mathbf{R})$ if and only if $|\hat{\psi}(\xi)| \leq Ce^{-L|\xi|^{1/2}}$ with some $L > 0$, $C > 0$. Thus if we take $\psi \notin \gamma_0^{(2)}(\mathbf{R})$ which is even then $\lambda^{-N}\hat{\psi}(\lambda)e^{iT\lambda^{1/2}R_0e^{i\theta_0}}$ is not bounded as $\lambda \to \infty$ for any $N \in \mathbf{N}$. Checking that

$$\lambda^{\kappa}\int w(y;\zeta_0)\phi(x_1)dx_1 \to c \neq 0$$

with a suitable choice of ϕ and $\kappa \in \mathbf{R}$ we could get a contradiction proving non local solvability of (3.19). Let $\alpha = b_0^{1/5}R_0^{-1/5}e^{-i\theta_0/5}$, $\beta = b_0^{-4/5}R_0^{4/5}e^{4i\theta_0/5}/3$ and note that it is enough to show $\int w(\alpha x_1 + \beta;\zeta_0)x_1^k dx_1 \neq 0$ for at least one $k = 0, 1, 2$. Put

$$v(\xi) = \int e^{-ix\xi}w(\alpha x + \beta;\zeta_0)dx$$

then $v(\xi)$ satisfies the equation

$$\left(i\alpha\frac{d}{d\xi} + \beta\right)^3 v(\xi) + \zeta_0\left(i\alpha\frac{d}{d\xi} + \beta\right)v(\xi) + \alpha^{-2}\xi^2 v(\xi) = 0$$

and

$$v^{(k)}(0) = (-i)^k\int w(\alpha x + \beta;\zeta_0)x^k dx.$$

So if $v^{(k)}(0) = 0$ for $k = 0, 1, 2$ then we would have $v(\xi) = 0$ so that $w(\alpha x + \beta;\zeta_0) = 0$ which is a contradiction.

4. Cones and factorization. Here we briefly verify that the propagation cone is not transversal to the triple manifold. Let $p(x,\xi) = \xi_0^3 - (\xi_1^2 + x_1^2\xi_n^2)\xi_0 - b_0x_1^3\xi_n^3$ be the symbol of the operator (1.1). p vanishes exactly of order 3 on $\Sigma_3 = \{x_1 = \xi_0 = \xi_1 = 0\}$ near $(0; 0, \dots, 1)$ if $|b_0| < \frac{2}{3\sqrt{3}}$. Fix $z \in \Sigma_3$ and take $\delta v = (-1, 0, \dots, 0; 0)$. Clearly $\delta v \in T_z\Sigma_3$ and, since $\sigma(\delta v, (\delta y, \delta\eta)) = -\delta\eta_0 \leq 0$ if $(\delta y, \delta\eta) \in \Gamma_z$, we have that $C_z \cap T_z\Sigma_3 \neq \emptyset$. On the other hand C_z cannot be completely contained in $T_z\Sigma_3$, because otherwise $T_z^{\sigma}\Sigma_3 \subset C_z^{\sigma}$ and this would imply that $\langle H_{\xi_0}, H_{\xi_1}, H_{x_1}\rangle \subset \overline{\Gamma_z}$, which is false. Therefore C_z is neither disjoint from nor totally inside $T_z\Sigma_3$. For the next item we change slightly the notations in order to simplify the treatment of a third degree equation naturally associated with the problem. Let us show that for our model no root is C^∞. Let $p = \tau^3 - 3(x^2 + \xi^2)\tau - 2bx^3$, with $0 < |b| < 1$. If p could be written like $p = (\tau - L(x,\xi))(\tau^2 + A(x,\xi)\tau + B(x,\xi))$ with C^∞ functions L, A, B, one then would get $A = L$, $L^2 - B = 3(x^2 + \xi^2)$ and $LB = 2bx^3$. This shows that we have $L = xL_1(x,\xi)$ or $B = xB_1(x,\xi)$ with C^∞ smooth L_1, B_1. If $B = xB_1$ then $L(0,\xi)^2 = 3\xi^2$ so that $L(0,\xi) = \sqrt{3}\xi$ or $L(0,\xi) = -\sqrt{3}\xi$ and hence $L = \pm\sqrt{3}\xi + xL_1$. From $LB = x(\pm\sqrt{3}\xi + xL_1)B_1 = -2bx^3$ we would have $B_1 = x^2B_2$ so that $B = x^3B_2$ which is incompatible with $LB = -2bx^3$. If $L = xL_1$ then from $L^2 - B = x^2L_1^2 - B = 3(x^2 + \xi^2)$ we would have $B = -3\xi^2 + x^2B_1$. Then from $LB = x(-3\xi^2 + x^2B_1)L_1 = -2bx^3$ we would have $L_1 = x^2L_2$ which is incompatible with $LB = -2bx^3$. This contradiction proves that p cannot be smoothly factorized.

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