A criterion for holomorphic families of Riemann surfaces to be virtually isomorphic

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Abstract: In this paper, we give a rigidity theorem for holomorphic disks associated to holomorphic families of Riemann surfaces, aiming for studying the geometry (the asymptotic boundary behaviors) of such holomorphic disks. Indeed, our rigidity theorem is described with distributions of associated holomorphic disks and orbits of monodromies at infinity.

Key words: Holomorphic family; Teichmüller space; Teichmüller modular group; Teichmüller distance.

1. Introduction.

1.1. Background. In a celebrated paper [4], Y. Imayoshi and H. Shiga gave an analytic proof of the finiteness theorem of holomorphic families of Riemann surfaces of fixed analytically finite type over a (fixed) Riemann surface. Their result gives an affirmative answer to the Shafarevich conjecture in the function field case (cf. [1], [2] and [7], see also [8]). To prove the finiteness theorem, they showed the *rigidity theorem* claiming that given two locally non-trivial holomorphic families of Riemann surfaces over a common base Riemann surface are isomorphic when they have the same monodromies. Thus, the monodromy (a topological data) of the family and the conformal structure of the base surface determines the holomorphic family of Riemann surfaces. Notice that they conclude the rigidity theorem with the assumption that the base surface is of class O_G.

Our aim in future is to understand the geometry (the behaviors) of holomorphic disks associated with holomorphic families of Riemann surfaces in the Teichmüller space. Motivated from this, instead of the assumption "coincidence of monodromies" in Imayoshi-Shiga's rigidity theorem, we propose here a geometric assumption for monodromies which implies that two families of Riemann surfaces are virtually isomorphic. In fact, the condition is described in terms of the asymptotic behaviors of orbits of the actions of monodromies on the Teichmüller space.

1.2. Main theorem. Let $\mathcal{T}_{g,n}$ be the Teichmüller space of Riemann surfaces of analytically finite type (g, n) with $2g - 2 + n > 0$. Let d_T be the Teichmüller distance on $\mathcal{T}_{g,n}$. Let $\mathrm{Mod}_{g,n}$ be the Teichmüller modular group acting on $\mathcal{T}_{g,n}$ isometrically (for details, see [5]). The *Gromov product* of the Teichmüller distance with basepoint $x_0 \in \mathcal{T}_{g,n}$ is defined by

$$\langle x \mid y \rangle_{x_0} = \frac{1}{2}(d_T(x_0, x) + d_T(x_0, y) - d_T(x, y))$$

for $x, y \in \mathcal{T}_{g,n}$. The Gromov product is a standard tool in the study of metric spaces (e.g. [3]).

Two holomorphic families $\mathcal{M}_i$ $(i = 1, 2)$ of Riemann surfaces of type (g, n) over a Riemann surface B is said to be *virtually isomorphic* if there is a finite covering $\pi' : B' \to B$ and a biholomorphic mapping $F : \tilde{M}_1 \to \tilde{M}_2$ between the total spaces of the pullback families $\tilde{\mathcal{M}}_j = (\pi')^* \mathcal{M}_j$ $(j = 1, 2)$ such that

$$\begin{array}{ccc} \tilde{M}_1 & \xrightarrow{F} & \tilde{M}_2 \\ \downarrow & & \downarrow \\ B' & \xrightarrow{id} & B' \end{array}$$

is an isomorphism as holomorphic families. The smallest degree among finite coverings $\pi' : B' \to B$ with the above condition is called the *virtual isomorphism degree* of $\mathcal{M}_1$ and $\mathcal{M}_2$.

The aim of this note is to show the following theorem.

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Theorem 1.1 (Weak Rigidity Theorem). *Let Γ be the Fuchsian group of a Riemann surface of class O_G acting on the unit disk $\mathbf{D}$. Let $\mathcal{M}_i$ ($i = 1, 2$) be a locally non-trivial holomorphic family of Riemann surface of type (g, n) over $\mathbf{D}/\Gamma$. Let $\rho_i\colon \Gamma \to \mathrm{Mod}_{g,n}$ be a monodromy of the family $\mathcal{M}_i$. Suppose that there are $x_1, x_2 \in \mathcal{T}_{g,n}$, and $\varphi \in \mathrm{Mod}_{g,n}$ such that for any $M > 0$, there is a finite set $C \subset \Gamma$ with*

$$(1) \qquad \langle \rho_1(\gamma)(x_1) \mid \varphi \circ \rho_2(\gamma)(x_2) \rangle_{x_0} > M$$

for all $\gamma \in \Gamma - C$. Then, $\mathcal{M}_1$ and $\mathcal{M}_2$ are virtually isomorphic, and the virtual isomorphism degree is at most $42(2g - 2 + n)$. Furthermore if a fiber of $\mathcal{M}_i$ has the trivial automorphism group, two families are isomorphic.

The condition (1) is independent of the choice of the basepoint since $|\langle x \mid y \rangle_{x_0} - \langle x \mid y \rangle_{x_0}| \le d_T(x_0, x_1)$ holds for any three points $x, y, x_1 \in \mathcal{T}_{g,n}$.

Remark 1.1. When $\rho_1(\gamma) = \varphi \circ \rho_2(\gamma) \circ \varphi^{-1}$ for $\gamma \in \Gamma$,

$$|\langle \rho_1(\gamma)(x_1) \mid \varphi \circ \rho_2(\gamma)(x_2) \rangle_{x_0} - d_T(x_0, \rho_1(\gamma)(x_1))| \le d_T(x_1, \varphi(x_2)).$$

Hence, when two monodromies ρ_1 and ρ_2 are conjugate, especially when $\rho_1 = \rho_2$, the monodromies satisfy the assumption (1) of the theorem.

Remark 1.2. The assumption (1) on the monodromies can be replaced with the following condition: There are $x_1, x_2 \in \mathcal{T}_{g,n}$, $\varphi \in \mathrm{Mod}_{g,n}$ and a measurable subset $E \subset \partial\mathbf{D}$ of positive measure such that for any $z_0 \in E$ there is a sequence $\{\gamma_n\}_{n=1}^{\infty}$ such that $\gamma_n(0) \to z_0$ nontangentially and

$$\langle \rho_1(\gamma_n)(x_1) \mid \varphi \circ \rho_2(\gamma_n)(x_2) \rangle_{x_0} \to \infty$$

as $n \to \infty$. This condition looks technical, but is weaker than the assumption in the theorem since the base surface is of class O_G.

2. Holomorphic families. Let $\hat{M}$ be a two-dimensional complex manifold and let C be a non-singular one-dimensional analytic subset of $\hat{M}$ or empty. Let B be a Riemann surface. Assume that there is a holomorphic mapping $\hat{\pi}\colon \hat{M} \to B$ such that (1) $\hat{\pi}$ is proper and of maximal rank at every point of $\hat{M}$, and (2) setting $M = \hat{M} - C$ and $\pi = \hat{\pi} \mid M$, the fiber $M_b = \pi^{-1}(b)$ ($b \in B$) is an irreducible analytic subset of M and is of fixed analytically finite type (g, n) as a Riemann surface. We call such a triple (M, π, B) a *holomorphic family of Riemann surfaces of type (g, n) over B.*

Let (M, π, B) be a holomorphic family of Riemann surfaces of type (g, n) with $2g - 2 + n > 0$. Let $\Pi\colon \tilde{B} \to B$ be the universal covering of B with the covering transformation group Γ. Then, there is a holomorphic mapping $\Phi\colon \tilde{B} \to \mathcal{T}_{g,n}$ and a homomorphism $\rho\colon \Gamma \to \mathrm{Mod}_{g,n}$ such that (1) the underlying surface of $\Phi(z)$ is conformally equivalent to the fiber $M_{\Pi(z)}$, and (2) Φ is Γ-equivariant in the sense that

$$\Phi \circ \gamma = \rho(\gamma) \circ \Phi$$

for all $\gamma \in \Gamma$. We call Φ and ρ a *representation of the family* (M, π, B) and the *monodromy with respect to the representation* Φ. Notice that for $\varphi \in \mathrm{Mod}_{g,n}$, the composition $\varphi \circ \Phi$ is also a representation of (M, π, B). In this case, the monodromy with respect to $\varphi \circ \Phi$ is obtained by the conjugation:

$$(2) \qquad \Gamma \ni \gamma \mapsto \varphi \circ \rho(\gamma) \circ \varphi^{-1} \in \mathrm{Mod}_{g,n}.$$

A holomorphic family is locally non-trivial if and only if a representation of the family is a non-constant holomorphic mapping. Furthermore, if a holomorphic family (M, π, B) is non-trivial, the universal covering surface of B is conformally equivalent to the unit disk $\mathbf{D}$ (see [4]).

Let B' be a Riemann surface and $\mathcal{M} = (M, \pi, B)$ a holomorphic family of Riemann surfaces. For a holomorphic covering mapping $f\colon B' \to B$, we define the *pullback bundle* $f^*\mathcal{M} = (\tilde{M}, \pi', B')$ is defined by

$$\tilde{M} = \{(b', x) \in B' \times M \mid f(b') = \pi(x)\}$$

and $\pi'(b', x) = b'$. When $B = \mathbf{D}/\Gamma$ and $B' = \mathbf{D}/\Gamma'$ with $\Gamma' \subset \Gamma$, the representation of $\tilde{\mathcal{M}}$ coincides with that of $\mathcal{M}$ (up to conjugation), and the monodromy of $\tilde{\mathcal{M}}$ is obtained by restricting that of M with respect to the representation to Γ_0.

3. Proof of the main theorem.

3.1. Rigidity theorem for holomorphic disks. In [6], the author discussed a version of Lusin-Priwaloff-Riesz theorem on holomorphic disks in Teichmüller space by using extremal length geometry on Teichmüller space (cf. Theorem 1 in [6]).

Theorem 3.1 (Rigidity of holomorphic disks). *Let f_1 and f_2 be holomorphic mappings from the unit disk $\mathbf{D}$ to $\mathcal{T}_{g,n}$. Suppose that there is a measurable set $E \subset \partial\mathbf{D}$ of positive linear measure with the following property: For any $z_0 \in E$, there is a sequence $\{z_n\}_{n=1}^{\infty} \subset \mathbf{D}$ such that $z_n \to z_0$ non-*

tangentially and $\langle f_1(z_n) \mid f_2(z_n)\rangle_{x_0} \to \infty$. *Then,* $f_1(z) = f_2(z)$ *for all* $z \in \mathbf{D}$.

3.2. Proof of the main theorem. Let us prove Theorem 1.1. Let $\Phi_i: \mathbf{D} \to \mathcal{T}_{g,n}$ be the representation of the holomorphic family $\mathcal{M}_i$ and ρ_i be the monodromy with respect to Φ_i $(i = 1, 2)$. Without loss of generality, we assume that φ is the identity element in $\mathrm{Mod}_{g,n}$.

Let $y_i = \Phi_i(0)$ for $i = 1, 2$. Then,

$$\begin{aligned}(3)\qquad &\langle \rho_1(\gamma)(y_1) \mid \rho_2(\gamma)(y_2)\rangle_{x_0}\\ &\geq \langle \rho_1(\gamma)(x_1) \mid \rho_2(\gamma)(x_2)\rangle_{x_0}\\ &\quad - d_T(x_1, y_1) - d_T(x_2, y_2)\end{aligned}$$

for all $\gamma \in \Gamma$. Let $z_0 \in \partial\mathbf{D}$ be a conical limit point of Γ. Take $\{\gamma_n\}_{n=1}^{\infty} \subset \Gamma$ such that $\gamma_n(0) \to z_0$ non-tangentially. From the proof of the rigidity theorem in [4] we assume that $\{\Phi_i(\gamma_n(0))\}_{i=1}^{\infty}$ is a divergence sequence in $\mathcal{T}_{g,n}$ for $i = 1, 2$. From the assumption and (3), we have

$$\begin{aligned}&\langle \Phi_1(\gamma_n(0)) \mid \Phi_2(\gamma_n(0))\rangle_{x_0}\\ &\quad = \langle \rho_1(\gamma_n)(y_1) \mid \rho_2(\gamma_n)(y_2)\rangle_{x_0}\\ &\quad \geq \langle \rho_1(\gamma_n)(x_1) \mid \rho_2(\gamma_n)(x_2)\rangle_{x_0} + O(1) \to \infty\end{aligned}$$

as $n \to \infty$. Since $\mathbf{D}/\Gamma$ is of class O_G, the conical limit set has full linear measure in $\partial\mathbf{D}$. From Theorem 3.1, we conclude that $\Phi_1 = \Phi_2$ on $\mathbf{D}$. From Lemma 3.1 proven below, there is a finite index subgroup Γ_0 of Γ such that $\rho_1 = \rho_2$ on Γ_0 and $[\Gamma : \Gamma_0] \leq 42(2g - 2 + n)$. Therefore, by virtue of Imayoshi-Shiga's rigidity theorem, two holomorphic families $\mathcal{M}_1$ and $\mathcal{M}_2$ are virtually isomorphic via a finite covering $\mathbf{D}/\Gamma_0 \to \mathbf{D}/\Gamma$ (cf. [4]).

3.3. A lemma. To complete the proof of Theorem 1.1, we shall show the following lemma:

Lemma 3.1. *Under the conditions in the above proof, we have the following*:

(a) *There is a finite index subgroup* Γ_0 *of* Γ *such that* $\rho_1 = \rho_2$ *on* Γ_0 *and* $[\Gamma : \Gamma_0] \leq 42(2g - 2 + n)$.

(b) *If the automorphism group of some fiber of* M_i *is trivial, then* $\rho_1 = \rho_2$ *on* Γ.

Proof. (a) For the simplicity, we let $\Phi = \Phi_1 = \Phi_2$ on $\mathbf{D}$. For $z \in \mathbf{D}$, let us denote by $\mathrm{Stab}(\Phi(z))$ the stabilizer subgroup of $\Phi(z)$ in $\mathrm{Mod}_{g,n}$. Set $H = \cap_{z\in\mathbf{D}}\mathrm{Stab}(\Phi(z))$ and $\psi_\gamma = \rho_2(\gamma) \circ \rho_1(\gamma)^{-1}$ for $\gamma \in \Gamma$. Then, for all $z \in \mathbf{D}$ and $\gamma \in \Gamma$,

$$\begin{aligned}\psi_\gamma(\Phi(z)) &= \rho_2(\gamma) \circ \rho_1(\gamma)^{-1}(\Phi(z))\\ &= \rho_2(\gamma) \circ \Phi(\gamma^{-1}(z))\\ &= \Phi(\gamma\gamma^{-1}(z)) = \Phi(z)\end{aligned}$$

and $\psi_\gamma \in H$. Notice that

$$\begin{aligned}\psi_{\gamma_1\gamma_2} &= \rho_2(\gamma_1\gamma_2) \circ \rho_1(\gamma_1\gamma_2)^{-1}\\ &= \rho_2(\gamma_1) \circ \rho_2(\gamma_2) \circ \rho_1(\gamma_2)^{-1} \circ \rho_1(\gamma_1)^{-1}\\ &= \psi_{\gamma_1} \circ (\rho_2(\gamma_1) \circ \psi_{\gamma_2} \circ \rho_2(\gamma_1)^{-1}).\end{aligned}$$

Define a homomorphism $\alpha: \Gamma \to \mathrm{Aut}(H)$ by $\alpha[\gamma](h) = \rho_2(\gamma) \circ h \circ \rho_2(\gamma^{-1})$. Then the operation on the semidirect product $H \rtimes_\alpha \Gamma$ is defined by

$$(h_1, \gamma_1) \cdot (h_2, \gamma_2) = (h_1\alpha[\gamma_1](h_2), \gamma_1\gamma_2).$$

From the above argument, we have a well-defined homomorphism

$$\beta: \Gamma \ni \gamma \mapsto (\psi_\gamma, \gamma) \in H \rtimes_\alpha \Gamma.$$

Let $G_0 = \{id\} \times \Gamma < H \rtimes_\alpha \Gamma$. Since H is contained in the stabilizer group of $\Phi(z)$ in $\mathrm{Mod}_{g,n}$, H is a finite group of index at most $2\pi|2 - 2g - n|/(\pi/21) = 42(2g - 2 + n)$ and G_0 is a finite index subgroup in $H \rtimes_\alpha \Gamma$ of index $|H|$. Therefore, we deduce

$$[\Gamma : \Gamma_0] \leq [H \rtimes_\alpha \Gamma : G_0] = |H| \leq 42(2g - 2 + n).$$

Notice from the definition that $\rho_2(\gamma) \circ \rho_1(\gamma)^{-1} = \psi_\gamma = id$ holds for all $\gamma \in \Gamma_0$.

(b) If the automorphism group of some fiber of M_i is trivial, the group H is the trivial group and hence $\Gamma_0 = \Gamma$ in the above construction. □

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