Weighted inequalities for spherical maximal operator

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Abstract: Given a set $E = (0,\infty)$, the spherical maximal operator $\mathcal{M}$ associated to the parameter set E is defined as the supremum of the spherical means of a function when the radii of the spheres are in E. The aim of this paper is to study the following inequality

$$\int_{\mathbf{R}^n} (\mathcal{M}f(x))^p \phi(x)dx \le B_p \int_{\mathbf{R}^n} |f(x)|^p \phi(x)dx, \tag{0.1}$$

holds for $p > \frac{2n}{n-1}$ with the continuous spherical maximal operator $\mathcal{M}$ and where the nonnegative function ϕ is in some weights obtained from the A_p classes. As an application, we will get the boundedness of vector-valued extension of the spherical means.

Key words: Spherical maximal operator; oscillatory integrals; A_p weights.

1. Introduction. The aim of this paper is to study the boundedness of spherical maximal operator corresponding to $E = (0,\infty)$ from $L^p(\phi)$ to $L^p(\phi)$, where the nonnegative function ϕ is in some weights obtained from the A_p classes. Given a function f, continuous and compactly supported, we consider for each $x \in \mathbf{R}^n$ and $t > 0$, the operator $S_t f(x) = \int_{\mathbf{S}^{n-1}} f(x - ty)d\sigma(y)$, where $d\sigma$ is the normalized Lebesgue measure over the unit sphere $\mathbf{S}^{n-1}$. Then $S_t f(x)$ is the mean value of f over the sphere of radius t centered at x and it defines a bounded operator in $L^p(\mathbf{R}^n)$ for $1 \le p \le \infty$. Consider now the spherical maximal operator given by

$$\mathcal{M}f(x) = \sup_{t>0} |S_t f(x)|.$$

Then $\mathcal{M}$ defines a bounded operator in $L^p(\mathbf{R}^n)$ if and only if $p > \frac{n}{n-1}$ with $n > 1$. This result was first proved by Stein [3,10], for $n \ge 3$ and later Bourgain showed that the same is true for $n = 2$, (see [2]). Other proof for $n = 2$ is due to Mockenhaupt, Seeger and Sogge [4]. In [1], Javier Duoandikoetxea and Luis Vega, they have studied the weighted inequalities of the type

$$\int_{\mathbf{R}^n} (\mathcal{M}f(x))^p \phi(x)dx \le B \int_{\mathbf{R}^n} |f(x)|^p \phi(x)dx,$$

with B depending only on p and ϕ and where the weight ϕ is a locally integrable nonnegative function. They proved that there are no weighted inequalities for $p \le \frac{n}{n-1}$, and have proven the most interesting result: $\mathcal{M}$ is bounded in $L^p(|x|^\alpha)$ if $1 - n < \alpha < (n-1)(p-1) - 1$. We shall use c as a constant independent of j in several spaces without mentioning it. In this paper, we shall consider the weights ϕ with the property

$$W = \{\phi : \mathcal{M}\phi(x) \le c\,\phi(x)\ a.e.\}, \tag{1.1}$$

where $\mathcal{M}$ is the spherical maximal operator. For example, $\phi(x) = |x|^\alpha$ $(1 - n < \alpha \le 0) \in W$. Under the above assumption (1.1) on the weights, we shall study the following weighted inequalities

$$\int_{\mathbf{R}^n} (\mathcal{M}f(x))^p \,\phi(x)\,dx \le B_p \int_{\mathbf{R}^n} |f(x)|^p \,\mathcal{N}\phi(x)\,dx, \tag{1.2}$$

where the operator $\mathcal{N}$ is defined by

$$\mathcal{N}\phi(x) = \sup_{1\le t\le 2} \int_{|y|=1} \phi(x + ty)|d\sigma(y)|, \tag{1.3}$$

where $|d\sigma|$ is the total variation measure on the sphere $|y| = 1$. In fact, in [1], they have studied the following weighted inequalities,

$$\int_{\mathbf{R}^n} (\mathcal{M}f(x))^p \,\phi(x)\,dx \le B_p \int_{\mathbf{R}^n} |f(x)|^p \,\mathcal{N}\phi(x)\,dx,$$

with some operator $\mathcal{N}$. They have suggested that the operator $\mathcal{N}$ is in some sense similar to $\mathcal{M}$ and at least one expects to have

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$$\mathcal{N}u \le c\,u \text{ for } u^s \in W \text{ with } s > 1.$$

Notice that, in our case, using Hölder inequality, we have,

$$\mathcal{N}\phi \le c\,\phi \text{ for } \phi^s \in W \text{ with } s > 1. \tag{1.4}$$

Here, one could now replace $\mathcal{N}\phi$ with ϕ at the right hand side of (1.2), reducing the result of (1.2) to a one-weight inequality i.e., the boundedness of $\mathcal{M}$ from $L^p(\phi)$ to $L^p(\phi)$. The main application was exactly the vector-valued extension of the classical Hardy-Littlewood maximal theorem as in [5] to spherical means.

1.1. The dyadic maximal operator. The proof of our main result, as well as many other arguments that involve explicitly (or implicitly) the Fourier transform, makes use of the division of the dual (frequency) space into dyadic spherical shells. Dyadic decomposition, whose ideas originated in the work of Littlewood and Paley, and others, will now be described in the form most suitable for us. Let ψ be a nonnegative radial function in $C_c^\infty(\mathbf{R}^n)$ supported in $\{\frac{1}{2} \le |\xi| \le 1\}$ such that $\sum_{j=1}^\infty \psi(2^{-j}\xi) = 1$ for $|\xi| \ge 1$. Define $\psi_j(\xi) = \psi(2^{-j}\xi)$ for $j > 0$, and $\phi_0(\xi) = 1 - \sum_{j=1}^\infty \psi_j(\xi)$. Denote by σ^j, μ the C^∞ functions given by

$$\widehat{(\sigma^j)}(\xi) = \widehat{(d\sigma)}(\xi)\,\psi_j(\xi) \text{ and } \widehat{\mu}(\xi) = \widehat{(d\sigma)}(\xi)\,\phi_0(\xi).$$

Let S_t^j and B_t be the operators defined by

$$S_t^j f(x) = \int_{\mathbf{R}^n} f(x - ty)\,\sigma^j(y)\,dy \text{ and}$$

$$B_t f(x) = \int_{\mathbf{R}^n} f(x - ty)\,\mu(y)\,dy.$$

Notice that B_t is pointwise majorized by a constant times the Hardy-Littlewood maximal operator M. Then,

$$\mathcal{M}f(x) \le \sum_{j=0}^\infty \sup_t |S_t^j f(x)| + c\,Mf(x).$$

1.2. Angular decomposition. We now discuss the second dyadic decomposition of the frequency-space that is needed for each dyadic operators S_t^j in $\mathbf{R}^n$. To do this, for fixed j, we first choose unit vectors ξ_j^ν, $\nu = 1, \ldots, N(j)$ such that

$$|\xi_j^\nu - \xi_j^{\nu'}| \ge C_0 2^{-\frac{j}{2}}, \nu \ne \nu',$$

for some positive constant C_0 and such that balls of radius $2^{-\frac{j}{2}}$ centered at ξ_j^ν cover $\mathbf{S}^{n-1}$. Note that

$$N(j) \approx 2^{j\frac{(n-1)}{2}}.$$

They give an essentially uniform grid on the unit sphere, with separation $2^{-\frac{j}{2}}$. Let Γ_j^ν denote the corresponding cone in the $\xi-$ space whose central direction is ξ_j^ν, i.e.,

$$\Gamma_j^\nu = \left\{\xi : \left|\frac{\xi}{|\xi|} - \xi_j^\nu\right| \le c\,2^{-\frac{j}{2}}\right\}.$$

Now, we introduce an associated partitions of unity $\mathbf{R}^n \backslash \{0\}$ that depend on scale j. Specifically, we choose C^∞ functions

$$\chi_\nu,\ \nu = 1, \ldots, N(j) \approx 2^{j\frac{n-1}{2}},$$

satisfying $\sum_\nu \chi_\nu = 1$ and having the additional properties,

(1) χ_ν's are to be homogeneous of degree zero and satisfy the uniform estimates,

$$|D^\alpha \chi_\nu(\xi)| \le c_\alpha\, 2^{\frac{j|\alpha|}{2}} \text{ for } \textit{every } \alpha \text{ if } |\xi| = 1. \tag{1.5}$$

(2) $\chi_\nu(\xi_j^\nu) \ne 0$ and χ_ν's are to have the natural support properties

$$\textit{i.e.}, \chi_\nu(\xi) = 0 \textit{ if } |\xi| = 1 \text{ and } |\xi - \xi_j^\nu| \ge c\,2^{-\frac{j}{2}}. \tag{1.6}$$

Using the homogeneous partitions of unity χ_ν, we make an angular decomposition of the operators by setting

$$\widehat{(\sigma_\nu^j)}(\xi) = \widehat{(d\sigma)}(\xi)\,\psi_j(\xi)\,\chi_\nu(\xi),$$

and define the corresponding operators

$$S_t^{j.\nu} f(x) = \int_{\mathbf{R}^n} f(x - ty)\sigma_\nu^j(y)dy.$$

1.3. A_p weights. In this section, we introduce weight class A_p $(1 < p < \infty)$ consisting of weights ϕ for which the Hardy-Littlewood maximal operator M is bounded in $L^p(\phi)$ (see García-Cuerva and Rubio de Francia, ch.4 [7]). The weights ϕ in A_p $(1 < p < \infty)$ were characterized by Mockenhoupt as

$$\sup_Q \left(\frac{1}{|Q|}\int_Q \phi\,dx\right)\left(\frac{1}{|Q|}\int_Q \phi^{-\frac{1}{p-1}}\,dx\right)^{p-1} \le c, \tag{1.7}$$

where the supremum is taken over all cubes (or balls) Q in $\mathbf{R}^n$. For the case $p = 1$, we refer to the weights which satisfy $M\phi(x) \le c\,\phi(x)\,a.e.\ x \in \mathbf{R}^n$, as A_1 weights. Now, we shall state the weighted versions of the Littlewood-Paley inequalities. Recall that L_ϕ^p $(1 < p < \infty)$ denotes the L^p space on $\mathbf{R}^n$ with measure $\phi(x)dx$. The following result (Theorem 1.1) for $\phi \in A_p$ is basically known. This was first proved by Kurtz ([6]) for ψ supported in an annulus. Here, we shall state the result for $\phi \in A_p$.

Theorem 1.1. *Let $1 < p < \infty$ and $\phi \in A_p$. Let $\psi_o \in \mathbf{C}_c^\infty$, have nonzero integral, and $\psi = \psi_0 - 2^{-n}\psi_0(\frac{\cdot}{2})$. Consider the square function operator S given by*

$$S(f)(x) = \left(\sum_j |\psi_j * f(x)|^2\right)^{\frac{1}{2}}$$
$$(f \in \mathscr{S},\ x \in \mathbf{R}^n,\ \hat{\psi}_j = \hat{\psi}(2^{-j}\cdot)).$$

Then,

$$\|S(f)\|_{L^p_\phi} \approx \|f\|_{L^p_\phi},\ \forall f \in \mathscr{S}. \tag{1.8}$$

Here, $\approx$ means the bilateral estimate with positive constants independent of f.

Remark 1.2. In fact, the condition $\phi^s \in W$, $s > 1$ implies that $\phi^s \in A_1$. This is because pointwise $M\phi(x) \le \mathcal{M}\phi(x)$, where M is the Hardy-Littlewood maximal operator.

Next, we will state our main result of this paper in (§, 2).

2. Main results.

Theorem 2.1. *Let $\mathcal{M}$ be a spherical maximal operator. Then, for $p > \frac{2n}{n-1}$, the inequality*

$$\int_{\mathbf{R}^n} (\mathcal{M}f(x))^p\, \phi(x)\, dx \le B_p \int_{\mathbf{R}^n} |f(x)|^p\, \mathcal{N}\phi(x)\, dx, \tag{2.1}$$

holds for $f \in \mathscr{S}$ and the operator $\mathcal{N}$ and the weights ϕ satisfy the condition as in (1.4)*, with constant B_p depending only on p.*

Theorem (2.1) has the following consequence on unweighted spherical means.

Corollary 2.2. *Let f be a bounded measurable function on $\mathbf{R}^n$, $n > 1$. Then the inequality*

$$\|\mathcal{M}f\|_{L^p(\mathbf{R}^n)} \le B_p\, \|f\|_{L^p(\mathbf{R}^n)}$$

holds whenever $p > \frac{2n}{n-1}$.

The proof of the Corollary follows immediately by taking $\phi \equiv 1$ in (2.1) of the above theorem (2.1).

Proof of Theorem 2.1. We shall prove that, for $p > \frac{2n}{n-1}$,

$$\int_{\mathbf{R}^n} \sup_{t>0} |S_t f(x)|^p\, \phi(x)\, dx \le B_p \int_{\mathbf{R}^n} |f(x)|^p\, \mathcal{N}\phi(x)\, dx. \tag{2.2}$$

Our proof will consist of three main steps. First we will decompose each S_t into dyadic operator S_t^j. Then we will use the method of Littlewood-Paley square function to deduce the general result for $t > 0$ from the inequality where the supremum is only taken over $t \in [1, 2]$. We will then use the method of stationary phase to expose the behavior of our operator $S_t^{j,\nu}$.

We now turn to the details. To obtain the inequality (2.2) by summing a geometric series, it is enough to prove the following: There exists a constant $\epsilon(p) > 0$ such that for $p > \frac{2n}{n-1}$,

$$\int_{\mathbf{R}^n} \sup_{1\le t\le 2} |S_t^j f(x)|^p\, \phi(x) dx \le c\, 2^{-j\,\epsilon(p)} \int_{\mathbf{R}^n} |f(x)|^p\, \mathcal{N}\phi(x)\, dx. \tag{2.3}$$

By rescaling, the inequality (2.3) will be true for $\sup\limits_{2^k \le t \le 2^{k+1}}$ with the same constant.

To see that (2.3) is enough, we need to use Littlewood-Paley operators L_k, which are defined by $\widehat{(L_k f)}(\xi) = \psi(2^{-k}|\xi|)\ \hat{f}(\xi)$, where ψ's are defined in (§, 1). It then must follow that there is an absolute constant C_0 such that, when $t \in [1, 2]$, we have

$$S_t^j f(x) = S_t^j\left(\sum_{|j-k|\le C_0} L_k f\right)(x).$$

Thus, if (2.3) held, then a scaling argument will give,

$$\begin{aligned}
&\int \sup_{t>0} |S_t^j f(x)|^p\, \phi(x)\, dx \\
&\le \sum_{l=-\infty}^{\infty} \int \sup_{t\in[2^l, 2^{l+1}]} \left| S_t^j \left(\sum_{|k+l-j|\le C_0} L_k f\right)(x)\right|^p \phi(x) dx \\
&\le c^p\, C_0\, 2^{-j\epsilon(p)\,p} \int \sum_{k=-\infty}^{\infty} |L_k f(x)|^p\, \mathcal{N}\phi(x)\, dx \\
&\le c^p\, C_0\, 2^{-j\epsilon(p)\,p} \int \left(\sum_{k=-\infty}^{\infty} |L_k f|^2\right)^{(\frac{p}{2})} \mathcal{N}\phi(x)\, dx.
\end{aligned}$$

In the last step we have used the fact that $p > \frac{2n}{n-1} > 2$. Now using the weighted inequalities (1.8) for the Littlewood-Paley decomposition [6], we finish our proof.

Next we claim that the inequality (2.3) in turn would follow from the estimates

$$(2.4)\qquad \int_{\mathbf{R}^n} \sup_{1\le t\le 2} |S_t^{j,\nu} f(x)|^p \phi(x)dx \le 2^{-j[p\frac{n-1}{2}+\epsilon(p)]} \int_{\mathbf{R}^n} |f(x)|^p \mathcal{N}\phi(x)\,dx.$$

To show that (2.4) implies (2.3), using Hölder inequality for sums, we get,

$$\begin{aligned}(2.5)\ \int_{\mathbf{R}^n} \sup_{1\le t\le 2} |S_t^{j} f(x)|^p\,\phi(x)\,dx &\le 2^{\frac{j(n-1)(p-1)}{2}} \sum_\nu \left[\int_{\mathbf{R}^n} \sup_{1\le t\le 2} |S_t^{j,\nu} f(x)|^p\,\phi(x)\,dx\right]\\ &\le 2^{\frac{j(n-1)(p-1)}{2}} \sum_\nu 2^{-j[p\frac{n-1}{2}+\epsilon(p)]} \int_{\mathbf{R}^n} |f(x)|^p \mathcal{N}\phi(x)\,dx\\ &= c\,2^{-j\,\epsilon(p)} \int_{\mathbf{R}^n} |f(x)|^p \mathcal{N}\phi(x)\,dx,\end{aligned}$$

where we use (2.4) in the second inequality. To prove (2.4), we shall use a Sobolev embedding to replace $\sup_{1\le t\le 2} |S_t^{j,\nu} f(x)|^p$ with

$$\int_1^2 |D_t^\beta S_t^{j,\nu} f(x)|^p dt,\ \beta > \frac{1}{p}.$$

Compute the norm for $\beta = 0$ and $\beta = 1$ and then interpolate. Using Hölder inequality, for $\beta = 0$ and $p > \frac{2n}{n-1}$, we have

$$\begin{aligned}(2.6)\qquad \int_{x\in\mathbf{R}^n} \int_1^2 |S_t^{j,\nu} f(x)|^p\,\phi(x)\,dx\,dt &= \int_{x\in\mathbf{R}^n} \int_1^2 \left|\int_{y\in\mathbf{R}^n} f(x-ty)\,\sigma_\nu^j(y)\,dy\right|^p \times \phi(x)\,dx\,dt\\ &\le \int_1^2 \left(\int_y \left|\sigma_\nu^j\left(\frac{y}{t}\right)\right|dy\right)^{p-1} \times \int_x\int_y t^{-n}\,|f(x-y)|^p \left|\sigma_\nu^j\left(\frac{y}{t}\right)\right|dy \times \phi(x)\,dx\,dt\\ &= \int_1^2 \left(\int_y \left|\sigma_\nu^j\left(\frac{y}{t}\right)\right|dy\right)^{p-1} \int_y |f(y)|^p \times \left[\int_x \phi(y+tx)\,|\sigma_\nu^j(x)|\,dx\right] dy\,dt\\ &\le \int_1^2 \left(\int_y \left|\sigma_\nu^j\left(\frac{y}{t}\right)\right|dy\right)^{p-1} dt \times \int_y |f(y)|^p\,\mathcal{N}\phi(y)\,dy.\end{aligned}$$

In the last step we have used the fact that $|d\sigma(x)| \ge |\sigma_\nu^j(x)|$, where $|\sigma_\nu^j(x)|$ is the total variation norm. Therefore, we have, $\int_x \phi(y+tx)\,|\sigma_\nu^j(x)|\,dx \le \mathcal{N}\phi(y)$.

Now our aim is to estimate the following integral $\int_y |\sigma_\nu^j(\frac{y}{t})|\,dy$. Using the property of Bessel's function, let us consider the following estimate $\widehat{(d\sigma)}(\xi) = e^{i|\xi|}\,a(\xi)(|\xi|)^{-\frac{n-1}{2}}$, for $|\xi| \in [2^{j-1}, 2^j]$, where a is $C^\infty(\mathbf{R}^n \setminus \{0\})$, homogeneous of degree zero (see [8,9]). We have,

$$\begin{aligned}\sigma_\nu^j\left(\frac{y}{t}\right) &= \int_\xi e^{i\langle\xi,y\rangle}\,t^n\,\widehat{(d\sigma)}(t\xi)\,\psi_j(t\xi)\,\chi_\nu(\xi)\,d\xi\\ &= \int_\xi e^{i\langle\xi,y\rangle}\,t^n\,e^{it|\xi|}\,a(\xi)\,(t|\xi|)^{-\frac{n-1}{2}}\,\psi_j(t\xi)\,\chi_\nu(\xi)\,d\xi\\ &= 2^{-\frac{j(n-1)}{2}}\,t^{\frac{n+1}{2}}\,F(y,t),\end{aligned}$$

where, $F(y,t) = \int_{|\xi|=2^{j-1}}^{2^j} e^{i\langle\xi,y\rangle}\,e^{it|\xi|}\,a'(\xi)\,\psi(t|\xi|2^{-j})\,\chi_\nu(\xi)\,d\xi$, where a' is homogeneous of degree zero. Since $\psi(0) = 0$, we have

$$\begin{aligned}F(y,t) &= \int_0^t \frac{\partial F}{\partial s}\,ds\\ &= \int_0^t \int_\xi e^{i\langle\xi,y\rangle}\,e^{is|\xi|}\,a'(\xi)\,(2^{-j}|\xi|)\,\psi'(s|\xi|2^{-j}) \times \chi_\nu(\xi)\,d\xi ds\\ &\quad + \int_0^t \int_{|\xi|=2^{j-1}}^{2^j} e^{i\langle\xi,y\rangle}\,(i|\xi|)\,e^{is|\xi|}\,a'(\xi) \times \psi(s|\xi|2^{-j})\,\chi_\nu(\xi)\,d\xi ds.\end{aligned}$$

Hence we have,

$$\int_y \left|\sigma_\nu^j\left(\frac{y}{t}\right)\right|dy = \int_y 2^{-\frac{j(n-1)}{2}} t^{\frac{(n+1)}{2}}\,|F(y,t)|\,dy \le 2^{-\frac{j(n-1)}{2}} t^{\frac{(n+1)}{2}}\,(I_1 + I_2).$$

Using integration by parts and change of variable formula to the following integral we get,

$$\begin{aligned}(2.7)\qquad I_1 &= \int_y \left|\int_0^t \int_\xi e^{i\langle\xi,y\rangle}\,e^{is|\xi|}\,a'(\xi)\,(2^{-j}|\xi|) \times \psi'(s|\xi|2^{-j})\,\chi_\nu(\xi)\,d\xi ds\right|dy\\ &= \int_y 2^{\frac{(n+1)j}{2}} \left|\int_\xi e^{i2^j\langle\xi,y\rangle}\,a'(\xi)\,\chi_\nu(\xi) \times \left(\int_{s=0}^{t|\xi|} e^{is2^j}\,\psi'(s)\,\frac{ds}{|\xi|}\right)d\xi\right|dy\\ &\le c\,2^{\frac{(n+1)j}{2}} \int_y \left|\frac{1}{(1+|2^j y|^2)^N}\right|dy\end{aligned}$$

$$= c\,2^{\frac{(n+1)j}{2}} 2^{-nj} \int_y \left| \frac{1}{(1+|y|^2)^N} \right| dy$$
$$\le c\,2^{-\frac{(n-1)j}{2}} \text{ if } N > \frac{(n+1)}{2},$$

where $a'(\xi)$ is homogeneous function of degree zero in ξ.

To simplify the writing of the estimates for I_2, we set $\bar{\xi} = \xi_j^\nu$, and the corresponding partitions of unity $\bar{\chi}(\xi) = \chi_\nu(\xi)$. Now choose axes in the $\xi-$ space so that ξ_1 is in the direction of $\bar{\xi}$ and $\xi' = (\xi_2, \ldots, \xi_n)$ is perpendicular to $\bar{\xi}$. With this, let, $L = I - 2^{2j}(\frac{\partial}{\partial \xi_1})^2 - 2^j \nabla_{\xi'}{}^2$ and $y = (y_1, y')$ and $\xi = (\xi_1, \xi')$. Next with the help of integration by parts formula (see, Stein, ch.9, [9]) and change of variable formula to the following integral we get,

$$(2.8) \qquad I_2 = \int_y \left| \int_0^t \int_\xi e^{i\langle \xi, y\rangle} (i|\xi|)\, e^{is|\xi|}\, a'(\xi) \times \psi(s|\xi|2^{-j})\, \bar{\chi}(\xi)\, d\xi ds \right| dy$$
$$= 2^j \int_y \left| \int_0^t \int_\xi e^{i[\langle \xi, y\rangle + s\xi_1]} \,[e^{is[|\xi| - \xi_1]}\, a'(\xi) \times \psi(s|\xi|2^{-j})\, \bar{\chi}(\xi)]\, d\xi ds \right| dy$$
$$= \int_y \left| \int_0^t \int_\xi \frac{2^j\, e^{i[\langle y, \xi\rangle + s|\xi|]}}{\{1 + 2^j|y_1 + s| + 2^{\frac{j}{2}}|y'|\}^{2N}} \times L^N[a'(\xi)\, \psi(s|\xi|2^{-j})\, \bar{\chi}(\xi)] d\xi ds \right| dy$$
$$\le c\,2^j\, 2^{-\frac{(n+1)j}{2}} \int_y \left| \frac{1}{\{1 + |y_1| + |y'|\}^{2N}} \right| dy$$
$$\le c\,2^{-\frac{(n-1)j}{2}}, \text{ if } N > \frac{n+1}{2}.$$

Therefore, using (2.7) and (2.8), we have

$$(2.9) \qquad \int_1^2 \left(\int_y \left| \sigma_\nu^j\left(\frac{y}{t}\right) \right| dy \right)^{p-1} dt \le c\,2^{-j(p-1)(n-1)}.$$

Hence, using (2.9) in (2.6) we get,

$$\int_{\mathbf{R}^n} \int_1^2 |S_t^{j,\nu} f(x)|^p\, \phi(x)\, dx dt \le c\,2^{-j(p-1)(n-1)} \int_{\mathbf{R}^n} |f(x)|^p\, \mathcal{N}\phi(x)\, dx.$$

For $\beta = 1$, we have $D_t^1 S_t^{j,\nu}$ which is essentially 2^j times an operator similar to $S_t^{j,\nu}$ so that the above estimate appears multiplied by 2^{jp}. By interpolation, we finally get

$$\int_{\mathbf{R}^n} \int_1^2 |D_t^\beta S_t^{j,\nu} f(x)|^p\, \phi(x)\, dx \le c\,2^{j(-(p-1)(n-1)+p\beta)} \int_{\mathbf{R}^n} |f(x)|^p\, \mathcal{N}\phi(x)\, dx.$$

Inequality (2.4) follows by taking β such that $-(p-1)(n-1) + p\,\frac{n-1}{2} + p\,\beta < -\epsilon(p)$, since $p > \frac{2n}{n-1}$. Hence the theorem. □

3. Application to some maximal inequalities. Inequality of this type (2.1) are important, since among other things, they can be used to derive the boundedness of vector-valued maximal operators. In this section, we extend the spherical maximal inequalities to the case of l^p- valued functions.

Theorem 3.1. *Let $f = (f_1, f_2, \ldots)$ be a sequence of functions on $\mathbf{R}^n$. From the sequence $\mathcal{M}f(x) = \{\mathcal{M}f_1(x), \mathcal{M}f_2(x) \ldots\}$ which k-th term is the spherical maximal function of f_k, we have, for $\frac{2n}{n-1} < r, p < \infty$,*

$$(3.1) \qquad \left\| \left(\sum_{k=1}^\infty |\mathcal{M}f_k(x)|^r \right)^{\frac{1}{r}} \right\|_{L^p} \le A_{r,p} \left\| \left(\sum_{k=1}^\infty |f_k|^r \right)^{\frac{1}{r}} \right\|_{L^p}.$$

Proof. We look separately at the cases $p = r$, $p < r$ and $p > r$. For the case $p = r$, inequality (3.1) follows immediately from the usual spherical maximal theorem, since

$$\int_{\mathbf{R}^n} \sum_{k=1}^\infty |\mathcal{M}f_k(x)|^r dx = \sum_{k=1}^\infty \int_{\mathbf{R}^n} |\mathcal{M}f_k(x)|^r dx \le A_r \sum_{k=1}^\infty \int_{\mathbf{R}^n} |f_k|^r dx = A_r \int_{\mathbf{R}^n} \sum_{k=1}^\infty |f_k|^r dx.$$

For $p < r$, we need some preliminary observation.

Definition. An operator T defined in $L^p(\mathbf{R}^n)$ is called linearizable if there exists a linear operator U defined in $L^p(\mathbf{R}^n)$ whose values are $B-$ valued functions (for some Banach space B) and such that $|Tf(x)| = \|Uf(x)\|_B$, $(f \in L^p(\mathbf{R}^n))$.

In our case, $\mathcal{M}f(x) = \sup_t |S_t f(x)|$, we take $B = L^\infty$, where $(S_t)_{t\in\mathbf{R}_{\ge 0}}$ is a sequence of linear operator. For the case $p < r$ will then follow from the following corollary (see [7], ch.5, corollary 1.23, pp. 482).

Corollary 3.2. *Let T be a linearizable operator which is bounded in $L^p(\mathbf{R}^n)$ for some $1 \le p < \infty$. If T is positive (in the sense that: $|f(x)| \le g(x)$*

a.e., implies $|Tf(x)| \leq Tg(x)$ *a.e.,) then, the following inequalities hold*:

$$\left\| \left(\sum_j |Tf_j|^r \right)^{\frac{1}{r}} \right\|_{L^p} \leq C \left\| \left(\sum_j |f_j|^r \right)^{\frac{1}{r}} \right\|_{L^p}, \quad (p \leq r \leq \infty).$$

To prove the remaining parts $p > r$ of our vector valued spherical maximal theorem, we are going to use duality. That is, we shall control the size of $\|(\sum_{k=1}^{\infty} |\mathcal{M} f_k(x)|^r)^{\frac{1}{r}}\|_{L^p}$ by estimating the integral

$$\int_{\mathbf{R}^n} \left(\sum_{k=1}^{\infty} |\mathcal{M} f_k(x)|^r \right) \phi(x)\, dx,$$

for all ϕ belonging to a suitable space of test functions.

Remark 3.3. We shall use the following observation to prove the remaining part of the proof. In fact the inequality (2.1) of our main theorem, holds for weighted simple function. Here, we consider $\phi = \sum_{i=0}^{m} \alpha_i \chi_{E_i}$ with each $|E_i| < \infty$, disjoint and $\alpha_i \in \mathbf{R}$. Therefore, we have if $x \notin E_i$, $\forall i = 0, \ldots, m$, then (2.1) holds automatically as $LHS = 0$, otherwise if $x \in E_i$ for some i, then $\phi^s \in W$, $s > 1$, since $\mathcal{M}\phi^s \leq c\,\phi^s$ holds, and also $\mathcal{N}\phi \leq c\,\phi$ holds, which gives the inequality (2.1).

Now using the weighted inequalities (2.1) for $r > \frac{2n}{n-1}$ and with the help of duality arguments, we have for positive functions ϕ and $f_1, f_2, \ldots, f_k, \ldots$

$$\begin{aligned} (3.2) \qquad & \int_{x \in \mathbf{R}^n} \left(\sum_{k=1}^{\infty} |\mathcal{M} f_k(x)|^r \right) \phi(x)\, dx \\ &= \sum_{k=1}^{\infty} \int_{x \in \mathbf{R}^n} (|\mathcal{M} f_k(x)|^r) \phi(x)\, dx \\ &\leq B_r \int_{x \in \mathbf{R}^n} \left(\sum_{k=1}^{\infty} |f_k(x)|^r \right) \mathcal{N}\phi(x)\, dx. \end{aligned}$$

If in (3.2), we let ϕ run over the space Σ of simple functions that vanish outside a set of finite measure (as in remark (3.3)), with $\|\phi\|_{L^q} \leq 1$, $\frac{2n}{n-1} < q \leq \infty$, we obtain,

$$(3.3) \qquad \left\| \sum_{k=1}^{\infty} |\mathcal{M} f_k(x)|^r \right\|_{L^{q'}} \leq B_{r,p} \left\| \sum_{k=1}^{\infty} |f_k(x)|^r \right\|_{L^{q'}}$$

where $\frac{1}{q} + \frac{1}{q'} = 1$. Using interpolation and above estimate (3.3), we get

$$\left\| \left(\sum_{k=1}^{\infty} |\mathcal{M} f_k(x)|^r \right)^{\frac{1}{r}} \right\|_{L^p} \leq B_{r,p} \left\| \left(\sum_{k=1}^{\infty} |f_k(x)|^r \right)^{\frac{1}{r}} \right\|_{L^p}$$

for $r \leq p < \infty$, which is the case $p > r$ in (3.1). Hence the theorem. $\square$

Remark 3.4. It would be interesting to know, whether the Theorem (2.1) holds for $p > \frac{n}{n-1}$. If yes, this will give the boundedness of the vector-valued extension of the spherical maximal operators for $\frac{n}{n-1} < r, p < \infty$.

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