Derivatives of meromorphic functions and sine function

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Abstract: In the paper, we take up a new method to prove the following result. Let \( f \) be a meromorphic function in the complex plane, all of whose zeros have multiplicity at least \( k+1 \) \((k \geq 2)\) and all of whose poles are multiple. If \( T(r,\sin z) \equiv o(T(r, f(z))) \) as \( n \to \infty \), then \( f^{(k)}(z) - \sin z \) has infinitely many zeros.

Key words: Meromorphic function; normal family; sine function.

1. Introduction. In his excellent paper [1], W. K. Hayman proved the following result.

Theorem A. Let \( f \) be a transcendental meromorphic function with finitely many zeros in \( \mathbb{C} \). Then \( f^{(k)} \) assumes every finite non-zero value infinitely often.

A natural problem arises: what can we say if “finite non-zero value” in Theorem A is replaced by \( f \) having infinitely many zeros and poles. In fact, the following result [2] was proved in 2015.

Theorem B. Let \( f \) be a transcendental meromorphic function in \( \mathbb{C} \), all but finitely many of whose zeros are multiple, and let \( \alpha(\neq 0) \) be a rational function. Then \( f' - \alpha \) has infinitely many zeros.

In 2008, Liu, Nevo and Pang proved the following result [3].

Theorem C. Let \( f(z) \) be a transcendental meromorphic function of finite order in \( \mathbb{C} \), and \( \alpha(z) = P(z) \exp Q(z) \neq 0 \), where \( P \) and \( Q \) are polynomials. Let also \( k \geq 2 \) be an integer. Suppose that

(a) all zeros of \( f \) have multiplicity at least \( k+1 \), except possibly finitely many, and
(b) \( \lim_{r \to \infty} \left( \frac{T(r, f)}{T(r, \alpha)} \right) = \infty \).

Then the function \( f^{(k)}(z) - \alpha(z) \) has infinitely many zeros. Moreover, in the case that \( p(f) \notin \mathbb{N} \), then the result holds with condition (b) only.

Clearly, \( \alpha(z) \) has only finitely many zeros and poles in Theorem B and Theorem C. Chen, Pang and Yang considered the case that \( \alpha(z) \) has infinitely many zeros and poles. In fact, the following result [4] was proved in 2015.

Theorem D. Let \( f \) be a nonconstant meromorphic function in \( \mathbb{C} \), all of whose zeros have multiplicity at least \( k+1 \) \((k \geq 2)\), except possibly finitely many. Let \( \alpha \) be a nonconstant elliptic function such that \( T(r, \alpha) = o(T(r, f)) \) as \( r \to \infty \). Then \( f^{(k)} - \alpha \) has infinitely many solutions (including the possibility of infinitely many common poles of \( f \) and \( \alpha \)).

Noting that \( \alpha(z) \) is a certain class of double-periodic function in Theorem D, it is a very interesting work to consider the case \( \alpha(z) \) is a certain class of single-periodic function. In this direction, we prove the following results with some new ideas.

Theorem 1.1. Let \( f \) be a meromorphic function of infinite order in \( \mathbb{C} \). Suppose that

(a) all zeros of \( f \) have multiplicity at least \( k+1 \) \((k \geq 2)\), except possibly finitely many, and
(b) all poles of \( f \) are multiple, except possibly finitely many.

Then \( f^{(k)}(z) - \sin z \) has infinitely many zeros.

Theorem 1.2. Let \( f \) be a meromorphic function of finite order in \( \mathbb{C} \). Suppose that

(a) all zeros of \( f \) have multiplicity at least \( k+1 \) \((k \geq 2)\), except possibly finitely many, and
(b) \( T(r, \sin z) = o(T(r, f(z))) \) as \( n \to \infty \) outside of a possible exceptional set of finite linear measure.

Then \( f^{(k)}(z) - \sin z \) has infinitely many zeros.

Remark. Theorem 1.1 and Theorem 1.2 still hold if \( \sin z \) is replaced by \( \cos z \).

Notation. Let \( \mathbb{C} \) be the complex plane and \( D \) be a domain in \( \mathbb{C} \). For \( z_0 \in \mathbb{C} \) and \( r > 0 \), we write \( \Delta(z_0, r) := \{ z : |z - z_0| < r \} \), \( \Delta := \Delta(0,1) \) and \( \Delta'(z_0, r) := \{ z : 0 < |z - z_0| < r \} \). Let \( V(z_0, \theta, A) := \)

doi: 10.3792/pjaa.91.129
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\[ \{ \arg(z - z_0) - \theta_0 < A \} \]
\[ \nabla(z_0, \theta_0, A) := \{ \arg(z - z_0) - \theta_0 \leq A \} \] and \( \Gamma(z_0, r) := \{ z : |z - z_0| = r \} \). Let \( n(r, f) \) denote the number of poles of \( f(z) \) in \( \Delta(0, r) \) (counting multiplicity). We write \( f_n \to f \) in \( D \) to indicate that the sequence \( \{ f_n \} \) converges to \( f \) in the spherical metric uniformly on compact subsets of \( D \) and \( f_n \to f \) in \( D \) if the convergence is in the Euclidean metric.

For \( f \) meromorphic in \( D \), set
\[ f^\#(z) := \frac{|f'(z)|}{1 + |f(z)|^2} \]
and
\[ S(D, f) := \frac{1}{\pi} \int_D |f^\#(z)|^2 \, dz \, dy. \]

The Ahlfors–Shimizu characteristic is defined by \( T_0(r, f) = \int_0^r S(t, f) \, dt \). Let \( T(r, f) \) denote the usual Nevanlinna characteristic function. Since \( T(r, f) - T_0(r, f) \) is bounded as a function of \( r \), we can replace \( T_0(r, f) \) with \( T(r, f) \) in the paper.

The order \( \rho(f) \) of the meromorphic function \( f \) is defined as
\[ \rho(f) := \lim_{r \to \infty} \frac{\log T(r, f)}{\log r} \quad \text{or} \quad \rho(f) := \lim_{r \to \infty} \frac{\log T_0(r, f)}{\log r}. \]

**2. Auxiliary results for the proof of Theorem 1.1.**

**Lemma 2.1.** Let \( \mathcal{F} \) be a family of functions meromorphic in \( D \), all of whose zeros have multiplicity at least \( k \), and suppose that there exists \( A \geq 1 \) such that \( |f^{(k)}(z)| \leq A \) whenever \( f(z) \neq 0 \). Then if \( \mathcal{F} \) is not normal at \( z_0 \in D \), there exist, for each \( 0 \leq \alpha \leq k \),

(a) points \( z_n \in D \), \( z_n \to z_0 \);

(b) functions \( f_n \in \mathcal{F} \); and

(c) positive numbers \( \rho_n \) such that
\[ \rho_n f_n(z_n + \rho_n z_0) \to g(\zeta) \quad \text{in} \quad C, \]
where \( g \) is a nonconstant meromorphic function in \( C \) such that \( g^\#(0) \leq A \alpha + 1 \). In particular, \( g \) has order at most 2.

This is the local version of [5, Lemma 2] (cf. [6, Lemma 1]; [7, pp. 216–217]). The proof consists of a simple change of variable in the result cited from [5]; cf. [8, pp. 299–300].

**Lemma 2.2** ([9, p. 12]). Let \( f(z) \) be a meromorphic function of infinite order in \( C \). Then there exist points \( a_n \to \infty \) and positive numbers \( \delta_n \to 0 \) such that \( f^\#(a_n) \to \infty \) and \( S(\Delta(a_n, \delta_n), f) \to \infty \).

**Lemma 2.3** ([10, Theorem 1 on p. 67]). Let \( k \geq 2 \) be an integer and let \( \{ f_n \} \) be a family of meromorphic functions in \( D \), all of whose poles are multiple and whose zeros all have multiplicity at least \( k + 1 \). Let \( \{ h_n \} \) be a sequence of holomorphic functions in \( D \) such that \( h_n \to h \) in \( D \), where \( h \neq 0 \) in \( D \). Suppose that for each \( n \), \( h \) and \( h_n \) have the same zeros with the same multiplicity and \( f_n^{(k)}(z) \neq h_n(z) \) for \( z \in D \). Then \( \{ f_n \} \) is normal in \( D \).

**Lemma 2.4** ([11, Theorem 1]). Let \( f \) be a meromorphic function in \( D \), and let \( a_1, a_2, a_3 \) be three distinct complex numbers. Assume that the number of zeros of \( \prod_{n=1}^k (f(z) - a_i) \) in \( D \) is \( \leq n \), where multiple zeros are counted only once. Then
\[ S(r, f) \leq n + A \frac{A}{1 - r}, \quad 0 \leq r < 1, \]
where \( A > 0 \) is a constant, which depends on \( a_1, a_2, a_3 \) only.

**Lemma 2.5.** Let \( \{ f_n \} \) be a family of meromorphic functions in \( \Delta(0, r) \). Suppose that

(a) \( f_n \to f \) in \( \Delta^*(0, r) \), where \( f(\neq 0) \) may be \( \infty \) identically, and
(b) there exists \( M > 0 \) such that \( n(\Delta(0, r), f_n^\#) \leq M \) for sufficiently large \( n \).

Then there exists \( M > 0 \) such that \( S(\Delta(0, r/4), f_n) < M \) for sufficiently large \( n \).

Proof. Without loss of generality, we may assume that \( r = 2 \) and \( z_0 = 0 \).

We consider the following two cases.

**Case 1.** \( \Delta \neq 1 \) and \( \Delta \neq 2 \) in \( \Delta(0, 2) \).

Obviously, \( \Delta(0, 1) \cap \Delta(0, 2) = \Delta(0, \frac{1}{2}) \neq 0 \). Thus there exists \( s \in (1, 2) \) such that \( \frac{1}{s} - 1 \) has no poles and zeros on \( \Gamma(s, 0, s) \). For sufficiently large \( n \), we have
\[ n(s, \frac{1}{f_n - 1}) - n(s, \frac{1}{f_n}) = n(s, \frac{1}{f_n - 1}) \]
\[ = n(s, \frac{1}{f_n - 1}) - n(s, \frac{1}{f_n - 1}) \]
\[ = \frac{1}{2\pi i} \int_{\Gamma(s, 0, s)} \frac{(\frac{1}{f_n - 1})'}{\frac{1}{f_n - 1}} \, dz = \frac{1}{2\pi i} \int_{\Gamma(s, 0, s)} \frac{(\frac{1}{f_n - 1})'}{\frac{1}{f_n - 1}} \, dz. \]

Observing that \( \frac{1}{2\pi i} \int_{\Gamma(s, 0, s)} \frac{(\frac{1}{f_n - 1})'}{\frac{1}{f_n - 1}} \, dz \) is an integer, we have for sufficiently large \( n \),
\[ \frac{1}{2\pi i} \int_{\Gamma(s, 0, s)} \frac{(\frac{1}{f_n - 1})'}{\frac{1}{f_n - 1}} \, dz = \frac{1}{2\pi i} \int_{\Gamma(s, 0, s)} \frac{(\frac{1}{f_n - 1})'}{\frac{1}{f_n - 1}} \, dz. \]

Set \( M_1 := \frac{1}{2\pi i} \int_{\Gamma(s, 0, s)} \frac{(\frac{1}{f_n - 1})'}{\frac{1}{f_n - 1}} \, dz + M_0 \). We have for sufficiently large \( n \),
\[ \left( 1, \frac{1}{f_n - 1} \right) \leq n(s, \frac{1}{f_n - 1}) \]
Clearly, $\Delta(a_n, \varepsilon_n) \subset \Delta(b_n, \tau_n)$, $b_n \to \infty$ and $\tau_n \to 0$. By (3.1), we have

$$S(\Delta(b_n, \tau_n), g) \to \infty \text{ as } n \to \infty.$$  

There exist integers $j_n$ and points $x_n \in (-\pi, \pi)$ such that $\widehat{x}_n = x_n - 2\pi j_n$. Taking a subsequence and renumbering, we may assume that $\widehat{x}_n \to \widehat{x}$. Clearly, $\widehat{x} \in [-\pi, \pi]$. Set

$$f_n(z) := f(z + b_n - \widehat{x}_n) \quad \text{and} \quad g_n(z) := g(z + b_n - \widehat{x}_n)$$

for $z \in E$, where

$$E := \{z \mid \text{Re } z \in (-2\pi, 2\pi) \text{ and } \text{Im } z \in (-2\pi, 2\pi)\}.$$  

By (3.2) and (3.3), we have

$$S(\Delta(\widehat{x}_n, \tau_n), f_n) \to \infty \text{ as } n \to \infty.$$  

Set $\tau_n^* := \tau_n + |\widehat{x}_n - \widehat{x}|$. Clearly, $\Delta(\widehat{x}_n, \tau_n) \subset \Delta(\widehat{x}^*, \tau_n^*)$ and $\tau_n^* \to 0$. By (3.4),

$$S(\Delta(\widehat{x}^*, \tau_n^*), g_n) \to \infty \text{ as } n \to \infty.$$  

Now, we have for sufficiently large $n$,

(a1) all zeros of $f_n$ have multiplicity at least $k + 1$ and all poles of $f_n$ are multiple in $E$,

(a2) $f_n^{(k)}(z) \neq \sin(z + iy^*)$ in $E$.

In fact, by (a), (b) and (3.3), (a1) holds for sufficiently large $n$. Since $f_n^{(k)}(z) - \sin z$ has at most finitely many zeros, (a2) holds for sufficiently large $n$ by (3.3).

By Lemma 2.3, $\{f_n\}$ is normal in $E$. Taking a subsequence and renumbering, we may assume that $f_n \Rightarrow f^*$ in $E$.

**Subcase 1.1.** $f^* \neq 0$.

Clearly, there exists $M_0 > 0$ such that $n(\Delta(\widehat{x}^*, 2), 1/f_n^*) < M_0$. By Hurwitz’ Theorem, $n(\Delta(\widehat{x}^*, 1), 1/f_n) < M_0$ for sufficiently large $n$. Thus, $n(\Delta(\widehat{x}^*, 1), 1/g_n) < M_0$ for sufficiently large $n$. Let $\delta \in (0, 1)$ such that $\sin(z + iy^*) \neq 0$ in $\Delta(\widehat{x}^*, \delta)$. Thus, $g_n \Rightarrow \frac{f^*}{\sin(z + iy^*)}$ in $\Delta(\widehat{x}^*, \delta)$. By Lemma 2.5, there exists $M > 0$ such that $S(\Delta(\widehat{x}^*, \delta/4), g_n) < M$ for sufficiently large $n$. This contradicts (3.5).

**Subcase 1.2.** $f^* \equiv 0$.

We see that for sufficiently large $n$,

$$0 \neq f_n^{(k)}(z) - \sin(z + iy^*) \Rightarrow -\sin(z + iy^*) \in E.$$  

By Hurwitz’ Theorem, $\sin(z + iy^*) \neq 0$ in $E$. Thus,

$$g_n(z) = \frac{f_n(z)}{\sin(z + iy^*)} \Rightarrow \frac{f^*(z)}{\sin(z + iy^*)} = 0 \in E.$$
Clearly, \(g^n_n(z) \to 0\) in \(E\), and hence
\[
S(\Delta(x^*, 1), g_n) = \frac{1}{\pi} \int_{\Delta(x^*, 1)} |g^n_n(z)|^2 \, dx \, dy \to 0.
\]
This contradicts (3.5)

**Case 2.** \(y^* = \pm \infty\).

We claim that there exists points \(t_n\) such that
\[
\text{Im} \, t_n \to \infty, \quad \frac{f(t_n)}{\sin t_n} \to 0 \quad \text{and} \quad \frac{f^{(k)}(t_n)}{\sin t_n} \to \infty.
\]
Set
\[
g_n(z) := g(z + a_n) \quad \text{for} \quad z \in \Delta.
\]
Since all zeros of \(g(z)\) have multiplicity at least \(k + 1\) (except possibly finite many), we have for sufficiently large \(n\), all zeros of \(g_n\) have multiplicity at least \(k + 1\) in \(\Delta\). By (3.1), we have
\[
g_n(0) \to \infty \quad \text{as} \quad n \to \infty.
\]
Thus, no subsequence of \(\{g_n\}\) is normal at 0. Using Lemma 2.1 for \(\alpha = k - (1/2)\), there exist points \(z_n \to 0\), positive numbers \(\rho_n \to 0\), and a subsequence of \(\{g_n\}\) (still denoted by \(\{g_n\}\)) such that
\[
G_n(\zeta) = g_n(z_n + \rho_n \zeta) \chi_{\bar{C}}(\zeta) \to G(\zeta) \quad \text{in} \quad C,
\]
where \(G\) is a nonconstant meromorphic function in \(C\), all of whose zeros have multiplicity at least \(k + 1\).

We claim that \(G^{(k)}(\zeta) \neq 0\). Otherwise, \(G(\zeta) = c_{k-1} z^{k-1} + c_{k-2} z^{k-2} + \cdots + c_0\), where \(c_0, c_1, \ldots, c_{k-1}\) are constants. Thus, either \(G \equiv 0\), or all zeros of \(G\) have multiplicity at most \(k - 1\). A contradiction.

Let \(\zeta_0\) be not a zero or pole of \(G(\zeta)\), and set \(t_n := a_n + z_n + \rho_n \zeta_0\). Noting that \(G^{(k)}(\zeta_0) \to G^{(k)}(\zeta_0)\) as \(n \to \infty\), we see that
\[
g^{(k)}(t_n) = g_n^{(k)}(z_n + \rho_n \zeta_0) = \rho_n^{k-i-(1/2)} G_n^{(i)}(\zeta_0)
\]
\[
= \begin{cases} 
0 & \text{for} \quad i = 0, 1, \ldots, k - 1.
\end{cases}
\]

Clearly, \(\frac{f(t_n)}{\sin t_n} = g(t_n) \to 0\). Since \(y_n \to \infty\) and \(|t_n - a_n| \to 0\), we have \(\text{Im} \, t_n \to \infty\), and hence \(1/2 < |\sin^{-1}(t_n)| < 2\) for sufficiently large \(n\). Thus we have
\[
\frac{f^{(k)}(t_n)}{\sin t_n} = \frac{(g(z) \sin z)^{(k)}}{\sin t_n} \bigg|_{z=t_n}
\]
\[
= \sum_{i=0}^{i=k} C_i \frac{g^{(i)}(z) \sin^{(k-i)}(z)}{\sin t_n} \bigg|_{z=t_n}
\]
\[
= \sum_{i=0}^{i=k} C_i \frac{g^{(i)}(t_n) \sin^{(k-i)}(z)}{\sin t_n} \to \infty.
\]

Without loss of generality, we may assume that \(\text{Im} \, t_n \to +\infty\). Set \(F_n(z) := \frac{f(z + t_n)}{\sin t_n}\) for \(z \in \Delta\). Now, we have for sufficiently large \(n\),
(b1) all zeros of \(F_n\) have multiplicity at least \(k + 1\)
and all poles of \(F_n\) are multiple in \(\Delta\),
(b2) \(F^{(k)}_n(z) \neq \frac{\sin(z + t_n)}{\sin t_n} \to \cos z - i \sin z\) in \(\Delta\).
In fact, (b1) holds by (a) and (b). Since \(F^{(k)}_n(z) - \sin z\) has at most finitely many zeros, (b2) holds for sufficiently large \(n\).

By Lemma 2.3, \(\{F_n\}\) is normal in \(\Delta\). However by (3.6), we have
\[
F_n(0) = \frac{f(t_n)}{\sin t_n} \to 0 \quad \text{and} \quad F^{(k)}_n(0) = \frac{f^{(k)}(t_n)}{\sin t_n} \to \infty.
\]
Hence, no subsequence of \(\{F_n\}\) is normal at \(z = 0\). This is a contradiction.

### 4. Auxiliary results for the proof of Theorem 1.2.

**Lemma 4.1** ([12, Theorem 1.2]). Let \(k \geq 2\) be an integer and \(f\) be a meromorphic function of finite order in \(C\). If \(f\) has infinitely many poles, then \(f^{(k)}\) has infinitely many zeros.

**Lemma 4.2.** Let \(f\) be a meromorphic function in \(C\), let \(R(\not 0)\) be a rational function, and let \(Q(z) = -z^m + c_{m-1}z^{m-1} + \cdots + c_0\), where \(m \geq 2\) is an integer and \(c_0, c_1, \ldots, c_{m-1}\) are constants. Suppose that \(f^{(k)}(z) = R(z) \exp(Q(z))\), where \(k \geq 2\) is an integer. Then for any given constant \(\delta \in (0, \frac{3\pi}{2m})\),
\[
f^{(k-1)}(z) = (1 + r(z)) \frac{R(z) \exp(Q(z))}{Q'(z)} + d_0,
\]
\[
f^{(k-2)}(z) = (1 + s(z)) \frac{R(z) \exp(Q(z))}{[Q'(z)]^2} + d_1 z + d_2
\]
\[\quad \text{in} \quad V(0, 0, \frac{3\pi}{2m} - \delta), \quad \text{where} \quad r(z) \quad \text{and} \quad s(z) \quad \text{are meromorphic in} \quad V(0, 0, \frac{3\pi}{2m} - \delta) \quad \text{and converge uniformly to} \quad 0 \quad \text{as} \quad z \to \infty, \quad d_0, \quad d_1 \quad \text{and} \quad d_2 \quad \text{are constants.}
\]

**Remark.** Lemma 4.2 is stated explicitly in [3, pp. 523–528], so we omit the proof.

### 5. Proof of Theorem 1.2.

We consider the following two cases.

**Case 1.** \(f\) has infinitely many poles.

Clearly, \(f(z) - \sin(z - \pi/2)\) has infinitely many poles. Thus by Lemma 4.1, \(f^{(k)}(z) - \sin z = (f(z) - \sin(z - \pi/2))^{(k)}\) has infinitely many zeros.

**Case 2.** \(f\) has finitely many poles.

Suppose that, to the contrary, \(f^{(k)}(z) - \sin z\) has only finitely many zeros. Clearly, \(f^{(k)}(z) - \sin z\)
has finitely many poles, so we have
\[(5.1) \quad (f(z) - \sin(z - k\pi/2))^{(k)} = f^{(k)}(z) - \sin z = T(z)e^{P(z)},\]
where \(T(z)(\neq 0)\) is a rational function and \(P(z)\) is a polynomial. By the condition (b) of Theorem 1.2, \(P(z)\) is a polynomial of degree \(\geq 2\).

We claim that \(f\) has infinitely many zeros. Otherwise, suppose that \(f\) has finitely many zeros. Then \(f(z) = T_0(z)e^{P_1(z)}\) and hence \(f^{(k)}(z) = T_1(z)e^{P_2(z)}\), where \(T_0(z)(\neq 0)\) and \(T_1(z)(\neq 0)\) are rational functions, \(P_1(z)\) is a polynomial. By (5.1),
\[(5.2) \quad T(z)e^{P(z)} + \sin z = T_1(z)e^{P_1(z)}.
\]
Since \(P(z)\) is a polynomial of degree \(\geq 2\), by (5.2), \(P_1(z)\) must have the same degree and the leading coefficient as \(P(z)\). We write (5.2) in the form
\[(5.3) \quad T(z) + \sin z e^{-P(z)} = T_1(z)e^{P_1(z)-P(z)}.
\]
By standard results in Nevanlinna theory and (5.3), we have
\[\rho(T(z) + \sin z e^{-P(z)}) = \rho(e^{-P(z)}) = \deg P(z),\]
and hence, by (5.1),
\[\rho(T_1(z) e^{P_1(z)-P(z)}) = \deg(P(z)).\]

This is a contradiction.

Set \(\lambda := \sqrt{\frac{-1}{a_m}}\), where \(a_m\) is the leading coefficient of \(P(z)\). Substituting \(z = \lambda \xi\) into (5.1), we obtain that
\[(5.4) \quad (g(\xi) - \sin(\lambda \xi - k\pi/2))^{(k)} = g^{(k)}(\xi) - \lambda^k \sin \lambda \xi = R(\xi)e^{Q(\xi)},\]
where \(g(\xi) = f(\lambda \xi), \quad Q(\xi) = P(\lambda \xi)\) and \(R(\xi) = \lambda^kT(\lambda \xi)\). Thus \(Q(\xi)\) has the following form
\[Q(\xi) = -\xi^m + c_{m-1}\xi^{m-1} + \cdots + c_0,\]
where \(m \geq 2\) is an integer and \(c_0, c_1, \cdots, c_{m-1}\) are constants.

Since \(f\) has infinitely many zeros, we can assume that \(g\) has infinitely many zeros \(\{\xi_n\}\), and all of them are of multiplicity at least \(k+1\). Thus we get
\[(5.5) \quad g(\xi_n) = g'(\xi_n) = \cdots = g^{(k)}(\xi_n) = 0.
\]
Let \(S\) be a subsequence of \(\{\xi_n\}\) (denote it also by \(\{\xi_n\}\)) such that \(\arg(\xi_n)\) converges to \(\alpha\). By (5.4) and (5.5), we have for all \(n\)
\[(5.6) \quad g^{(k)}(\xi_n) = R(\xi_n)\exp(Q(\xi_n)) + \lambda^k \sin \lambda \xi_n = 0.
\]
If \(\alpha \notin \bigcup_{n=0}^{m-1} i\pi \in [-\pi m, \frac{\pi}{m}]\), then \(R(\xi_n)e^{Q(\xi_n)} + \lambda^k \sin \lambda \xi_n \to \infty\), which contradicts (5.6).

Without loss of generality, we may assume that \(\alpha \in \left[ -\frac{\pi}{m}, \frac{\pi}{m} \right]\).

By (5.4) and Lemma 4.2,
\[(5.7) \quad g^{(k-1)}(\xi_n) = (1 + r(\xi_n)) \frac{R(\xi_n)\exp(Q(\xi_n))}{Q'(\xi_n)} + d_1 - \lambda^{k-1} \cos \lambda \xi_n = 0,
\]
and then
\[(5.8) \quad g^{(k-2)}(\xi_n) = (1 + s(\xi_n)) \frac{R(\xi_n)\exp(Q(\xi_n))}{Q''(\xi_n)} + d_2 \xi_n + \frac{d_3}{\lambda} - \lambda^{k-2} \sin \lambda \xi_n = 0,
\]
where \(r(\xi)\) and \(s(\xi)\) are meromorphic in \(V(0, 0, \frac{\pi}{m})\) and converge uniformly to 0 as \(\xi \to \infty\), \(d_1, d_2\) and \(d_3\) are constants. Eliminating \(\sin \lambda \xi_n\) from (5.6) and (5.8), we have for all \(n\)
\[(5.9) \quad R(\xi_n)\exp(Q(\xi_n)) = \lambda^2 (d_2 \xi_n + d_3)Q^3(\xi_n) + \lambda^2 + t(\xi_n),
\]
where \(t(\xi) = \lambda^3 s(\xi)\). Clearly, \(t(\xi)\) are meromorphic in \(V(0, 0, \frac{\pi}{m})\) and converge uniformly to 0 as \(\xi \to \infty\). Noting \(\sin^2 \lambda \xi_n + \cos^2 \lambda \xi_n = 1\), we have by (5.6) and (5.7),
\[(5.10) \quad \lambda^2 \left( 1 + s(\xi_n) \frac{R(\xi_n)\exp(Q(\xi_n))}{Q''(\xi_n)} + d_1 \right)^2 + [R(\xi_n)\exp(Q(\xi_n))]^2 = \lambda^{2k},
\]
for all \(n\). Eliminating \(R(\xi_n)\exp(Q(\xi_n))\) from (5.9) and (5.10), we have for all \(n\)
\[(5.11) \quad |\lambda(d_2 \xi_n + d_3)Q^2(\xi_n)|^2 + |\lambda^2(1 + r(\xi_n))(d_2 \xi_n + d_3)Q'(\xi_n)|^2 + \lambda^2 \exp(Q^2(\xi_n) + \lambda^2 + t(\xi_n))|^2 = 0.
\]
The coefficient of the highest power of \(\xi_n\) in (5.11) is \(\lambda^2 d_2^2 m^4\), so we have \(d_2 = 0\). Thus (5.11) has been reduced into the following form
\[(5.12) \quad |\lambda d_3 Q^2(\xi_n)|^2 + |\lambda^2 d_1(1 + r(\xi_n))Q'(\xi_n)|^2 + \lambda^2(1 + t(\xi_n))^2 = 0.
\]
The coefficient of the highest power of \(\xi_n\) in (5.12) is \((d_1^2 + \lambda^2 d_2^2 - \lambda^{2k-2})m^4\), so we have
\[(5.13) \quad d_1^2 + \lambda^2 d_2^2 - \lambda^{2k-2} = 0.
\]
Thus we have for all \(n\)
\[(5.14) \quad -2\lambda^2 d_1 d_3(1 + r(\xi_n))Q^3(\xi_n) + \lambda^2 d_1^2(1 + r(\xi_n))^2 + 2d_1^2(\lambda^2 + t(\xi_n)) = 0.
\]
\[-2\lambda^{2k-2}(\lambda^2 + t(\xi_n))Q'(\xi_n)\]
\[-2\lambda^2d_1d_3(1 + r(\xi_n))(\lambda^2 + t(\xi_n))Q'(\xi_n)\]
\[\quad + (d_1^2 - \lambda^{2k-2})(\lambda^2 + t(\xi_n))^2 = 0.\]

The coefficient of the highest power of $\xi_n$ in (5.14) is $-2\lambda^2d_1d_3(1 + r(\xi_n))$, so we have
\[(5.15) \quad d_1d_3(1 + r(\xi_n)) = 0 \text{ for all } n.\]

Noting that $d_2 = 0$ and $R(\xi_n)\exp(Q(\xi_n)) \neq 0$ for sufficiently large $n$, we have $d_3 \neq 0$ by (5.9). Since $1 + r(\xi_n) \to 1$ as $n \to 0$, we get $d_1 = 0$ by (5.15). Thus (5.14) has been reduced into the following form
\[(5.16) \quad \lambda^4 d_2^2(1 + r(\xi_n))^2 - 2\lambda^{2k-2}(\lambda^2 + t(\xi_n))Q'(\xi_n)\]
\[\quad - \lambda^{2k-2}(\lambda^2 + t(\xi_n))^2 = 0.\]

Clearly, we must have
\[(5.17) \quad \lambda^4 d_2^2(1 + r(\xi_n))^2 - 2\lambda^{2k-2}(\lambda^2 + t(\xi_n))\]
\[\quad \to \lambda^4d_2^2 - 2\lambda^{2k} = 0.\]

Thus $d_2^2 = 2\lambda^{2k-4}$ and then $d_2^2 + \lambda^2d_3^2 - \lambda^{2k-2} = \lambda^{2k-2} \neq 0$, which contradicts (5.13).

Acknowledgments. The authors would like to express his gratitude to the referee for his very helpful and detailed comments, which have significantly improved the presentation of this paper. This work was supported by the National Natural Science Foundation of China (No. 11371139, No. 11401381), China Postdoctoral Science Foundation (No. 2015M571726) and the Project of Sichuan Provincial Department of Education (No. 15ZB0172).

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