On some Hasse principles for algebraic groups over global fields

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Abstract: We consider certain local-global principles related with some splitting problems for connected linear algebraic groups over global fields. The main tools are certain reciprocity results due to Prasad and Rapinchuk, Harder's Hasse principle for homogeneous projective spaces of reductive groups for number fields and their extensions to global function fields.

Key words: Splitting field; tori; unipotent groups.

1. Introduction. Let k be a field, G a smooth affine algebraic group (i.e. a linear algebraic group) defined over k. A well-known Hasse Principle (in fact Albert–Hasse–Noether's Theorem) (cf. e.g. [Pi]), for central simple algebras (CSA) says that if k is a global field, $V := V_k$ is the set of all places of k, and if a central simple algebra A over k is split over k_v (i.e., $A \simeq M_n(k_v)$, the $n \times n$-matrix algebra over k_v for some n) for all $v \in V$, then A is already split over k, $A \simeq M_n(k)$. There are many other well-known similar results (local–global principles) in other contexts, say Hasse–Minkowski Theorem for quadratic forms, Landherr Theorem for hermitian forms, etc. We may ask, if there is any corresponding result for algebraic groups with a suitable notion of splitting. Recall that (cf. [B, Chap. V, 15.1], [CGP, A.1.2]) a connected solvable algebraic k-group G is *k-split* if there exists a composition series $G = G_0 > G_1 > \cdots > G_{n-1} > G_n = \{1\}$ such that $G_i/G_{i+1} \simeq \mathbf{G_a}$ or $\mathbf{G}_m$, for all $0 \leq i \leq n-1$. Also (cf. [B, Chap. V, 18.6], [CGP, A.4]), a connected reductive k-group G is *k-split* if G has a maximal torus which is defined and split over k. More generally, one says that a smooth connected affine algebraic k-group G is *pseudo-k-split* (or *pseudo-split over k*) if G has a maximal torus which is defined and split over k, see [CGP, Def. 2.3.1]. Here we would like to consider the notion of splitting which really combines the case of solvable and reductive groups as in [T2]. Thus we say that a connected affine algebraic k-group G is *k-split*, or *split over k*, if its unipotent radical $R_u(G)$ is defined and split over k, and the reductive quotient group $G/R_u(G)$ is defined and split over k. Likewise, we say that a smooth affine k-group G is *quasi-split over k* (or *k-quasi-split*) if $R_u(G)$ is defined over k and there exists a Borel subgroup B of $G/R_u(G)$ defined over k.

It is well-known that (see [Ti], [Sat], [Sp, Chap. 15–17]) one can associate to each reductive algebraic group over a field the so-called Tits index. It is the Dynkin diagram of the given group equipped with certain action of the absolute Galois group. It is very useful that one can study the splitness of the given group via its Tits index. In this note we are interested in certain local-global principles related with some splitting properties of the given connected affine groups, related with the Tits index of these groups in some connection with a Hasse principle for homogeneous spaces. A full detailed proof will be published elsewhere.

2. A reciprocity law for algebraic groups over global fields.

2.1. As a main tool in the study of several local-global principles to appear in the sequel we make use of the following results due to Prasad and Rapinchuk [PR] and its extension in [T2].

Recall the following setup. Let G be an (absolutely) almost simple group defined over a field k. Let G_0 be a quasi-split inner k-form of G. Let $\Delta(G,k)$ be the Tits index over k and $\Delta(G,k)_d$ the set of all circled (i.e., distinguished) vertices of $\Delta(G,k)$. There is a so-called $*$-action of $\Gamma := Gal(k_s/k)$ on $\Delta(G,k)$. Denote by Ω_i the Γ-orbits on $\Delta(G_0,k)$, $i = 1, 2, \ldots, r$.

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2.1.1. Theorem ([PR, Theorem 1], [T2, Theorem 1]). *With above notation, assume that k is a global field and G_0 is simply connected. Fix a non-archimedean valuation v_0 of k and assume that there are given k_v-forms, which are inner twists G_v of G_0 for all $v \in V \setminus \{v_0\}$, such that for almost all v, G_v is quasi-split over k_v.*
a) There exists a k-form, which is an inner twist G of G_0 and is k_v-isomorphic to G_v for all $v \in V \setminus \{v_0\}$.
b) If an isotropic k-form G satisfying a) as above exists then there exists an index i, $1 \le i \le r$, such that $\Omega_i \subset \Delta(G_v, k_v)_d$ for all $v \in V \setminus \{v_0\}$ and the k-rank of G is less or equal to the number of orbits satisfying the above inclusion.
c) Let L be the minimal splitting field of G_0. Assume that v_0 is not split in L if $[L : k] = 2$. Then there exists an isotropic k-form G as in a) if there is some orbit Ω_i satisfying b), and there exists a k-form G whose k-rank is equal to the total number of such orbits.

Regarding the uniqueness of the global forms with prescribed local forms as above, we have the following

2.1.2. Theorem (Cf. [PR, Theorem 3], [T2, Theorem 4]). *Let G_0 be an absolutely almost simple simply connected group defined and quasi-split over a global field k, $\bar{G}_0$ the adjoint k-group corresponding to G_0, F_0 the center of G_0 and v_0 a non-archimedean valuation of k. Assume that for all $v \neq v_0$, there are given local k_v-groups G_v which are inner twists of G_0, and consider the k-form G of G_0, which is locally k_v-isomorphic to G_v for all $v \neq v_0$.*
1) The k-form G of G_0 is unique if and only if the localization map

$$\alpha : \mathrm{H}^1_{flat}(k, \bar{G}_0) \to \oplus_{v \neq v_0} \mathrm{H}^1_{flat}(k_v, \bar{G}_0)$$

is injective.
2) α is injective if and only if the following localization map

$$\beta : \mathrm{H}^2_{flat}(k, F_0) \to \oplus_{v \neq v_0} \mathrm{H}^2_{flat}(k_v, F_0)$$

is injective.
3) Let L be the minimal splitting field of G_0, P=L (resp. P is a cubic extension of k contained in L) if $[L : k] \neq 6$ (resp. $[L : k] = 6$, i.e., G_0 is of trialitarian type ${}^6D_4$). Then β is injective if and only if v_0 is not split in P.
4) In general, the uniqueness may not hold and there are only finitely many k-isomorphism classes of above indicated such k-forms G.

(Here $\mathrm{H}^1_{flat}(k, \cdot)$ stands for flat cohomology of algebraic groups.)

As a first application of Theorems 2.1.1–2.1.2, we give an extension (to the case of function field) of a result which due to Harder in the case of number field. In [Ha1] the following Hasse principle for projective homogeneous spaces was proved for number fields.

Theorem A ([Ha1, Satz 4.3.3]). *Let X be a projective homogeneous space of a semisimple group G, all are defined over a number field k. Then the Hasse principle holds for X.*

One should note that the proof given in [Ha1] is only sketched and relies on some other arguments (due to Kneser) related with regular semisimple classes in the case of characteristic 0 (number field). Later on some other proofs were given (see [Bo]), where the main tool used is the theory of non-abelian H^2. Altogether, the proof given in [Ha1] and also the another ones given in [Bo] (using the non-abelian H^2) do not seem to extend to the case of positive characteristic. In this section, we describe yet another proof, which also proves the same result in the case of global function fields, the case that previous proofs do not seem to cover. We have the following

Theorem B. *Let X be a projective homogeneous space of a semisimple group G, all are defined over a global function field k. Then the Hasse principle holds for X.*

Theorem A and Theorem B can be combined to yield the following

2.1.3. Theorem. *Let k be a global field, G a connected linear algebraic group, supposed to be reductive if $char.k > 0$ and let X be a projective homogeneous space of G. Then the Hasse principle holds for X.*

The proof of Theorem 2.1.3 is reduced to proving the following equivalent statement.

2.1.4. Proposition. *Let G be an almost simple group defined over a global field k. If G has a parabolic k_v-subgroup P_v of type $\Theta = \Omega_{i_1} \cup \cdots \cup \Omega_{i_s}$ for all places v of k, then it does so over k.*

2.1.5. Remarks. 1) Theorems 2.1.1–2.1.2 are a kind of reciprocity law for "splitting pattern" of almost simple algebraic groups over global fields. Notice that the proofs of 2.1.1 and 2.1.2 make use of deep results on arithmetic and cohomology of the global fields, culminated in various duality theorems like Tate–Nakayama Theorem, Tate–Poitou

Theorem and local-global class field theory.
2) The proof of Theorem 2.1.3 presented above gives in the case of number fields a new proof of classical result of Harder.
3) Theorem 2.1.3 has also been proved in [CGPa, Corol. 5.7] for fields k of geometric type.

3. A local–global principle related with splitting problems. In this and the next sections we consider some applications (of the results presented in previous section) to some local–global problems related with splitting problems.

3.1. Let notation be as in Section 2. We consider the following problem.

3.1.1. *Assume that a connected smooth affine algebraic group G is L_v-split (resp., L_v-quasi-split) for all $v \in V$, where L_v/k_v is a Galois extension with its Galois group Γ_v belonging to a certain class of groups $\mathcal{C}$. Is it true that G is also split (resp. quasi-split) over a Galois extension L/k with its Galois group Γ also belonging to $\mathcal{C}$? If not, what is the obstruction?*

Here we consider, among the others, the most common class $\mathcal{C}$ of groups such as (pro-)cyclic, (pro-)metacyclic, (pro-)p-, (pro-)nilpotent, or (pro-)solvable groups (cf. also [Sa], [T1]).
In this note we consider above question in the simplest case, where $\Gamma_v = \{1\}$ for all v, i.e., k_v are the (quasi-)splitting field for G for all v. In other words, the first question we try to answer is

3.1.2. *Given that a smooth affine algebraic k-group G is (quasi-)split locally everywhere. Is G already (quasi-)split over k? If not, what is the obstruction?*

Further questions will be discussed in Section 6, after we have given an answer to 3.1.2. Recall that for absolutely almost simple groups over global fields, above question has been considered in [PR] and [T2] (see Theorems 2.1.1–2.1.2) where v does not run over all V, but it runs only over the set $V \setminus \{v_0\}$, with v_0 some fixed non-archimedian place. There were given also some obstruction related with the uniqueness of the global forms in question (see Section 2 for more details).

4. Some reductions to partial cases.

4.1. Solvable case. The first class of groups we are considering is that of solvable algebraic groups. By [Co], there exists a unique maximal connected normal k-split subgroup G_{split} for a given connected solvable k-group G. Thus G is k-split if and only if $G = G_{split}$.
We have the following

4.1.1. Theorem. *Let k be a global field and let G be a solvable k-group. Then G is split over k if and only if G is so over all k_v, $v \in V$.*

We need the following in the proof

4.1.2. Theorem ([Co, Thm. 5.4])**.** *Let k be a field. With above notation, G/G_{split} is a central extension of a k-wound unipotent group U by a k-anisotropic torus T.*

4.1.3. Lemma ([Co, Lemma 5.7])**.** *Let k be a field, U a k-split unipotent group and M an algebraic k-group of multiplicative type. Then any exact sequence*

$$1 \to M \to G \to U \to 1$$

is uniquely split, i.e., we have $G = M \times U$.

4.2. Reductive case. We have the following local-global principle for the splitting.

4.2.1. Theorem. *Let k be a global field and let G be a connected reductive k-group. Then G is split over k if and only if G is so over all k_v, $v \in V$.*

We have two proofs of this result. The first one makes use of Prasad–Rapinchuk's result and its extension to function fields (Theorems 2.1.1–2.1.2) and also Harder's Theorem (and its extension, Theorem 2.1.3) to prove our result. The second one avoids of using 2.1.1–2.1.4 and is more elementary, by making use of only standard facts of algebraic groups and Hasse principles for forms over global fields (see [Sch, Chap. X]). In the proof, we will make a frequent use of the following

4.2.1.1. Theorem ([Ha2, Korollar 1])**.** *If G is an absolutely almost simple group of type different from type* A*, defined over a global function field k, then G is k-isotropic.*

5. Quasi-splitting.

5.1. Let G be a smooth affine algebraic group over a global field k. It is well-known that if G is a connected reductive group, then for almost all $v \in V$, G is quasi-split over k_v, i.e., G has a Borel subgroup defined over k_v. However with our notion of quasi-split groups introduced above, it is not true for general groups. A natural question arises as follows:
If G is quasi-split over k_v for all v, is then G also quasi-split over k?
It is clear that we may assume that G is reductive. Denote by $\mathcal{B}_G$ the variety of Borel subgroups of G. It is well-known that $\mathcal{B}_G$ is defined over k and rational over $\bar{k}$. Thus the question is reduced to the

following Hasse principle for $\mathcal{B}_G$:
Does $\mathcal{B}_G$ have a k-point if it does so over all k_v?
We have the following

5.1.1. Theorem. *Let k be a global field and let G be a connected smooth affine group defined over k. If G is quasi-split over k_v for all v, then so is G over k.*

First proof. We are reduced to the case of reductive groups and then to (absolutely) almost simple k-groups. By Theorem 2.1.3, the variety $\mathcal{B}_G$ has a k-point. □

Second proof. We do not use Theorems 2.1.1–2.1.4 here. Instead, we will make only use of an idea which Kneser employs in his proof of strong approximation theorem as in [Kn]. First we can reduce as above to the case G is semisimple over k and may reduce further to the case, where G is an absolutely almost simple k-group. We notice that the assertion of 5.1.1 is true if G is of inner type, since then G is in fact split over k_v, thus we may apply results of Section 4 to see that G is also split also over k. So we assume that G is of outer type (of Dynkin type A, D or E). Let G_1 be a quasi-split k-group, which is an inner form of G over k and let $\xi \in \mathrm{H}^1_{flat}(k, Ad(G_1))$ be the element corresponding to G, where $Ad(G_1)$ denotes the adjoint group of G_1. Let $\pi : \tilde{G}_1 \to Ad(G_1)$ be the canonical central k-isogeny $\tilde{F} := Cent(\tilde{G}_1)$. Denote by $\tilde{B} = \tilde{T}B_u$ a Borel k-subgroup of $\tilde{G}_1$, where $\tilde{G}_1$ is the simply connected covering of G_1, $\tilde{T}$ the maximal k-torus of $\tilde{B}$ containing a maximal k-split subtorus $\tilde{S}$ of $\tilde{G}_1$. Set $B = \tilde{B}/\tilde{F},\ T = \tilde{T}/\tilde{F},\ S = \pi(\tilde{S})$. Then it is known that $Z_G(\tilde{S}) = \tilde{T}$. We have the following commutative diagram with exact rows

$$\begin{array}{ccccccc}
\mathrm{H}^1_{flat}(k,\tilde{T}) & \xrightarrow{\pi} & \mathrm{H}^1_{flat}(k,T) & \xrightarrow{\Delta} & \mathrm{H}^2_{flat}(k,\tilde{F}) & \xrightarrow{\theta} & \mathrm{H}^2_{flat}(k,\tilde{T}) \\
\downarrow \beta & & \downarrow \alpha & & \downarrow = & & \\
\mathrm{H}^1_{flat}(k,\tilde{G}_1) & \xrightarrow{\pi'} & \mathrm{H}^1_{flat}(k,Ad(G_1)) & \xrightarrow{\Delta} & \mathrm{H}^2_{flat}(k,\tilde{F}) & &
\end{array}$$

and similar diagram over k_v. Let $\zeta = \Delta(\xi) \in \mathrm{H}^2_{flat}(k, \tilde{F})$. Since locally everywhere G is k_v-quasi-split, it means that the restriction of ξ via $res_v : \mathrm{H}^1_{flat}(k, Ad(G_1)) \to \mathrm{H}^1_{flat}(k_v, Ad(G_1))$ belongs to the image of the map $\alpha_v : \mathrm{H}^1_{flat}(k_v, T) \to \mathrm{H}^1_{flat}(k_v, Ad(G_1))$, for all v. Therefore $\theta(\zeta)$ has locally trivial image everywhere. We need the following well-known

5.1.2. Lemma. *Let $\tilde{G}$ be an absolutely almost simple simply connected quasi-split group defined over a field k, $\tilde{S}$ a maximal k-split torus of $\tilde{G}$, and $\tilde{T} = Z_{\tilde{G}}(\tilde{S})$. Then $\tilde{T}$ is k-isomorphic to a direct product of induced tori. In particular, if k is a global field then $\tilde{T}$ satisfies cohomological Hasse principle in degree 2.*

Therefore, by 5.1.2, $\theta(\zeta)$ is trivial, thus $\zeta = \Delta(t), t \in \mathrm{H}^1_{flat}(k, T)$, which is what we need. □

6. Some further applications.

6.1. In this section, we consider some applications related to the local–global behavior of the relative rank (dimension of maximal split subtorus) of a given connected reductive group G defined over a global field k.
Let T be a maximal k-torus of G, T_s the maximal k-split subtorus of T, $T = T_aT_s$, an almost direct product, where T_a is anisotropic k-subtorus of T. Let $s := dim(T_s), a := dim(T_a),\ r := rank_k(G)$, the k-rank of G, $n := s + a = dim(T)$, the rank of G. We say that T is of type (a, s). It is clear that $r \geq s$. For each place v of k, denote $r_v := rank_{k_v}(G)$. Then it is clear that $r_v \geq r$ for all v. There are natural questions related with the behavior of r_v:

6.1.1. *a) Is it true that if for some non-negative integer c and for all v, we have $r_v = c$, then $r = c$?*
b) Is it true that if $r_v > 0$ for all v then so is r?
c) Is it true that if k is a global field and if G has a maximal k_v-torus of type (a, s) over k_v for all places v of k, then so does G over k?
d) Is it true that $min_v\, r_v = r$?

6.1.2. Remarks. 1) It should be mentioned that this question is closely related to questions we considered in previous sections. Namely, if G has a maximal torus T of type $(0, n)$ over a field k, then it means that G is split over k. Therefore the question has an affirmative answer in this case.
2) If G has maximal k_v-tori of type $(1, n-1)$ for all places, then perhaps the best we would say is that G is isotropic over k_v for all places v. In fact, if the semi-simple part of G has at least two almost simple components, then we can construct without difficulty an example of a semisimple group G defined over a global field k such that G is isotropic over k_v for all places v but G is anisotropic over k (see the results below with more precise statements). Therefore 6.1.1 truly makes sense only when we restrict ourselves to the case where G is an absolutely almost simple k-group.

6.2. Theorem. *Let k be a global field, G an absolutely almost simple k-group, and c a non-negative integer.*
a) If $r_v = c$ for all v, then $r = c$.

b) Let G be of Dynkin type different from ${}^1\mathrm{A}_n$, or ${}^1\mathrm{E}_6$ (and k is a real number field). If $r_v > 0$ for all v then $r > 0$.

c) In the remaining cases ${}^1\mathrm{A}_n$ or ${}^1\mathrm{E}_6$, there are global fields k and almost simple k-groups of the corresponding type, for which the local-global principle for isotropy does not hold.

6.2.1. Remark. It follows from above that questions 6.1.1, c)–d) also have negative answers.

7. Existence of rational points on homogeneous spaces. One of the main steps in proving Theorem 2.1.3 is the proof of certain local-global principle for lifting (namely, the lifting of a class of cohomology which is locally liftable). Some of the general results have been proved by Rapinchuk and Borovoi (cf. [Bo]) for number fields. We give some analogs in the case of function fields. We have

7.1. Theorem. 1) (Cf. [Bo, 6.4] for local fields of char. 0). *Let k be a local or global function field and $1 \to G_1 \to G_2 \to G_3 \to 1$ an exact sequence of reductive k-groups, where G_1 is connected and $G_1^{tor} = 1$. Then the induced map $\mathrm{H}^1_{flat}(k, G_2) \to \mathrm{H}^1_{flat}(k, G_3)$ is surjective.*

2) (Cf. [Bo, 6.10] for number fields) *Let k be a global function field, $1 \to G_1 \to G_2 \to G_3 \to 1$ an exact sequence of reductive k-groups, where G_1 is connected with $dim(G_1^{tor}) \leq 1$. If a class of cohomology from $\mathrm{H}^1_{flat}(k, G_3)$ locally is liftable to $\mathrm{H}^1_{flat}(k, G_2)$ then it is so globally.*

7.2. Let G be a connected reductive group, X a homogeneous G-space, all defined over a global field k. In [Bo], a very general results have been proved in the case charcateristic 0, regarding the existence of rational points on X over local or global fields; in particular, Hasse principle of X has been proved under some conditions on the stabilizers of X in G in the case k is a number field. We extend some of these results to the case char. $p > 0$, but under a stronger condition on the stabilizers.

7.3. Theorem (Cf. [Bo, Thm. 7.2, Thm. 7.3] for char. 0 case). *Let k be a field, G a smooth connected (supposed reductive if char.$k > 0$) group, X a right G-homogeneous space, all defined over k, such that for some point $x \in X$, the stabilizer $\bar{H} := Stab(x)$ of x is connected and reductive.*

1) *Then one can associate to the pair (G, X) a gerbe $\mathcal{X}$ with its band $\mathcal{L} := S(\mathcal{X}) := lien(\mathcal{X})$ represented by a connected reductive k-group H and a k-torus $T_{\mathcal{L}}$.*

2) *Assume that k is a local non-archimedean field and one of the following conditions holds:*
i) $\mathrm{H}^2_{flat}(k, T_{\mathcal{L}}) = 0$;
ii) The k-torus $T_{\mathcal{L}}$ is k-anisotropic;
iii) $T_{\mathcal{L}} = 1$.
If $\mathrm{H}^1(k, G) = 0$, then $X(k) \neq \emptyset$.

3) *Assume that k is a global field and one of the following conditions holds*:
i) $\text{Ш}^2(k, T_{\mathcal{L}}) = 0$;
ii) $T_{\mathcal{L}}$ is k_v-anisotropic for some place v;
iii) $T_{\mathcal{L}} = 1$;
iv) $T_{\mathcal{L}}$ is an induced k-torus;
v) $T_{\mathcal{L}}$ is a k-torus split over a cyclic extension of k;
vi) $dim(T_{\mathcal{L}}) \leq 1$.
If $\text{Ш}^1(G) = 0$, then the Hasse principle holds for X, i.e., if $X(k_v) \neq \emptyset$ for all places v of k, then $X(k) \neq \emptyset$. In particular, 1) *and* 2) *above hold if G is a quasi-trivial group* (*supposed to be reductive if char. $k > 0$*).

We derive the following corollaries.

7.3.1. Corollary (Cf. [Bo, Corol. 7.4, 7.6] for number field case). *Let k be a global field and the notation be as above. Assume that G is an absolutely almost simple simply connected k-group, X a k-homogeneous space under G with a stabilizer $H = G^\sigma$, where σ is a semisimple automorphism of G, G^σ the set of all fixed points of σ. If $dim(H/[H, H]) \leq 1$, then the Hasse principle holds for X.*

7.3.2. Corollary (Cf. [Bo, Corol. 7.4] for number field case). *Let k be a global field and let the notation be as above. Assume that the condition 7.3, 3iii) holds. If either $X(k_v) \neq \emptyset$ for all archimedean places v of k or k has no real embeddings, then $X(k) \neq \emptyset$.*

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