A second-order time-discretization scheme for a system of nonlinear Schrödinger equations

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Abstract: We study a linear semidiscrete-in-time finite difference method for the system of nonlinear Schrödinger equations that is a model of the interaction of non-relativistic particles with different masses. The main aim is to show that the scheme is second-order convergent.

Key words: Time-discretization; finite difference method; nonlinear Schrödinger equation; error analysis.

1. Introduction and main results. We consider the following system of nonlinear Schrödinger equations:

$$(1)\qquad \begin{cases} i\partial_t u + \alpha\Delta u = \lambda\bar{u}v, \ t \geq 0, \ x \in \mathbf{R}^d, \\ i\partial_t v + \beta\Delta v = \mu u^2, \ t \geq 0, \ x \in \mathbf{R}^d, \\ u(0,x) = u_0(x), \ v(0,x) = v_0(x), \ x \in \mathbf{R}^d, \end{cases}$$

where u and v are complex-valued functions, Δ is the Laplacian in $\mathbf{R}^d$, α and β are positive constants, and λ and μ are complex constants. This system is a model of the interaction of a non-relativistic particles with different masses.

The mathematical study for (1) is well developed. Throughout this paper, we suppose that

$$s > d/2, \ s : \text{integer}.$$

For any $s_1 \geq s$, there exists a constant $T^* = T^*(u_0, v_0) \in (0, \infty]$ such that the system (1) admits a unique maximal solution

$$(u, v) \in C^{s_1}([0, T^*); H^{s_1}(\mathbf{R}^d))^2,$$

for any $(u_0, v_0) \in H^{s_1}(\mathbf{R}^d)^2$; see, e.g., Cazenave [2]. Moreover, the asymptotic profiles of solutions of (1) are studied, for example, in [4].

In this paper, we are concerned with a time discretization method for (1). As is well-known, we need to consider implicit schemes to obtain stable numerical solutions for Schrödinger equations. In particular, the Crank-Nicolson scheme is useful and widely applied, since it is stable and second-order convergent. However, if we apply the Crank-Nicolson scheme to a nonlinear Schrödinger equation, we encounter a nonlinear elliptic equation at each time step as the resulting equation in order to maintain the second-order convergence (cf. [3]). As a consequence, we meet another difficulty for solving nonlinear elliptic equations. This can be quite time-consuming when the size of a fully discretized problem is very large. In this connection, Besse's relaxation scheme ([1]) is a method worthy of note. He considers a nonlinear Schrödinger equation and studies a linear scheme by considering both the main time step t_n and a dual one $t_{n+1/2}$. Here, by a linear scheme, we mean a time discretization method whose resulting equations consist of linear elliptic equations. His relaxation scheme is shown to be convergent but the proof of the second-order convergence is still open at present.

In this paper, we propose a linear scheme for (1) that is motivated by the relaxation scheme. The main contribution of this paper is to show that it is actually second-order convergent. As stated above, we restrict our attention within a time discretization scheme and not discuss space discretizations. However, the resulting equations of our scheme is linear so that the standard space discretization methods, for example, the finite difference, finite element, spectral methods are readily applicable. Furthermore, our time discretization scheme and our method of convergence analysis can be applied to nonlinear Schrödinger and wave equations with

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power-nonlinearities by converting these equations into suitable systems. Although those applications are of interest, herein we consider only (1) in order to present our idea as clearly as possible. As a matter of fact, fully discrete numerical schemes for (1) and those equations under various boundary conditions will be studied in forthcoming papers. Moreover, numerical examples will be reported there.

Now let us state the time discretization scheme for (1) to be considered. Let h be a time step size. We propose the following scheme for (1).

$$(2)\quad \begin{cases} i\dfrac{u^{n+\frac{3}{2}}-u^{n+\frac{1}{2}}}{h}+\alpha\Delta\dfrac{u^{n+\frac{3}{2}}+u^{n+\frac{1}{2}}}{2} \\ \qquad\qquad = \lambda\overline{\dfrac{u^{n+\frac{3}{2}}+u^{n+\frac{1}{2}}}{2}}v^{n+1}, \\ i\dfrac{v^{n+1}-v^{n}}{h}+\beta\Delta\dfrac{v^{n+1}+v^{n}}{2}=\mu(u^{n+\frac{1}{2}})^2 \end{cases}$$

for $n = 0, 1, 2, \dots$ Namely the first and the second equations of (1) are discretized at $t_{n+1} = (n+1)h$ and $t_{n+\frac{1}{2}} = (n+\frac{1}{2})h$, respectively.

The scheme (2) consists of two linear equations in $u^{n+3/2}$ and v^{n+1} at each time step. More specifically, the first equation of (2) is equivalently written as

$$K_{n+1}\begin{pmatrix} u^{n+\frac{3}{2}} \\ \bar{u}^{n+\frac{3}{2}} \end{pmatrix} = \begin{pmatrix} \left(1+i\dfrac{\alpha h}{2}\Delta\right)u^{n+\frac{1}{2}}-i\dfrac{\lambda h}{2}\bar{u}^{n+\frac{1}{2}}v^{n+1} \\ \left(1-i\dfrac{\alpha h}{2}\Delta\right)\bar{u}^{n+\frac{1}{2}}+i\dfrac{\lambda h}{2}u^{n+\frac{1}{2}}\bar{v}^{n+1} \end{pmatrix},$$

where

$$K_{n+1} = \begin{pmatrix} 1-i\dfrac{\alpha h}{2}\Delta & i\dfrac{\lambda h}{2}v^{n+1} \\ -i\dfrac{\lambda h}{2}\bar{v}^{n+1} & 1+i\dfrac{\alpha h}{2}\Delta \end{pmatrix}.$$

Since the operator K_{n+1} is defined in terms of the solution v^{n+1}, it is not certain that K_{n+1} is invertible at this stage. However, as we will state in Proposition 1 and Theorem 2 below, the scheme (2) has a unique solution in $t_{n+3/2} < T^*$ for a suitably chosen h so that K_{n+1} is actually invertible.

Below, we use the usual Lebesgue spaces $L^2 = L^2(\mathbf{R}^d)$, $L^\infty = L^\infty(\mathbf{R}^d)$ and Sobolev spaces $H^k = H^k(\mathbf{R}^d)$ for an integer k together with their standard norms. We write as $\|\cdot\|_{L^2} = \|\cdot\|_{L^2(\mathbf{R}^d)}$ and $\|\cdot\|_{H^s} = \|\cdot\|_{H^s(\mathbf{R}^d)}$.

First, we state the following local stability result which plays an important role.

Proposition 1. *Let a, $b \in H^s$, and choose an $R \geq \|a\|_{H^s} + \|b\|_{H^s}$. Then there exists a constant $T_R > 0$ such that, if $h \in (0, T_R/2]$, the scheme* (2) *with $(u^{\frac{1}{2}}, v^0) = (a, b)$ admits a unique solution $\{(u^{n+\frac{1}{2}}, v^n)\}_n$ and the solution satisfies*

$$(3)\qquad \|u^{n+\frac{1}{2}}\|_{H^s} + \|v^n\|_{H^s} \leq 2R$$

for all $n \in \mathbf{N}$ with $nh \leq T_R$. The constant T_R depends only on R, λ, μ, s and d.

It should be kept in mind that, since $h \in (0, T_R/2]$, the set $\{n \geq 1 \mid nh \leq T_R\}$ is not an empty set. This proposition will be proved in Section 2, after having prepared a few preliminary results.

We are now in a position to state the main result of this paper.

Theorem 2. *Let u_0, $v_0 \in H^{s+6}$, and let $T^* = T^*(u_0, v_0)$ be the maximal existence time of the solution (u, v) of* (1) *as mentioned before. Then (u, v) further satisfies*

$$(4)\qquad (u, v) \in \bigcap_{k=0}^{3} C^k([0, T^*); (H^{s+6-2k}))^2.$$

Let $T \in (0, T^*)$ be arbitrary, and set $M^* = \max_{0\leq k\leq 3}\{M_k\}$, where

$$(5)\qquad M_k = \max_{t\in[0,T]}\{\|\partial_t^k u(t)\|_{H^{s+6-2k}} + \|\partial_t^k v(t)\|_{H^{s+6-2k}}\} \quad (k = 0, 1, 2, 3).$$

Moreover, let $\{(u^{n+\frac{1}{2}}, v^n)\}_n$ be the solution of (2) *with initial condition*

$$(6)\qquad u^{\frac{1}{2}} = u_0 + \frac{ih}{2}(\alpha\Delta u_0 - \lambda\overline{u_0}v_0),\ v^0 = v_0.$$

Then, there exist positive constants h_0 and K_0, which depend only on $\alpha, \beta, \lambda, \mu, T$ and M_0, such that the problem (2) *is solvable and the solution $\{(u^{n+\frac{1}{2}}, v^n)\}_n$ satisfies*

$$(7)\qquad \|u(t_{n+\frac{1}{2}}) - u^{n+\frac{1}{2}}\|_{H^s} + \|v(t_n) - v^n\|_{H^s} \leq K_0 h^2$$

for all $h \in (0, h_0]$ and $n \in \mathbf{N}$ satisfying $(n+1)h \leq T$.

2. Proof of Proposition 1. First, we collect preliminary results. We introduce operators, for positive constants p and q,

$$A_p = \left(I + i\frac{ph}{2}\Delta\right)\left(I - i\frac{ph}{2}\Delta\right)^{-1},$$

$$B_q = \left(I - i\,\frac{qh}{2}\,\Delta\right)^{-1}$$

which are primarily defined on L^2, where I denotes the identity operator in L^2. Applying the Fourier transformation, we can deduce the following lemma.

Lemma 1. *1. A_p is a unitary operator on H^s and we can write*

$$A_p = \left(I - i\,\frac{ph}{2}\,\Delta\right)^{-1}\left(I + i\,\frac{ph}{2}\,\Delta\right).$$

2. B_q is a bounded operator on H^s.

The following estimates are readily obtainable from Taylor's theorem.

Lemma 2. *1. Let $f(t) \in C^3([0,T];H^s)$, $h > 0$, $t+h \in [0,T]$, and $t-h \in [0,T]$. Then we have*

$$\|h^{-1}(f(t+h) - f(t-h)) - 2\partial_t f(t)\|_{H^s} \le \frac{1}{3}\,\|f\|_{C^3([0,T];H^s)}h^2.$$

2. Let $f(t) \in C^2([0,T];H^s)$, $h \ge 0$, $t+h \in [0,T]$, and $t-h \in [0,T]$. Then we have

$$\|f(t+h) + f(t-h) - 2f(t)\|_{H^s} \le \|f\|_{C^2([0,T];H^s)}h^2.$$

We will make use of the well-known inequality.

Lemma 3. *There exists a positive constant C which depends only on d and s such that*

$$\|uv\|_{H^s} \le C\|u\|_{H^s}\|v\|_{H^s} \quad (u, v \in H^s).$$

Now we can state the following proof.

Proof of Proposition 1. It is based on the contraction mapping principle. Let $a, b \in H^s$ be arbitrary and set $R \ge \|a\|_{H^s} + \|b\|_{H^s}$.

First, the equation (2) with $(u^{\frac{1}{2}}, v^0) = (a, b)$ can be written in the following form:

$$(8)\quad \begin{cases} u^{n+\frac{3}{2}} = A_\alpha^{n+1}a \\ \qquad - i\lambda h \displaystyle\sum_{j=0}^{n} A_\alpha^{n-j}B_\alpha \overline{\frac{u^{j+\frac{3}{2}} + u^{j+\frac{1}{2}}}{2}}\, v^{j+1}, \\ v^{n+1} = A_\beta^{n+1}b \\ \qquad - i\mu h \displaystyle\sum_{j=0}^{n} A_\beta^{n-j}B_\beta (u^{j+\frac{1}{2}})^2 \end{cases}$$

for $n = 0, 1, 2, \dots$

For the time being, we fix $N \in \mathbf{N}$ and set $\hat{N} = \{1, 2, \dots, N\}$. Then, we consider a Banach space

$$\mathcal{X}_N = \{\{(w^{n+\frac{1}{2}}, \hat{w}^n)\}_{n\in\hat{N}} \mid w^{n+\frac{1}{2}}, \hat{w}^n \in H^s \quad (n \in \hat{N})\}$$

with the norm

$$\|\{(w^{n+\frac{1}{2}}, \hat{w}^n)\}_n\|_{\mathcal{X}_N} = \sup_{n\in\hat{N}}(\|w^{n+\frac{1}{2}}\|_{H^s} + \|\hat{w}^n\|_{H^s}),$$

for $\{(w^{n+\frac{1}{2}}, \hat{w}^n)\}_n \in \mathcal{X}_N$. We introduce $\mathcal{T} : \mathcal{X}_N \to \mathcal{X}_N$ by setting

$$(9)\qquad \{(\tilde{u}^{n+\frac{1}{2}}, \tilde{v}^n)\}_n = \mathcal{T}\{(u^{n+\frac{1}{2}}, v^n)\}_n,$$

where

$$(10)\quad \begin{cases} \tilde{u}^{n+\frac{3}{2}} = A_\alpha^{n+1}a \\ \qquad - i\lambda h \displaystyle\sum_{j=0}^{n} A_\alpha^{n-j}B_\alpha \overline{\frac{u^{j+\frac{3}{2}} + u^{j+\frac{1}{2}}}{2}}\, v^{j+1}, \\ \tilde{v}^{n+1} = A_\beta^{n+1}b \\ \qquad - i\mu h \displaystyle\sum_{j=0}^{n} A_\beta^{n-j}B_\beta \Big(u^{j+\frac{1}{2}}\Big)^2 \end{cases}$$

for $n = 0, 1, \dots, N-1$. Here, we set $u^{\frac{1}{2}} = a$.

Below, we will show that $\mathcal{T}$ is a contraction operator from a closed ball $\mathcal{B}_{2R}$ into itself, with a suitably chosen h, where

$$\mathcal{B}_{2R} = \{\{(w^{n+\frac{1}{2}}, \hat{w}^n)\}_n \in \mathcal{X}_N \mid \|\{(w^{n+\frac{1}{2}}, \hat{w}^n)\}_n\|_{\mathcal{X}_N} \le 2R\}.$$

First, let $\{(u^{n+\frac{1}{2}}, v^n)\}_n \in \mathcal{B}_{2R}$, and set $\{(\tilde{u}^{n+\frac{1}{2}}, \tilde{v}^n)\}_n = \mathcal{T}\{(u^{n+\frac{1}{2}}, v^n)\}_n$. By using Lemmas 1 and 3, we have

$$\begin{aligned} \|\tilde{u}^{n+\frac{3}{2}}\|_{H^s} &\le \|a\|_{H^s} \\ &\quad + Ch\sum_{j=0}^{n}(\|u^{j+\frac{3}{2}}\|_{H^s} + \|u^{j+\frac{1}{2}}\|_{H^s})\|v^{j+1}\|_{H^s} \\ &\le \|a\|_{H^s} + CNhR^2, \\ \|\tilde{v}^{n+1}\|_{H^s} &\le \|b\|_{H^s} + Ch\sum_{j=0}^{n}\|u^{j+\frac{1}{2}}\|_{H^s}^2 \\ &\le \|b\|_{H^s} + CNhR^2, \end{aligned}$$

for $n = 0, 1, \dots, N-1$. Here and in what follows, the generic positive constants depending only on λ, μ, s and d are denoted by C. The value of C may change in the same context. Hence there exists a positive constant C_1, which depends only on d, s, λ and μ, such that

$$\|\tilde{u}^{n+\frac{3}{2}}\|_{H^s} + \|\tilde{v}^{n+1}\|_{H^s} \le R + C_1NhR^2$$

for all $n = 0, 1, \dots, N-1$. Therefore, if

$$C_1NhR \le 1 \tag{11}$$

then we have $\|\{(\tilde{u}^{n+\frac{1}{2}}, \tilde{v}^n)\}_n\|_{\mathcal{X}_N} \le 2R$, which implies that $\mathcal{T}(\mathcal{B}_{2R}) \subset \mathcal{B}_{2R}$.

Next, let $\{(u_1^{n+\frac{1}{2}}, v_1^n)\}_n$, $\{(u_2^{n+\frac{1}{2}}, v_2^n)\}_n \in \mathcal{B}_{2R}$, and let

$$\{(\tilde{u}_j^{n+\frac{1}{2}}, \tilde{v}_j^n)\}_n = \mathcal{T}\{(u_j^{n+\frac{1}{2}}, v_j^n)\}_n. \quad (j = 1, 2).$$

Then, we have

$$\begin{aligned}
&\|\tilde{u}_1^{n+\frac{3}{2}} - \tilde{u}_2^{n+\frac{3}{2}}\|_{H^s} \\
&\quad \le Ch \sum_{j=0}^{n} \{(\|u_1^{j+\frac{3}{2}} - u_2^{j+\frac{3}{2}}\|_{H^s} \\
&\qquad + \|u_1^{j+\frac{1}{2}} - u_2^{j+\frac{1}{2}}\|_{H^s})\|v^{j+1}\|_{H^s} \\
&\qquad + (\|u_2^{j+\frac{3}{2}}\|_{H^s} + \|u_2^{j+\frac{1}{2}}\|_{H^s})\|v_1^{j+1} - v_2^{j+1}\|_{H^s}\} \\
&\quad \le CNhR\|\{(u_1^{k+\frac{1}{2}}, v_1^k)\}_k - \{(u_2^{k+\frac{1}{2}}, v_2^k)\}_k\|_{\mathcal{X}_N},
\end{aligned}$$

and

$$\begin{aligned}
&\|\tilde{v}_1^{n+1} - \tilde{v}_2^{n+1}\|_{H^s} \\
&\quad \le Ch \sum_{j=0}^{n} (\|u_1^{j+\frac{1}{2}}\|_{H^s} + \|u_2^{j+\frac{1}{2}}\|_{H^s})\|u_1^{j+\frac{1}{2}} - u_2^{j+\frac{1}{2}}\|_{H^s} \\
&\quad \le CNhR\|\{(u_1^{k+\frac{1}{2}}, v_1^k)\}_k - \{(u_2^{k+\frac{1}{2}}, v_2^k)\}_k\|_{\mathcal{X}_N}
\end{aligned}$$

for $n = 0, 1, \ldots, N-1$. Hence there exists a positive constant C_2, which depends only on d, s, λ and μ, such that

$$\begin{aligned}
&\|\tilde{u}_1^{n+\frac{3}{2}} - \tilde{u}_2^{n+\frac{3}{2}}\|_{H^s} + \|\tilde{v}_1^{n+1} - \tilde{v}_2^{n+1}\|_{H^s} \\
&\quad \le C_2NhR\|\{(u_1^{k+\frac{1}{2}}, v_1^k)\}_k - \{(u_2^{k+\frac{1}{2}}, v_2^k)\}_k\|_{\mathcal{X}_N}
\end{aligned}$$

for all $n = 0, 1, \ldots, N-1$. Therefore, if

$$C_2NhR \le \frac{1}{2},$$

then we have

$$\begin{aligned}
&\|\mathcal{T}\{(u_1^{k+\frac{1}{2}}, v_1^k)\}_k - \mathcal{T}\{(u_2^{k+\frac{1}{2}}, v_2^k)\}_k\|_{\mathcal{X}_N} \\
&\quad \le \frac{1}{2}\|\{(u_1^{k+\frac{1}{2}}, v_1^k)\}_k - \{(u_2^{k+\frac{1}{2}}, v_2^k)\}_k\|_{\mathcal{X}_N},
\end{aligned}$$

which implies that $\mathcal{T} : \mathcal{B}_{2R} \to \mathcal{B}_{2R}$ is a contraction mapping. At this stage, we define T_R as

$$T_R = \min\left\{\frac{1}{C_1R}, \frac{1}{2C_2R}\right\}.$$

Moreover, from now on, choose N as $N = \max\{n \mid n \ge 1, nh \le T_R\}$. Then, the mapping $\mathcal{T}$ turns out to be a contraction mapping of $B_{2R} \to B_{2R}$. As a result, $\mathcal{T}$ has a unique fixed point $\{(u^{n+\frac{1}{2}}, v^n)\}_{n\in\hat{N}}$ which obviously satisfies (10) and (3) for $1 \le n \le N$. This completes the proof of Proposition 1. □

3. Proof of Theorem 2. Let u_0, $v_0 \in H^{s+6}$ and let $\{(u^{n+\frac{1}{2}}, v^n)\}_n$ be the solution of (2) with initial condition (6). Then there exists a positive constant C^* which depends s, d and α, such that

$$\|u^0\|_{H^s} + \|v^0\|_{H^s} \le C^*M^*(1 + M^*).$$

Put $M' := \max\{M^*, C^*(M^*+1)M^*\}$. From Proposition 1, there exists a constant $T_{M'} > 0$, which depends only on R, λ, μ, s and d, a unique solution $\{(u^{n+\frac{1}{2}}, v^n)\}_n$ of (2) with initial condition (6) satisfies

$$\|u^{n+\frac{1}{2}}\|_{H^s} + \|v^n\|_{H^s} \le 2M'$$

for all $n \in \mathbf{N}$ with $nh \le T_{M'}$. We define

$$\nu_h = \sup\{n \in \mathbf{N} \mid \|u^{n+\frac{1}{2}}\|_{H^s} + \|v^n\|_{H^s} \le 3M'\}.$$

We divide the proof into two steps.

Step 1. First, we show that there exist positive constants h_1 and K_0, which depend only on T and M_0, such that the estimate (7) holds for all $h \in (0, h_1]$ and $n \in \mathbf{N}$ satisfying

$$(n+1)h \le T, \quad n \le \nu_h. \tag{12}$$

We define N_h as

$$N_h = \min\{[T/h] - 1, \nu_h\},$$

where $[T/h]$ denotes the integer part of T/h.

For $n = 0, 1, 2, \ldots$, we set

$$\theta^{n+\frac{1}{2}} = u(t_{n+\frac{1}{2}}) - u^{n+\frac{1}{2}}, \quad \rho^n = v(t_n) - v^n.$$

Then we have

$$\theta^{n+\frac{3}{2}} - \theta^{n+\frac{1}{2}} - i\frac{\alpha h}{2}\Delta(\theta^{n+\frac{3}{2}} + \theta^{n+\frac{1}{2}}) = ih\Phi^{n+1},$$

or equivalently,

$$\theta^{n+\frac{3}{2}} = A_\alpha\theta^{n+\frac{1}{2}} + ihB_\alpha\Phi^{n+1},$$

where $\Phi^{n+1} = \phi_1^{n+1} + \phi_2^{n+1} + \phi_3^{n+1}$,

$$\phi_1^{n+1} = i\left\{\partial_t u(t_{n+1}) - \frac{u(t_{n+\frac{3}{2}}) - u(t_{n+\frac{1}{2}})}{h}\right\},$$

$$\phi_2^{n+1} = \alpha\Delta\left\{u(t_{n+1}) - \frac{u(t_{n+\frac{3}{2}}) + u(t_{n+\frac{1}{2}})}{2}\right\},$$

$$\phi_3^{n+1} = -\lambda\left\{\overline{u(t_{n+1})}v(t_{n+1}) - \frac{\overline{u^{n+\frac{3}{2}}+u^{n+\frac{1}{2}}}}{2}v^{n+1}\right\}.$$

It follows from Lemma 1 that

$$\|\theta^{n+\frac{3}{2}}\|_{H^s} \le \|\theta^{n+\frac{1}{2}}\|_{H^s} + h\|\Phi^{n+1}\|_{H^s}.$$

Next, we estimate $\|\Phi^{n+1}\|_{H^s}$. First, from Lemma 2,

$$\|\phi_1^{n+1}\|_{H^s} \le CM_3h^2, \quad \|\phi_2^{n+1}\|_{H^s} \le CM_2h^2$$

for $n = 0, 1, \ldots, N_h - 1$, where M_2 and M_3 are constants defined by (5).

Moreover, since

$$\begin{aligned}
&\overline{u(t_{n+1})}v(t_{n+1}) - \frac{\overline{u^{n+\frac{3}{2}}+u^{n+\frac{1}{2}}}}{2}v^{n+1}\\
&= \left\{\overline{u(t_{n+1})} - \frac{\overline{u(t_{n+\frac{3}{2}})+u(t_{n+\frac{1}{2}})}}{2}\right\}v(t_{n+1})\\
&\quad + \left\{\frac{\overline{u(t_{n+\frac{3}{2}})+u(t_{n+\frac{1}{2}})}}{2} - \frac{\overline{u^{n+\frac{3}{2}}+u^{n+\frac{1}{2}}}}{2}\right\}v(t_{n+1})\\
&\quad + \frac{\overline{u^{n+\frac{3}{2}}+u^{n+\frac{1}{2}}}}{2}\{v(t_{n+1}) - v^{n+1}\},
\end{aligned}$$

it follows from Lemma 2 that

$$\begin{aligned}
\|\phi_3^{n+1}\|_{H^s} &\le CM_2h^2\|v(t_{n+1})\|_{H^s}\\
&\quad + C(\|u(t_{n+\frac{3}{2}}) - u^{n+\frac{3}{2}}\|_{H^s}\\
&\quad + \|u(t_{n+\frac{1}{2}}) - u^{n+\frac{1}{2}}\|_{H^s})\|v(t_{n+1})\|_{H^s}\\
&\quad + C(\|u^{n+\frac{3}{2}}\|_{H^s} + \|u^{n+\frac{1}{2}}\|_{H^s})\\
&\quad \times \|v(t_{n+1}) - v^{n+1}\|_{H^s}\\
&\le CM'(M_2h^2 + \|\theta^{n+\frac{3}{2}}\|_{H^s} + \|\theta^{n+\frac{1}{2}}\|_{H^s}\\
&\quad + \|\rho^{n+1}\|_{H^s})
\end{aligned}$$

for $n = 0, 1, \ldots, N_h - 1$. Thus, we obtain

$$\begin{aligned}
\|\Phi^{n+1}\|_{H^s} &\le CM'h^2 + CM'(\|\theta^{n+\frac{3}{2}}\|_{H^s}\\
&\quad + \|\theta^{n+\frac{1}{2}}\|_{H^s} + \|\rho^{n+1}\|_{H^s}),
\end{aligned}$$

and consequently, for $n = 0, 1, \ldots, N_h - 1$,

$$\begin{aligned}
\text{(13)}\quad \|\theta^{n+\frac{3}{2}}\|_{H^s} &\le \|\theta^{n+\frac{1}{2}}\|_{H^s} + h\|\Phi^{n+1}\|_{H^s}\\
&\le \|\theta^{n+\frac{1}{2}}\|_{H^s} + CM'h^3\\
&\quad + CM'h(\|\theta^{n+\frac{3}{2}}\|_{H^s} + \|\theta^{n+\frac{1}{2}}\|_{H^s}\\
&\quad + \|\rho^{n+1}\|_{H^s}).
\end{aligned}$$

Similarly, we have

$$\rho^{n+1} - \rho^n - i\,\frac{\beta h}{2}\,\Delta(\rho^{n+1} + \rho^n) = ih\Psi^{n+\frac{1}{2}},$$

or equivalently,

$$\rho^{n+1} = A_\beta\rho^n + ihB_\beta\Psi^{n+\frac{1}{2}},$$

where $\Psi^{n+\frac{1}{2}} = \psi_1^{n+\frac{1}{2}} + \psi_2^{n+\frac{1}{2}} + \psi_3^{n+\frac{1}{2}}$,

$$\begin{aligned}
\psi_1^{n+\frac{1}{2}} &= i\left\{\partial_t v(t_{n+\frac{1}{2}}) - \frac{v(t_{n+1}) - v(t_n)}{h}\right\},\\
\psi_2^{n+\frac{1}{2}} &= \beta\Delta\left\{v(t_{n+\frac{1}{2}}) - \frac{v(t_{n+1}) + v(t_n)}{2}\right\},\\
\psi_3^{n+\frac{1}{2}} &= -\mu\{(u(t_{n+\frac{1}{2}}))^2 - (u^{n+\frac{1}{2}})^2\}.
\end{aligned}$$

Again, from Lemma 2, we have

$$\|\psi_1^{n+\frac{1}{2}}\|_{H^s} \le CM_3h^2, \quad \|\psi_2^{n+\frac{1}{2}}\|_{H^s} \le CM_2h^2$$

for $n = 0, 1, \ldots, N_h - 1$. Moreover, we have

$$\begin{aligned}
\|\psi_3^{n+\frac{1}{2}}\|_{H^s} &\le C(\|u(t_{n+\frac{1}{2}})\|_{H^s} + \|u^{n+\frac{1}{2}}\|_{H^s})\\
&\quad \times \|u(t_{n+\frac{1}{2}}) - u^{n+\frac{1}{2}}\|_{H^s}\\
&\le CM'\|\theta^{n+\frac{1}{2}}\|_{H^s}
\end{aligned}$$

for $n = 0, 1, \ldots, N_h - 1$. Thus, we obtain

$$\begin{aligned}
\text{(14)}\quad \|\rho^{n+1}\|_{H^s} &\le \|\rho^n\|_{H^s} + h\|\Psi^{n+\frac{1}{2}}\|_{H^s}\\
&\le \|\rho^n\|_{H^s} + CM^*h^3 + CM'h\|\theta^{n+\frac{1}{2}}\|_{H^s}
\end{aligned}$$

for $n = 0, 1, \ldots, N_h - 1$.

Summing up estimates (13) and (14), we deduce

$$\begin{aligned}
&\|\theta^{n+\frac{3}{2}}\|_{H^s} + \|\rho^{n+1}\|_{H^s}\\
&\quad \le \|\theta^{n+\frac{1}{2}}\|_{H^s} + \|\rho^n\|_{H^s} + C_3M'h^3\\
&\qquad + C_4M'h(\|\theta^{n+\frac{3}{2}}\|_{H^s} + \|\rho^{n+1}\|_{H^s}\\
&\qquad + \|\theta^{n+\frac{1}{2}}\|_{H^s} + \|\rho^n\|_{H^s}),
\end{aligned}$$

where C_3 and C_4 denote positive constants depending only on d, s, α, β, λ and μ. Therefore

$$\begin{aligned}
&(1 - C_4M'h)(\|\theta^{n+\frac{3}{2}}\|_{H^s} + \|\rho^{n+1}\|_{H^s})\\
&\quad \le (1 + C_4M'h)(\|\theta^{n+\frac{1}{2}}\|_{H^s} + \|\rho^n\|_{H^s}) + C_3M'h^3
\end{aligned}$$

for $n = 0, 1, \ldots, N_h - 1$.

At this stage, we define

$$h_1 = \frac{1}{2C_4M'}$$

and we assume that $h \in (0, h_1]$. Then, we have

$$\|\theta^{n+\frac{3}{2}}\|_{H^s} + \|\rho^{n+1}\|_{H^s}$$
$$\le (1 + 4C_4M'h)(\|\theta^{n+\frac{1}{2}}\|_{H^s} + \|\rho^n\|_{H^s}) + 2C_3M'h^3$$
$$\le e^{4C_4M'h}(\|\theta^{n+\frac{1}{2}}\|_{H^s} + \|\rho^n\|_{H^s}) + 2C_3M'h^3$$

for $n = 0, 1, \dots, N_h - 1$. Thus, we have

$$\|\theta^{n+\frac{1}{2}}\|_{H^s} + \|\rho^n\|_{H^s}$$
$$\le e^{4C_2M'nh}(\|\theta^{\frac{1}{2}}\|_{H^s} + \|\rho^0\|_{H^s})$$
$$+ 2C_1M'h^3 \sum_{j=0}^{n-1} e^{4C_2M'jh}$$
$$\le e^{4C_2M'T}\|\theta^{\frac{1}{2}}\|_{H^s} + 2C_1M'Te^{4C_2M'T}h^2$$

for $n \in \mathbf{N}$ satisfying (12).

In view of the regularity property (4), we have

$$\partial_t u(0) = i(\alpha\Delta u_0 - \lambda\bar{u}_0 v_0).$$

Hence, using the Taylor theorem, we have

$$\theta^{\frac{1}{2}} = u(t_{\frac{1}{2}}) - u^{\frac{1}{2}}$$
$$= \left\{u(0) + \frac{h}{2}\,\partial_t u(0) + \int_0^{\frac{h}{2}}\left(\frac{h}{2} - \tau\right)\partial_\tau^2 u(\tau)\,d\tau\right\}$$
$$- \left\{u_0 + i\,\frac{h}{2}\,(\alpha\Delta u_0 - \lambda\overline{u_0}v_0)\right\}$$
$$= \int_0^{\frac{h}{2}}\left(\frac{h}{2} - \tau\right)\partial_\tau^2 u(\tau)\,d\tau.$$

This gives

$$\|\theta^{\frac{1}{2}}\|_{H^s} \le \int_0^{\frac{h}{2}}\left(\frac{h}{2} - \tau\right)\|\partial_\tau^2 u(\tau)\|_{H^s}\,d\tau \le \frac{M'}{8}\,h^2.$$

Therefore, taking

$$K_0 = \frac{M'}{8}\,e^{4C_2M'T} + 2C_1M'Te^{4C_2M'T},$$

we have shown that the desired estimate (7) holds for all $h \in (0, h_1]$ and $n \in \mathbf{N}$ satisfying (12).

Step 2. We set

$$h_0 = \min\left\{h_1, \sqrt{\frac{M'}{2K_0}}, \frac{1}{2}\,T_{\frac{3}{2}M'}\right\},$$

where $T_{\frac{3}{2}M'}$ is the constant introduced in Proposition 1 with $R = \frac{3}{2}\,M'$.

We prove

$$[T/h] - 1 \le \nu_h \quad (\forall h \in (0, h_0]) \tag{15}$$

by showing a contradiction. Thus, we assume that there exists $h \in (0, h_0]$ such that

$$[T/h] - 1 > \nu_h.$$

Then, we have $N_h = \nu_h$ and, since $h_0 \le h_1$, in view of Step 1,

$$\|u(t_{n+\frac{1}{2}}) - u^{n+\frac{1}{2}}\|_{H^2} + \|v(t_n) - v^n\|_{H^s} \le K_0h^2$$

for all $n = 1, \dots, \nu_h$. Moreover, since $(\nu_h + 1)h \le T$, it follows from the definition of M' that

$$\max_{n=1,\dots,\nu_h}(\|u(t_{n+\frac{1}{2}})\|_{H^s} + \|v(t_{n+\frac{1}{2}})\|_{H^s}) \le M'.$$

Combining those inequalities, we get

$$\|u^{n+\frac{1}{2}}\|_{H^s} + \|v^n\|_{H^s} \le M' + K_0h^2$$

for all $n = 1, \dots, \nu_h$. In particular, since $h \in (0, h_0]$, we have

$$\|u^{\nu_h}\|_{H^s} + \|v^{\nu_h}\|_{H^s} \le M' + K_0h^2 \le \frac{3}{2}\,M'.$$

Then, we apply Proposition 1 with $a = u^{\nu_h+\frac{1}{2}}$, $b = v^{\nu_h}$ and $R = \frac{3}{2}\,M'$ to obtain

$$\|u^{\nu_h+\frac{3}{2}}\|_{H^s} + \|v^{\nu_h+1}\|_{H^s} \le 3M'.$$

This contradicts the definition of ν_h. Therefore, (15) actually holds true. That is, we have $N_h = [T/h] - 1$ for all $h \in (0, h_0]$. Hence, by the result of Step 1, we see that the desired estimate (7) holds for all $h \in (0, h_0]$ and $n \in \mathbf{N}$ satisfying $(n + 1)h \le T$. This completes the proof of Theorem 2. □

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