Linearized product of two Riemann zeta functions

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Abstract: In this paper, we elucidate the well-known Wilton's formula for the product of two Riemann zeta functions. A proof of Wilton's expression for product of two zeta functions was given by M. Nakajima in [5] using the Atkinson dissection. On the similar line we derive Wilton's formula using the Riesz sum of the order $\kappa = 1$.

Key words: Wilton's formula; Riesz sums; Atkinson dissection; Dirichlet series; Riemann zeta function.

1. Introduction. In [5], M. Nakajima derived an expression for the product of two Dirichlet series. From this expression he gives a proof for the well-known Wilton's formula [6] and Bellman's formula [2] for the product of two Riemann zeta functions in particular. In this method, he uses the Atkinson dissection [1].

The Atkinson dissection involves splitting of the double sum $\sum_{m,n=1}^{\infty}$ as

$$\sum_{m,n=1}^{\infty} = \sum_{m=1}^{\infty}\sum_{n<m} + \sum_{m=n=1}^{\infty} + \sum_{n=1}^{\infty}\sum_{m<n}.$$

Nakajima in [5], splits the double sum by the following method

$$\sum_{m,n} = \sum_{m=1}^{\infty}\sum_{n\leq m}{}' + \sum_{n=1}^{\infty}\sum_{m\leq n}{}'$$

where $\sum'$ means that the corresponding term in the summation is to be halved.

The Riesz means, introduced by M. Riesz, have been studied in connection with summability of Fourier series and of Dirichlet series [3] and [4]. Given an increasing sequence $\{\lambda_k\}$ of reals and a sequence $\{\alpha_k\}$ of complex numbers, the Riesz sum of order κ is defined by

$$\begin{aligned} A^{\kappa}(x) = A^{\kappa}_{\lambda}(x) &= \sum_{\lambda_k\leq x}{}'(x-\lambda_k)^{\kappa}\alpha_k \\ &= \kappa\int_0^x (x-t)^{\kappa-1} A_{\lambda}(t)\mathrm{d}t \\ &= \int_0^x (x-t)^{\kappa}\,\mathrm{d}A_{\lambda}(t), \end{aligned}$$

where $A_{\lambda}(x) = A^0_{\lambda}(x) = \sum_{\lambda_k\leq x}{}'\alpha_k$, and the prime on the summation sign means that when $\lambda_k = x$, the correponding term is to be halved.

Consider the Dirichlet series $\varphi(s)$ and $\Phi(s)$ defined as

$$\varphi(s) = \sum_{n=1}^{\infty}\frac{\alpha_n}{\lambda_n^s},\ \sigma>\sigma_{\varphi} \text{ and } \Phi(s) = \sum_{n=1}^{\infty}\frac{a_n}{\gamma_n^s},\ \sigma>\sigma_{\Phi},$$

where $\{\lambda_n\}$ and $\{\gamma_n\}$ are increasing sequences of real numbers and α_n and a_n are complex numbers. Assume that they are continued to meromorphic functions over the whole plane and that they satisfy the growth condition

$$\varphi(\sigma+it) \ll (|t|+1)^{s_{\varphi}(\sigma)},\ \ \Phi(\sigma+it) \ll (|t|+1)^{s_{\Phi}(\sigma)}$$

in the strip $-b<\sigma<c$ $(b>0, c>0)$. In case of the Riemann zeta function, $s_{\zeta}(-b) = \frac{1}{2}+b$.

In this paper, we consider an integral of the following form for Dirichlet series Φ and φ, (for $c>0$ and $\kappa\geq 0$),

$$\begin{aligned} &\mathcal{F}_{(c)}(\varphi(u),\Phi(v)) = \mathcal{F}^{\kappa}_{(c)}(\varphi(u),\Phi(v);x) \\ &\quad = \frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(w)}{\Gamma(w+\kappa+1)}\varphi(u+w)\Phi(v-w)x^{w+\kappa}\,\mathrm{d}w, \end{aligned}$$

and its counterpart $\mathcal{F}_{(c)}(\Phi(v),\varphi(u))$ under the condition

$$\Re u > \sigma_{\varphi}+c,\quad \Re v>\sigma_{\Phi}+c.$$

In the next section, we consider Φ and φ as the Riemann zeta functions in particular. The Atkinson dissection is the special case of the Riesz sum $A^{\kappa}(x)$ with $\kappa=0$ in the sense that

(1) $\mathcal{F}^{\kappa}_{(c)}(\varphi(u),\Phi(v)) + \mathcal{F}^{\kappa}_{(c)}(\Phi(v),\varphi(u))$

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$$= \frac{1}{\Gamma(\kappa+1)} \sum_{m=1}^{\infty} a_m \gamma_m^{-v-\kappa} \sideset{}{'}\sum_{\lambda_n \le \gamma_m x} \alpha_n \lambda_n^{-u} (\gamma_m x - \lambda_n)^{\kappa} + \frac{1}{\Gamma(\kappa+1)} \sum_{n=1}^{\infty} \alpha_n \lambda_n^{-u-\kappa} \sideset{}{'}\sum_{\gamma_m \le \lambda_n x} a_m \gamma_m^{-v} (\lambda_n x - \gamma_m)^{\kappa}$$

implies

$$\sum_{m,n=1}^{\infty} \alpha_m \lambda_m^{-u} a_n \gamma_n^{-v} = \sum_{m=1}^{\infty} \sum_{n<m} \alpha_m \lambda_m^{-u} a_n \gamma_n^{-v} + \sum_{m=n}^{\infty} \alpha_n \lambda_n^{-u} a_n \gamma_n^{-v} + \sum_{n=1}^{\infty} \sum_{m<n} \alpha_m \lambda_m^{-u} a_n \gamma_n^{-v} \quad \text{for } \kappa = 0,\ x = 1. \tag{2}$$

We prove the well-known Wilton's formula by taking $\kappa = 1$. The Wilton's formula can be stated as follows:

Theorem 1. *For $\Re u > -1$, $\Re v > -1$, $\mathrm{Re}(u+v) > 0$ and $u + v \neq 2$, we have*

$$\zeta(u)\zeta(v) = \zeta(u+v-1)\left(\frac{1}{u-1} + \frac{1}{v-1}\right) + 2(2\pi)^{u-1} \sum_{n=1}^{\infty} \sigma_{1-u-v}(n) n^{u-1} u \int_{2\pi n}^{\infty} x^{-u-1} \sin x \, \mathrm{d}x + 2(2\pi)^{v-1} \sum_{n=1}^{\infty} \sigma_{1-u-v}(n) n^{v-1} v \int_{2\pi n}^{\infty} x^{-v-1} \sin x \, \mathrm{d}x. \tag{3}$$

2. Proof of the Theorem. For $\Re(u) > 1 + c'$ and $\Re(v) > 1 + c$, with $c, c' > 0$, we consider the following integral with $0 < \mathrm{Re}\, u - 1 < b < \frac{5}{2}$ and $0 < \mathrm{Re}\, v - 1 < b < \frac{5}{2}$,

$$\mathcal{F}^{\kappa}_{(-b)}(\zeta(u), \zeta(v); x) = \frac{1}{2\pi i} \int_{(-b)} \frac{\Gamma(w)}{\Gamma(w+\kappa+1)} \zeta(u+w)\zeta(v-w) x^{w+\kappa} dw.$$

Applying functional equation for $\zeta(u+w)$ we have

$$\mathcal{F}^{\kappa}_{(-b)}(\zeta(u), \zeta(v); x) = \frac{1}{2\pi i} \int_{(-b)} \frac{\Gamma(w)}{\Gamma(w+\kappa+1)} \frac{\pi^{u+w}}{\pi^{\frac{1}{2}}} \frac{\Gamma\left(\frac{1-u-w}{2}\right)}{\Gamma\left(\frac{u+w}{2}\right)} \zeta(1-u-w)\zeta(v-w) x^{w+\kappa} dw.$$

For simplicity of the expressions let

$$\frac{\Gamma(w)}{\Gamma(w+\kappa+1)} x^{w+\kappa} = S_{\kappa}(w),$$

and

$$\zeta(1-u-w)\zeta(v-w) = f(w).$$

Multiplying numerator and denominator by the factor $\Gamma(1 - \frac{u+w}{2})$, the right hand side of the above will be

$$\frac{1}{2\pi i} \int_{(-b)} S_{\kappa}(w) \frac{\pi^{u+w}\Gamma\left(\frac{1-u-w}{2}\right)\Gamma\left(1 - \frac{u+w}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{u+w}{2}\right)\Gamma\left(1 - \frac{u+w}{2}\right)} f(w) dw$$

$$= \frac{1}{2\pi i} \int_{(-b)} S_{\kappa}(w) \frac{\pi^{u+w} 2^{-(1-u-w)}}{\sqrt{\pi}\pi} 2\sqrt{\pi} \sin\left(\frac{\pi}{2}(u+w)\right) \Gamma(1-u-w) f(w) dw$$

$$= 2 \frac{1}{2\pi i} \int_{(-b)} S_{\kappa}(w) f(w) (2\pi)^{u+w-1} \sin\left(\frac{\pi}{2}(u+w)\right) \Gamma(1-u-w) dw.$$

By change of variables and assuming $\Re u < b$, we have

$$\frac{1}{\pi i} \int_{(b)} S_{\kappa}(-z) f(-z) (2\pi)^{u-z-1} \sin\left(\frac{\pi}{2}(u-z)\right) \Gamma(1-u+z) dz$$

$$= \frac{1}{\pi i} \int_{(b)} S_{\kappa}(-z) (2\pi)^{u-z-1} \sin\left(\frac{\pi}{2}(u-z)\right) \sum_{n=1}^{\infty} \sigma_{1-u-v}(n) n^{u-z-1} \Gamma(1-u+z) dz$$

$$= \frac{1}{\pi i} \sum_{n=1}^{\infty} \sigma_{1-u-v}(n) (2n\pi)^{u-1} \int_{(b)} S_{\kappa}(-z) \sin\left(\frac{\pi}{2}(u-z)\right) (2n\pi)^{-z} \Gamma(1-u+z) dz.$$

Taking the order $\kappa = 1$.

$$\mathcal{F}^{1}_{(-b)}(\zeta(u), \zeta(v); x) = \frac{1}{\pi i} \sum_{n=1}^{\infty} \sigma_{1-u-v}(n) (2n\pi)^{u-1} \int_{(b)} S_1(-z) \sin\left(\frac{\pi}{2}(u-z)\right) (2n\pi)^{-z} \Gamma(1-u+z) dz$$

$$= \frac{1}{\pi i} \sum_{n=1}^{\infty} \sigma_{1-u-v}(n) (2n\pi)^{u-1} \int_{(b)} S_1(-z) (2n\pi)^{-z} \left\{\frac{e^{i\frac{\pi}{2}(u-z)} - e^{-i\frac{\pi}{2}(u-z)}}{2i}\right\} \Gamma(1-u+z) dz.$$

Let

$$h_b(u; x) = \frac{1}{2\pi i} \int_{(b)} \frac{x^{-z+1}(2n\pi)^{-z}}{z(z-1)} e^{i\frac{\pi}{2}(u-z-1)} \Gamma(1-u+z) dz.$$

Similarly, let

$$g_b(u; x) = \frac{1}{2\pi i} \int_{(b)} \frac{x^{-z+1}(2n\pi)^{-z}}{z(z-1)} e^{-i\frac{\pi}{2}(u-z-1)} \Gamma(1-u+z) dz.$$

Differentiating above integral $h_b(u; x)$ with respect to x, we get

$$h_b(u;x)' = -\frac{1}{2\pi i}\int_{(b)} \frac{x^{-z}(2n\pi)^{-z}}{z} e^{i\frac{\pi}{2}(u-z-1)}\Gamma(1-u+z)dz = -\frac{1}{2\pi i} e^{i\frac{\pi}{2}(u-1)} \int_{(b)} \frac{(2\pi nxe^{i\frac{\pi}{2}})^{-z}}{z}\Gamma(1-u+z)dz.$$

Now, shifting the path of integration to the left, we get

$$h_b(u;x)' = -\frac{1}{2\pi i} e^{i\frac{\pi}{2}(u-1)} \int_{(b_1)} \frac{(2\pi nxe^{i\frac{\pi}{2}})^{-z}}{z}\Gamma(1-u+z)dz.$$

Note that $h_b(u;x)' + g_b(u;x)'$ is absolutely convergent on $\frac{1}{2}+b_1 < \Re(u),\ 0 < \Re(u-1) < b_1$. Therefore, using the formula for incomplete Gamma function, which is given as

$$\frac{1}{2\pi i}\int_{(c)} x^{-s}\Gamma(s+\alpha)\frac{ds}{s} = \Gamma(\alpha, x) \quad (c>0, \Re x > 0)$$

$$\frac{1}{2\pi i}\int_{(c)} (ix)^{-s}\Gamma(s+\alpha)\frac{ds}{s} = \Gamma(\alpha, ix)$$

$$\left(c>0, \Re\alpha < \frac{1}{2}-c,\ x \in R\right)$$

where

$$\Gamma(\alpha, x) = \int_x^\infty t^{\alpha-1}e^{-t}dt \qquad (|\arg\alpha| < \pi)$$

is the incomplete gamma function.
Now we have,

$$h_b(u;x)' + g_b(u;x)' = e^{-i\frac{\pi}{2}(1-u)}\Gamma(1-u, 2\pi nxe^{\frac{i\pi}{2}}) + e^{i\frac{\pi}{2}(1-u)}\Gamma(1-u, -2\pi nxe^{\frac{i\pi}{2}}) = 2\int_{2\pi nx}^\infty t^{-u}\cos tdt \quad (\Re u > 0)$$

where we used the following formula

$$\int_u^\infty x^{\alpha-1}\cos xdx = \frac{1}{2}e^{-i\frac{\pi}{2}\alpha}\Gamma(\alpha, iu) + \frac{1}{2}e^{i\frac{\pi}{2}\alpha}\Gamma(\alpha, -iu).$$

Hence, we obtain

$$(\mathcal{F}^1_{(-b)}(\zeta(u),\zeta(v);x))' = -\sum_{n=1}^\infty \sigma_{1-u-v}(n)(2\pi n)^{u-1}2\int_{2\pi nx}^\infty t^{-u}\cos tdt.$$

Note that the differentiated series is absolutely convergent.
Now, using the residue theorem,

$$\mathcal{F}^1_{(c)}(\zeta(u),\zeta(v);x) = \mathcal{F}^1_{(-b)}(\zeta(u),\zeta(v);x) + x\zeta(u)\zeta(v) - \zeta(u-1)\zeta(v+1) + \frac{\zeta(u+v-1)x^{2-u}}{(u-2)(u-1)}.$$

Differentiating with respect to x gives,

$$(\mathcal{F}^1_{(c)}(\zeta(u),\zeta(v);x))' = (\mathcal{F}^1_{(-b)}(\zeta(u),\zeta(v);x))' + \zeta(u)\zeta(v) - \frac{\zeta(u+v-1)x^{1-u}}{(u-1)}.$$

Also, we know that

$$(\mathcal{F}^1_{(c)}(\varphi(u),\Phi(v);x))' = \sum_{m=1}^\infty a_m m^{-v}{\sum_{n\le mx}}'\alpha_n n^{-u}.$$

Taking $x=1$ with $a_m = 1$, $\alpha_n = 1$, we have

$$\sum_{m=1}^\infty m^{-v}{\sum_{n\le m}}' n^{-u} = -2\sum_{n=1}^\infty \sigma_{1-u-v}(n)(2\pi n)^{u-1}u\int_{2\pi n}^\infty t^{-u-1}\sin tdt + \zeta(u)\zeta(v) - \frac{\zeta(u+v-1)}{(u-1)}.$$

Similarly,

$$\sum_{n=1}^\infty n^{-u}{\sum_{m\le n}}' m^{-v} = -2\sum_{n=1}^\infty \sigma_{1-u-v}(n)(2\pi n)^{v-1}v\int_{2\pi n}^\infty t^{-v-1}\sin tdt + \zeta(u)\zeta(v) - \frac{\zeta(u+v-1)}{(v-1)}.$$

Adding the above two equations, we get

$$\zeta(u)\zeta(v) = -2\sum_{n=1}^\infty \sigma_{1-u-v}(n)(2\pi n)^{u-1}u\int_{2\pi n}^\infty t^{-u-1}\sin tdt - 2\sum_{n=1}^\infty \sigma_{1-u-v}(n)(2\pi n)^{v-1}v\int_{2\pi n}^\infty t^{-v-1}\sin tdt + 2\zeta(u)\zeta(v) - \zeta(u+v-1)\left\{\frac{1}{v-1}+\frac{1}{u-1}\right\}$$

and hence

$$\zeta(u)\zeta(v) = \zeta(u+v-1)\left\{\frac{1}{v-1}+\frac{1}{u-1}\right\}$$

$$+ 2(2\pi)^{u-1} \sum_{n=1}^{\infty} \sigma_{1-u-v}(n) n^{u-1} u \int_{2\pi n}^{\infty} t^{-u-1} \sin t dt$$
$$+ 2(2\pi)^{v-1} \sum_{n=1}^{\infty} \sigma_{1-u-v}(n) n^{v-1} v \int_{2\pi n}^{\infty} t^{-v-1} \sin t dt.$$

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