Numerical Godeaux surfaces with an involution in positive characteristic

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Abstract: A numerical Godeaux surface X is a minimal surface of general type with $\chi(\mathcal{O}_X) = K_X^2 = 1$. Over $\mathbf{C}$ such surfaces have $p_g(X) = h^1(\mathcal{O}_X) = 0$, but $p_g = h^1(\mathcal{O}_X) = 1$ also occurs in characteristic $p > 0$. Keum and Lee [9] studied Godeaux surfaces over $\mathbf{C}$ with an involution, and these were classified by Calabri, Ciliberto, and Mendes Lopes [4]. In characteristic $p \geq 5$, we obtain the same bound $|\mathrm{Tors}\, X| \leq 5$ as in characteristic 0, and we show that the quotient X/σ of X by its involution is rational, or is birational to an Enriques surface. Moreover, we give explicit examples in characteristic 5 of quintic hypersurfaces Y with an action of each of the group schemes G of order 5, and having extra symmetry by $\mathrm{Aut}\, G \cong \mathbf{Z}/4\mathbf{Z}$, hence by the *holomorph* $H_{20} = \mathrm{Hol}\, G = G \rtimes \mathbf{Z}/4\mathbf{Z}$ of G.

Key words: Godeaux surface; involution; positive characteristic; action of group scheme.

1. Introduction. Godeaux surfaces are surfaces over $\mathbf{C}$ of general type with the smallest invariants $p_g = q = 0$ and $K_X^2 = 1$. Information on the torsion groups of numerical Godeaux surfaces was obtained by Bombieri, Miyaoka, and Reid. It is known that $\mathrm{Tors}\, X$ has order at most 5 and $\mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$ is impossible [1,17,19]. Simply connected examples were first constructed by Barlow in 1982, and Lee and Park [13] gave a more recent construction. Godeaux surfaces with an involution over $\mathbf{C}$ were studied by Keum and Lee [9], and subsequently Calabri, Ciliberto, and Mendes Lopes [4] classified the possibilities for the quotient space of a Godeaux surface by its involution, proving that it is either rational or birational to an Enriques surface.

Lang [11] showed Godeaux surfaces exist in every characteristic. In his treatment $\mathrm{Pic}\, X$ is reduced, $\mathrm{Pic}^\tau X = \mathbf{Z}/5\mathbf{Z}$, and X is a quotient of a quintic hypersurface Y in $\mathbf{P}^3$ by an action of the multiplicative group scheme $\boldsymbol{\mu}_5$. A minimal surface X of general type over $\mathbf{C}$ with $K_X^2 = 1$ and $\chi(\mathcal{O}_X) = 1$ has $p_g(X) = h^1(\mathcal{O}_X) = 0$, but $p_g(X) = h^1(\mathcal{O}_X) = 1$ can also happen in characteristic $p = 2, 3$ and 5 [15]. These Godeaux surfaces are called nonclassical, and have nonreduced $\mathrm{Pic}\, X$.

Miranda [16] constructed a Godeaux surface with nonreduced Picard scheme in characteristic 5 via a Godeaux-like construction. In a similar way, Liedtke constructed an action of the additive group scheme $\boldsymbol{\alpha}_5$ on a quintic in characteristic 5 [15] by a nowhere zero additive vector field.

In these three cases, $\mathrm{Pic}^\tau X$ is isomorphic to $\mathbf{Z}/5\mathbf{Z}$, $\boldsymbol{\mu}_5$ and $\boldsymbol{\alpha}_5$ respectively, and Pic^τ determines a finite flat morphism $\varphi\colon Y \to X$ which is a torsor over X under the group scheme $(\mathrm{Pic}^\tau X)^\vee$, where $G^\vee$ denotes the Cartier dual group scheme of G. We obtain the same bound $|\mathrm{Tors}\, X| \leq 5$ as in characteristic 0, and we show that the quotient X/σ of X by its involution is rational, or is birational to an Enriques surface. We study the three families in characteristic 5 due to Lang [11], Miranda [16], and Liedtke [15] with $\mathrm{Pic}^\tau X$.

We show explicit examples of quintic hypersurface Y having symmetry by $\mathrm{Aut}\, G \cong \mathbf{Z}/4\mathbf{Z}$ which is the *holomorph* $H_{20} = \mathrm{Hol}\, G = G \rtimes \mathbf{Z}/4\mathbf{Z}$ of G to give an involution on examples in each family in characteristic 5.

2. Godeaux surfaces in positive characteristic.

2.1. Notation and basic results. We work over an algebraically closed field k of characteristic $p \neq 2$. Recall the following definitions.

$\chi(\mathcal{O}_X) := \sum_{i=0}^n (-1)^i h^i(\mathcal{O}_X)$
$b_i^{\mathrm{et}} := \dim H^i_{\mathrm{et}}(X, \mathbf{Q}_l)$
$e(X) := \chi_{\mathrm{top}}(X) := \sum_{i=0}^n (-1)^i b_i^{\mathrm{et}}(X)$
$\omega_X :=$ dualizing sheaf of X
$p_g := h^2(X, \mathcal{O}_X) = \dim H^0(X, \omega_X)$
$q := \dim \mathrm{Alb}\, X$
$T_X := \mathcal{H}om_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$

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$\mathrm{Pic}^\tau X :=$ subscheme of $\mathrm{Pic}\,X$ of numerically trivial Cartier divisors

$W_2(k) :=$ ring of second Witt vectors of k.

Proposition 2.1 (Proposition 1, [15]). *Let X be a minimal surface of general type with $K_X^2 = 1$. Then the following equalities and inequalities hold*:

$$1 \le \chi(\mathcal{O}_X) \le 3, \quad p_g(X) \le 2, \quad h^1(\mathcal{O}_X) \le 1,$$
$$b_1(X) = 0, \quad |\pi_1^{\mathrm{et}}(X)| \le 6.$$

In particular, if $h^1(\mathcal{O}_X) \ne 0$, then X has nonreduced Picard scheme, which can happen only in positive characteristic.

Definition 2.2. A *numerical Godeaux surface* is a minimal surface X of general type over an algebraically closed field with $K_X^2 = 1$ and $\chi(\mathcal{O}_X) = 1$. In this paper we abbreviate *numerical Godeaux surface* to *Godeaux surface*.

Theorem 2.3 (Corollary 1, [15]). *Nonclassical Godeaux surfaces can exist only in characteristic $2 \le p \le 5$.*

2.2. Tors X. Let G be a subgroup scheme of order n in $\mathrm{Pic}^\tau X$. Then there is a finite morphism $\varphi: Y \to X$ that is a nontrivial $G^\vee$-torsor. If the cover is purely inseparable, Y may be singular, but is still an irreducible Gorenstein surface [6]. If $\varphi: Y \to X$ is a $\boldsymbol{\mu}_p$-torsor then

$$\varphi_*\mathcal{O}_Y = \bigoplus_{0 \le i \le p-1} L_i,$$

where $L_0 = \mathcal{O}_X$, $L_1 \in \mathrm{Pic}\,X$ is a line bundle with $L_1^{\otimes p} \cong \mathcal{O}_X$, and $L_i \cong L_1^{\otimes i}$.

If $\varphi: Y \to X$ is a $\boldsymbol{\alpha}_p$-torsor then $\varphi_*\mathcal{O}_Y$ is a successive extension of sheaves isomorphic to $\mathcal{O}_X$ [6, Proposition I.1.7]. And the equalities

$$\chi(\mathcal{O}_Y) = p\chi(\mathcal{O}_X) \quad \text{and} \quad K_Y^2 = pK_X^2 \tag{2.1}$$

hold as for a finite degree n Galois étale cover [15].

Proposition 2.4. *Let X be a minimal surface of general type over an algebraically closed field k. Suppose characteristic $p \ge 5$. If $K_X^2 = 1$ and $\chi(\mathcal{O}_X) = 1$, then $|\mathrm{Pic}^\tau X| \le 5$.*

Proof. The proof is similar to Reid [18]. Let $\varphi: Y \to X$ be the $G^\vee$-torsor associated to $G = \mathrm{Pic}^\tau X$ of order n. Since $\mathrm{char}\,k \ne 2, 3$, the Noether inequality $K^2 \ge 2p_g - 4$ and (2.1) imply $|\mathrm{Pic}^\tau X| \le 6$.

Suppose $|\mathrm{Pic}^\tau X| = 6$. There is 6-to-1 étale cover $\varphi: Y \to X$ with $p_g(Y) = 5$, $K_Y^2 = 6$. Then Y is a Horikawa surface with $h^1(\mathcal{O}_Y) = 0$. The canonical map is a double cover $\varphi_{K_Y}: Y \to Z$ and restricts to a g_6^3 on a general $C \in |K_Y|$. The classical Clifford theorem on an irreducible Gorenstein curve says that C is hyperelliptic [5]. The canonical image Z is an irreducible surface of degree 3 spanning $\mathbf{P}^4$ [6, Proposition 0.1.2 (iii)], [14, Theorem 2.3], and Z is either $\mathbf{F}_1$ embedded in $\mathbf{P}^4$ as the cubic scroll or the cone over a rational normal curve of degree 3 in $\mathbf{P}^4$ [14, Theorem 3.3]. In either case, the Horikawa double cover induces a biregular involution, and the composite $p \circ \varphi := f$ (where p is the projection $p: F \to \mathbf{P}^1$) is a canonically defined pencil of curves $f: Y \to \mathbf{P}^1$ with fibers of genus 2. The Horikawa double cover induces a biregular involution since we work in characteristic $\ne 2$, and the surface Y has a canonically defined pencil of curves of genus 2. This contradicts the free action of $\mathbf{Z}/3\mathbf{Z}$. $\square$

Remark 2.5. Proposition 2.4 implies that if $\mathrm{Pic}^\tau X$ contains a nontrivial subgroup scheme of odd order, then $\mathrm{Pic}^\tau X$ has no 2-torsion.

3. Numerical Godeaux surfaces with an involution in odd characteristic. Let X be a smooth Godeaux surface in positive characteristic $p \ne 2$ with an involution σ. The quotient double cover $\pi: X \to T := X/\sigma$ fits in the diagram

$$\begin{array}{ccc} V & \xrightarrow{\ \varepsilon\ } & X \\ {\scriptstyle\widetilde{\pi}}\downarrow & & \downarrow{\scriptstyle\pi} \\ W & \xrightarrow{\ \eta\ } & T \end{array} \quad . \tag{3.1}$$

Given an involution σ on X, its fixed locus is the union of a smooth curve R and n isolated fixed points $p_1, \ldots, p_n$. The singularities of T are canonical and the adjunction formula gives $K_X \equiv \pi^*K_T + R$. In diagram (3.1), let ε be the blowup of X at n isolated fixed points in X of the action of σ. The quotient map π induces a double cover $\widetilde{\pi}$, where η is the minimal resolution of the n ordinary double points of T. We set $E_i := \varepsilon^*(p_i)$, $R_0 := \varepsilon^*(R)$ on V and $C_i := \widetilde{\pi}(E_i)$, $B_0 := \widetilde{\pi}(R_0)$ on the smooth surface W. The C_i are n disjoint -2-curves.

The map $\widetilde{\pi}$ is a finite flat double cover with branch locus $\widetilde{B} := B_0 + \sum_{i=1}^n C_i$. Thus there exists a line bundle L on W for which $2L \equiv \widetilde{B}$ and $\widetilde{\pi}_*\mathcal{O}_V = \mathcal{O}_W \oplus L^{-1}$. Later in Lemma 3.6, we assume in addition that K_X is ample, so that H_m has no -2-curves other than the four C_i.

Proposition 3.1. *Let X be a minimal surface of general type in odd characteristic with an involution σ. Then*:

(i) *$2K_W + B_0$ is nef and big*;

(ii) $(2K_W + B_0)^2 = 2K_X^2$;

(iii) $K_W(K_W + L) \le 0$;

(iv) *The Kodaira dimension $\kappa(W) \le 0$.*

Proof. (i) and (ii) follow from $\widetilde{\pi}^*(2K_W + B_0) = \varepsilon^*(2K_X)$. Part (iii) is clear by formula (ii). For (iv), since the Kodaira dimension κ is a birational invariant, consider $\varphi\colon D \subset W \to D' \subset W_{\min} := W'$, where $D \in |2K_W + B_0|$ and $D' = \varphi_* D$. Suppose by contradiction that $\kappa(W) \geq 1$. Then $DK_W \leq 0$, which implies $D'K_{W'} \leq 0$. For $m \gg 0$ we have $D'mK_{W'} = D'(M + F)$, where M is the moving part and F the fixed part. Then $D'K_{W'} > 0$, which contradicts $D'K_{W'} \leq 0$. Hence $\kappa(W) \leq 0$. □

3.1. Vanishing theorem for $2K_W + L$. The Kodaira vanishing theorem and its extension due to Kawamata and Viehweg may fail in positive characteristic [7]. However, under additional assumptions, notably lifting to $W_2(k)$, the Kawamata–Viehweg vanishing theorem does hold.

Assumption 3.2. We fix the notation used in Theorem 3.3. X denotes a d-dimensional projective smooth variety over a perfect field k. Let $E = \sum_{j=1}^m E_j$ be a reduced simple normal crossing divisor on X. Assume that $E \subset X$ has a lifting $\widetilde{E} = \sum_{j=1}^m \widetilde{E}_j \subset \widetilde{X}$ to $W_2(k)$.

Theorem 3.3 (Corollary 3.8, [8])**.** *Let X be projective over a Noetherian affine scheme and let D be an ample* $\mathbf{Q}$*-divisor on X such that* $\operatorname{Supp}(D - [D]) \subseteq \operatorname{Supp}(E)$*. Assume that $E \subset X$ admits a lifting $\widetilde{E} \subset \widetilde{X}$ to $W_2(k)$. Then, if $i + j > d =$* $\dim X$ *and if $p > d$, we have*

$$H^i(X, \Omega_X^j(\log E)(-E - [-D])) = 0. \tag{3.2}$$

Proposition 3.4 ([12])**.** *Let X be an algebraic surface with isolated normal singularities, $\pi\colon V \to X$ its minimal resolution, and E the reduced exceptional divisor. If X has a cyclic quotient singularity of type $\frac{1}{n}(1, n-1)$, we assume that n is coprime to p. Then we have equality $\pi_* T_V(-\log E) = \pi_* T_V = T_X$.*

We keep the notation of diagram (3.1), $C := \sum_{i=1}^n C_i$ and $H_m := K_W + L - (\frac{1}{2} + \frac{1}{m})C$.

Lemma 3.5. *H_m is an ample* $\mathbf{Q}$*-divisor for* $m \gg 0$.

Proof. Let $N := K_W + L - \frac{1}{2}\sum_{i=1}^n C_i$, then $N = \frac{1}{2}(2K_W + B_0)$ in (3.1). $\widetilde{\pi}^* N = \frac{1}{2}\varepsilon^*(2K_X)$ is a nef and big divisor on V. For $s \gg 0$ the linear system $|sN|$ is basepoint free and the associated morphism is birational, and contracts exactly C_i. Hence $N = \eta^* A$ for some ample $\mathbf{Q}$-divisor A, and then $L + \nu(\varepsilon^* K_X) = L + \nu(\widetilde{\pi}^* N)$ is ample for $\nu \gg 0$ by [10, Proposition 1.45]. Therefore H_m is ample for $m \gg 0$. □

Consider the following sequence:

$$0 \to T_W(-\log C) \to T_W \to \bigoplus N_{C_i|W} \to 0. \tag{3.3}$$

Lemma 3.6. *(W, C) lifts over $W_2(k)$.*

Proof. (3.3) gives the long exact sequence:

$$\begin{aligned} \cdots &\to H^1(T_W) \to H^1(\bigoplus N_{C_i|W}) \\ &\to H^2(T_W(-\log C)) \to H^2(T_W) \to 0. \end{aligned}$$

By results of Lee and Nakayama in [12], and by Section 1 of Burns and Wahl [3], the morphism $H^1(T_W) \to \bigoplus H^1(N_{C_i|W})$ is surjective. If W is rational or an Enriques surface, then $H^2(T_W) = 0$ holds in any characteristic except possibly 2. Hence $H^2(T_W(-\log C)) = 0$ by the above exact sequence, so that (W, C) lifts to $W_2(k)$ by [20, Lemma 4.1]. □

Lemma 3.7. *Let W, L be as above. Then $H^i(W, \mathcal{O}_W(2K_W + L)) = 0$ for $i > 0$.*

Proof. By Lemma 3.5, H_m is ample for $m \gg 0$. Now (W, C) lifts over $W_2(k)$ by Lemma 3.6. Applied to Theorem 3.3, this gives the vanishing

$$H^i(W, \mathcal{O}_W(2K_W + L)) = 0 \quad \text{for } i > 0.$$

□

By Riemann–Roch on surfaces, and $\widetilde{\pi}_*\mathcal{O}_V = \mathcal{O}_W \oplus L^{-1}$, we get $\chi(\mathcal{O}_V) = 2\chi(\mathcal{O}_W) + \frac{1}{2}L(K_W + L)$, which holds in odd characteristic.

Theorem 3.8. *Let X, V and W be as above, and n the number of fixed points of σ. Then:*

(i) $\chi(\mathcal{O}_W) = 1$, *and* (ii) $n \geq 5$.

Proof. By the Riemann–Roch theorem, Lemma 3.7 and the standard double cover formula

$$\begin{aligned} 0 \leq h^0(2K_W + L) &= \chi(2K_W + L) \\ &= \chi(\mathcal{O}_W) + \tfrac{1}{2}(2K_W + L)(K_W + L) \\ &= \chi(\mathcal{O}_V) - \chi(\mathcal{O}_W) + K_W(K_W + L). \end{aligned}$$

Suppose by contradiction that $\chi(\mathcal{O}_W) \leq 0$. Then,

$$\begin{aligned} K_V^2 &= 2(K_W + L)^2 \\ &= 2((K_W + L)K_W + 2\chi(\mathcal{O}_V) - 4\chi(\mathcal{O}_W)) \\ &= 2(h^0(K_W + L) + \chi(\mathcal{O}_V) - 3\chi(\mathcal{O}_W)) \geq 2. \end{aligned}$$

A contradiction, and hence $\chi(\mathcal{O}_W) = 1$. From the standard double cover formula, (i), and by Proposition 3.1 (iii), we have $(K_W + L)^2 \leq -2$. So that $K_V^2 = \widetilde{\pi}^*(K_W + L)^2 \leq -4$, hence $n \geq 5$. □

Corollary 3.9. *Either W is rational or its minimal model is birational to an Enriques surface.*

Proof. From Theorem 3.8 (i), Proposition 3.1 (iv) and the Table of possible invariants for surfaces with $\kappa = 0$ in [2], $p_g(W) = h^1(\mathcal{O}_W) = 0$. The result thus follows from the classification of surfaces. □

Lemma 3.10. *$H^i(W, \mathcal{O}_W(2K_W + L)) = 0$ for $i > 0$, hence $h^0(W, \mathcal{O}_W(2K_W + L)) = 0$.*

Proof. By Theorem 3.8 (i), $\chi(\mathcal{O}_W) = 1$ and $L(K_W + L) = -2$. Therefore $\chi(2K_W + L) \leq 0$. Thus $H^i(2K_W + L) = 0$ if $i > 0$, hence $h^0(W, \mathcal{O}_W(2K_W + L)) = 0$. □

Corollary 3.11. *Let φ be the bicanonical map of X. Then*:

(i) *φ is composed with σ*;

(ii) $(K_W + L)K_W = 0$;

(iii) $n = 5$.

Proof. Lemma 3.10 gives (i). Vanishing for $2K_W + L$ and (i) give $h^0(2K_W + L) = K_W(K_W + L) = 0$, $K_V^2 = \widetilde{\pi}^*(K_W + L)^2 = 2(K_W + L)^2$, and $n = K_X^2 - K_V^2 = 1 - 2(K_W + L)^2 = 5$. □

Lemma 3.12. *Let $f\colon X \to E$ be a double cover with X a nonsingular surface of general type and E birational to an Enriques surface in characteristic $p \neq 2$. Then f^*K_E is a nontrivial torsion element in $\operatorname{Pic} X$. Equivalently, if $K \to E$ is the K3 double cover, then the fiber product $Y = X \times_E K$ is irreducible.*

Proof. Consider

$$(3.4)\qquad \begin{array}{ccc} Y & \longrightarrow & X \\ \downarrow & & \downarrow f \\ K & \xrightarrow{\ \varphi\ } & E \end{array}.$$

We can assume that $X \to E$ is the quotient by an involution, so E has only $\frac{1}{2}(1,1)$ singularities. The ramification locus of $X \to E$ is a nonzero divisor D_E. For otherwise $K_X = f^*K_E$ is numerically zero, which contradicts X of general type. Consider the fiber product $Y = X \times_E K$ as the composite $Y \to K \to E$. Here $\varphi\colon K \to E$ is the K3 double cover. Also $Y \to K$ is ramified in the divisor $D_K = \varphi^* D_E$ by fiber product. Now $D_K > 0$ and therefore Y is an irreducible surface. By base change, $Y \to X$ is the double cover corresponding to f^*K_E, so that $f^*K_E \neq 0$ in $\operatorname{Pic} X$ if and only if Y is irreducible. □

Corollary 3.13. *Let X be a Godeaux surface with 5-torsion in characteristic 5. Then the birational type of the quotient space of X by an involution cannot be an Enriques surface.*

Proof. Assume that the quotient of X is an Enriques surface W. We may assume W is minimal. An Enriques surface W in characteristic $\neq 2$ has K_W a 2-torsion class. Therefore the algebraic fundamental group $\pi_1^{\mathrm{et}}(W)$ is isomorphic to $\mathbf{Z}/2\mathbf{Z}$. The fundamental group is a birational invariant of surfaces with at worst rational singularities. There is an étale 2-to-1 cover $f\colon S \to W_{\min}$ with S a K3 surface in any characteristic [2]

$$(3.5)\qquad \begin{array}{ccccc} \widetilde{X} & \longrightarrow & X & \xrightarrow{\ 2:1\ } & W_{\min} \\ \downarrow & & \downarrow & \nearrow & \\ S & \underset{2:1}{\dashrightarrow} & W & \longrightarrow & W_{\min}. \end{array}$$

If the quotient of X by its involution is birational to an Enriques surface W, the pullback of the 2-torsion class K_W defines a nontrivial 2-torsion class on X by Lemma 3.12, so $|\operatorname{Pic}^{\tau} X|$ must have even order. This contradicts Remark 2.5. □

4. Godeaux surfaces in characteristic 5 with an involution. The Godeaux surfaces in characteristic 5 due to Lang [11], Miranda [16], and Liedtke [15] are constructed as quotients $X = Y/G$ of a quintic hypersurface $Y \subset \mathbf{P}^3$ by a group scheme G of order 5 action freely. Here if $G = \mathbf{Z}/5\mathbf{Z}$, *free* means that G acts without fixed points. In the inseparable cases $\boldsymbol{\mu}_5$ or $\boldsymbol{\alpha}_5$, it means that G acts by a nowhere zero vector field. They prove the nonsingularity of X by using Bertini's theorem for a very ample linear system on $\mathbf{P}^3/G$. Instead, in each case, we give an explicit example of Y having symmetry by $\operatorname{Aut} G \cong \mathbf{Z}/4\mathbf{Z}$, hence by the *holomorph* $H_{20} = \operatorname{Hol} G = G \rtimes \mathbf{Z}/4\mathbf{Z}$ of G. For the two inseparable cases, the nonsingularity of X involves a nonclassical calculation: as we show in 4.4, Y has exactly 11 singular points of type A_4.

4.1. The case $\mathbf{G = Z/5Z}$. Miranda [16] takes the linear map σ given by the matrix:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

He constructs a quintic surface Y invariant under $\langle\sigma\rangle$ using the subspace V of quintic forms generated by norms $N(l) = \prod_{i=0}^{4} \sigma^i(l)$ of linear forms l; these forms define an embedding $\mathbf{P}^3/\langle\sigma\rangle \subset \mathbf{P}(V)$, and his X is a hyperplane section.

Lemma 4.1. *The linear automorphisms*

$$A = \begin{pmatrix} 1 & 3 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

generate an action of $\operatorname{Hol} G = \mathbf{Z}/5\mathbf{Z} \rtimes \mathbf{Z}/4\mathbf{Z}$ on $\mathbf{P}^3$. Clearly A, B^2 generate an action of D_{10}.

Proof. One checks directly that $A^5 = 1$, $B^4 = 1$ and $BAB^{-1} = A^2$. □

Proposition 4.2. *There exists a nonsingular hypersurface Y in $\mathbf{P}^3$ invariant under* $\operatorname{Hol} G$ *action.*

Proof. Set

$$f := x^5 + 3x^3yw + 2x^3z^2 + 3x^2y^2z + 2x^2zw^2 - xy^4 - xy^2w^2 + 2xz^4 + 3xw^4 + 2y^3zw + 3y^2z^3 + yzw^3.$$

One checks that f is invariant under A and B, and the quintic surface $Y \subset \mathbf{P}^3$ defined by $f = 0$ is nonsingular. (This is easy by computer algebra, but it can also be done by hand.) □

Now the quotient $Y \to X$ is an étale $\mathbf{Z}/5\mathbf{Z}$ cover of a Godeaux surface X with $p_g(X) = h^1(\mathcal{O}_X) = 1$ and $\pi_1 = \mathbf{Z}/5\mathbf{Z}$, and the $\operatorname{Hol} G$ action on Y descends to a $\mathbf{Z}/4\mathbf{Z}$ action on X, so in particular an involution.

4.2. The case $G = \boldsymbol{\mu}_5$. Lang's Godeaux surfaces [11] satisfy $p_g(X) = h^1(\mathcal{O}_X) = 0$, and work in all characteristics. The group scheme $\boldsymbol{\mu}_5$ acts on $\mathbf{P}^3$ by $\varepsilon(x_i) \to \varepsilon^i x_i$ and $\mathbf{P}^3/\boldsymbol{\mu}_5$ is nonsingular except at the 4 coordinate points. If Y does not pass through these points, the $\boldsymbol{\mu}_5$ action on $\mathbf{P}^3$ restricts to a free action on Y. The general hyperplane $X = Y/\boldsymbol{\mu}_5$ is a nonsingular Godeaux surface.

Lemma 4.3. *Let $A = \boldsymbol{\mu}_5$ act on $\mathbf{P}^3$ with coordinates x, y, z, w by $\frac{1}{5}(1, 2, 4, 3)$. The permutation $B = (x, y, z, w)$ of S_4 defines a linear map of $\mathbf{P}^3$ that normalizes the $\boldsymbol{\mu}_5$ action, and generates an action of the semidirect product group scheme* $\operatorname{Hol} \boldsymbol{\mu}_5 = \boldsymbol{\mu}_5 \rtimes \mathbf{Z}/4\mathbf{Z}$. *Then $\langle A, B^2\rangle$ is a dihedral group scheme D_{10}.*

Proof. The 4-cycle $B = (x, y, z, w)$ corresponds to the generator $\varepsilon \mapsto \varepsilon^2$ of $\operatorname{Aut} \boldsymbol{\mu}_5 \cong \mathbf{Z}/4\mathbf{Z}$. One checks that $BAB^{-1} = A^2$. □

Proposition 4.4. *There exists a hypersurface $Y_5 \subset \mathbf{P}^3$ invariant under* $\operatorname{Hol} \boldsymbol{\mu}_5$ *such that the quotient $X = Y/\boldsymbol{\mu}_5$ is a nonsingular Godeaux surface.*

Proof. Set

$$f = x^5 + y^5 + z^5 + w^5 + 2(x^3zw + xy^3w + xyz^3 + yzw^3) + 3(x^2y^2z + x^2yw^2 + xz^2w^2 + y^2z^2w).$$

Clearly Y is invariant under $\operatorname{Hol} \boldsymbol{\mu}_5$ and D_{10}. See 4.4 for the nonsingularity of X. □

4.3. The case $G = \boldsymbol{\alpha}_5$. Liedtke [15] uses the vector field $\delta := y\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$ to generate an $\boldsymbol{\alpha}_5$ action on $\mathbf{P}^3$. Let V be the vector space of elements of degree 5 in the fixed ring of δ. The morphism $\varphi\colon \mathbf{P}^3 \to \mathbf{P}(V)$ can be identified with the quotient map $\mathbf{P}^3 \to \mathbf{P}^3/\boldsymbol{\alpha}_5$, at least outside $[1 : 0 : 0 : 0]$. Its general hyperplane is a nonsingular Godeaux surface $X = Y/\boldsymbol{\alpha}_5$ quotient of a δ-invariant quintic $Y_5 \subset \mathbf{P}^3$.

Lemma 4.5. *The following matrices generate an action of* $\operatorname{Hol} \boldsymbol{\alpha}_5 = \boldsymbol{\alpha}_5 \rtimes \mathbf{Z}/4\mathbf{Z}$ *on* $\mathbf{P}^3$:

$$A = \begin{pmatrix} 1 & 3t & 3t^2 & t^3 \\ 0 & 1 & 2t & t^2 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

such that $\langle A\rangle \cong \boldsymbol{\alpha}_5$ with $t^5 = 0$, $\langle B\rangle \cong \mathbf{Z}/4\mathbf{Z}$. Moreover A, B^2 generate a dihedral group scheme D_{10}.

Proof. The matrix A with $t^5 = 0$ defines a group scheme $\boldsymbol{\alpha}_5$. One sees that $BAB^{-1} = A^2$ as in Lemma 4.1. To prove $D_{10} = \langle A, B^2\rangle$ is clear. □

Proposition 4.6. *There exists a hypersurface $Y_5 \subset \mathbf{P}^3$ invariant under* $\operatorname{Hol} \boldsymbol{\mu}_5$ *with quotient $Y/\boldsymbol{\alpha}_5$ a nonsingular Godeaux surface.*

Proof. Set

$$f := x^5 + 2xy^2w^2 + xyz^2w + 2xz^4 + 2xw^4 - y^3zw + y^2z^3 - yzw^3 - z^3w^2.$$

Y is invariant under $\boldsymbol{\alpha}_5$, $\mathbf{Z}/4\mathbf{Z}$ and hence $\operatorname{Hol} G$. As in the $\boldsymbol{\mu}_5$ case, we show in 4.4 that X is nonsingular. □

4.4. Nonsingularity of the quotient X. A quintic Y with an inseparable group action as in Proposition 4.4 and 4.6 must be singular. In fact a nonsingular quintic Y has $e(Y) = c_2(Y) = 55$. But a nonsingular surface with an everywhere nonzero vector field has $c_2(Y) = 0$. Alternatively, if Y is an inseparable cover of a nonsingular Godeaux surface X, then X and Y are homeomorphic in the étale topology, so $e(Y) = e(X) = 11$. We define the *singular subscheme* of Y by $V(J) \subset Y$, where $J(f) := (f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}, \frac{\partial f}{\partial w})$ is the Jacobian ideal.

Lemma 4.7. *Let f be either of the invariant quintic polynomials of Proposition* 4.4 *and* 4.6. *Then* $\dim V(J) = 0$, $\deg V(J) = 55$ *and* $\deg V(J)_{\mathrm{red}} = 11$.

Proof. Computer algebra. (A Magma script is available on request.) □

Corollary 4.8. *Y has* 11 *singularities of type A_4 (of analytic type $xy = z^5$), and X is nonsingular.*

Proof. Lemma 4.7 says that $V(J)$ is supported at 11 distinct singular points of Y. $\boldsymbol{\mu}_5$ or $\boldsymbol{\alpha}_5$ act freely on $\mathbf{P}^3$ except at the fixed coordinate points and $V(J)$ is invariant under these G actions. Define $\mathcal{J} = J \cdot \mathcal{O}_{\mathbf{P}^3}$ to be the sheaf of ideals generated by J, and $\mathcal{J}_i$ its stalks at the 11 singular points. Then $\mathcal{J} = \bigcap_{i=1}^{11} \mathcal{J}_i$. Each $\mathcal{J}_i$ is G-invariant, so that $V(\mathcal{J}_i)$ contains an orbit of G. From Lemma 4.7, each

$V(\mathcal{J}_i)$ coincides with the G-orbit of P_i, which is a subscheme of length 5. Hence $\mathcal{O}_{\mathbf{P}^3}/\mathcal{J}_i \cong k[G] \cong k[z]/z^5$. We choose local regular coordinates x, y, z in the local ring $\mathcal{O}_{\mathbf{P}^3,P_i}$, so that $V(\mathcal{J}_i) = (x, y, z^5)$. Thus $x, y \in \mathcal{J}_i$, and after a coordinate change $f = xy - z^5 +$ higher order terms.

The group scheme G acts by an everywhere nonzero p-closed vector field D, and $D(f) = 0$. It follows that $D = a_0(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}) + b\frac{\partial}{\partial z}$, where $a_0 \in k[\![x, y, z]\!]$ and b is unit. We want to arrange that $Dx = Dy = 0$ after coordinate change. Set

$$\xi = x(1 + \alpha_1 z + \cdots + \alpha_4 z^4),\ \eta = y(1 + \cdots + \alpha_4 z^4)^{-1}.$$

We take $\alpha_1, \ldots, \alpha_4 \in \mathcal{O}_{\mathbf{P}^3,P_i}$ then $D(\xi), D(\eta) \in (z^4)$. This coordinate change gives $D = az^4(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}) + b\frac{\partial}{\partial z}$. $Dx = axz^4$, so that $D^5 x = 4!b^4(ax) + \cdots = \alpha \cdot axz^4$. $D^5(x) = D^4(az^4 x)$ includes the term $a \cdot 4! \cdot b^4 x$ and other terms in $a \cdot m^2$, where m is a maximal ideal. But $D^5(x) = cD(x) = caz^4 x$ with $c = 0$ or 1, a is divisible by z. Since $\mathcal{O}_{\mathbf{P}^3,P_i}$ is UFD, hence $a = 0$. Similarly for $D(y)$, then the vector field acting on z only by $z \to \alpha z + \beta$ with $\alpha^5 = 1$, $\beta^5 = 0$. Thus $x, y \in \mathcal{O}_{X,Q_i}$ are regular functions on the quotient $X = Y/G$ and the ideal they generate in $\mathcal{O}_{Y,P}$ is $\mathcal{J}\mathcal{O}_Y$. Therefore x, y generate the maximal ideal of $\mathcal{O}_{X,Q}$, which implies X is nonsingular. □

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