A note on Julia-Carathéodory Theorem for functions with fixed initial coefficients

By Adam Lecko and Barbara Uzar

Department of Analysis and Differential Equations, University of Warmia and Mazury, ul. Słoneczna 54, 10-710 Olsztyn, Poland

(Communicated by Masaki Kashiwara, m.j.a., Nov. 12, 2013)

Abstract: Sharpened version of the Julia-Carathéodory Theorem for functions with fixed initial coefficients is proved.

Key words: Julia-Carathéodory Theorem; Julia functions; Schwarz functions.

1. Introduction. The famous Julia-Carathéodory Theorem says that every analytic self-mapping of the unit disk $\mathbf{D} := \{z \in \mathbf{C} : |z| < 1\}$ having the angular limit 1 at 1 has the angular derivative at 1 which is either a positive real number or infinity. In this note we generalize the so-called General Boundary Lemma due to Osserman [11, p. 3515] giving sharp lower bound of the angular derivative of self-mappings of $\mathbf{D}$ depending on their initial coefficients. Some examples of self-mappings of $\mathbf{D}$ complete the result.

Let $\overline{\mathbf{C}} := \mathbf{C} \cup \{\infty\}$, $\overline{\mathbf{D}} := \{z \in \mathbf{C} : |z| \le 1\}$ and $\mathbf{T} := \partial\mathbf{D}$. Let $\mathcal{A}$ denote the class of analytic functions in $\mathbf{D}$ and $\mathcal{B}$ its subclass of all self-mappings of $\mathbf{D}$.

For $\xi \in \mathbf{D}$ let

$$\varphi_\xi(z) := \frac{z-\xi}{1-\overline{\xi}z}, \quad z \in \mathbf{C} \setminus \{1/\overline{\xi}\}.$$

For $z \in \mathbf{D}$ we have

$$\varphi_\xi(z) = -\xi + (1-|\xi|^2)\sum_{k=1}^{\infty}\overline{\xi}^{k-1}z^k. \tag{1.1}$$

It is well known that for each $\xi \in \mathbf{D}$, $\varphi_\xi \in \mathcal{B}$ is a conformal automorphism of $\overline{\mathbf{D}}$.

The angular limit and the angular derivative at $\zeta \in \mathbf{T}$ of a function $f \in \mathcal{A}$ will be denoted as $f_\angle(\zeta)$ (also as $\angle f(\zeta)$) and as $f'_\angle(\zeta)$, respectively. Recall (see e.g. [12, p. 79]) that f has the angular derivative at $\zeta \in \mathbf{T}$ if $f_\angle(\zeta)$ exists and is finite, and the angular limit

$$f'_\angle(\zeta) := \angle\lim_{z\to\zeta}\frac{f(z)-f_\angle(\zeta)}{z-\zeta}$$

exists.

The class $\mathcal{B}_1$ of the so-called *Julia functions* defined below was introduced in [7] (see also [8, p. 21]). Further studies of Julia functions and related idea of the boundary subordination introduced by the first author of this paper were continued in [9].

Definition 1.1. Let

$$\mathcal{B}_1 := \{\omega \in \mathcal{B} : \omega_\angle(1) = 1\}$$

be the class of Julia functions. For $\lambda \in (0, \infty]$ let

$$\mathcal{B}_1(\lambda) := \left\{\omega \in \mathcal{B}_1 : \omega'_\angle(1) = \lambda\right\}.$$

For $\omega \in \mathcal{B}_1$ let

$$\Lambda(\omega) := \sup\left\{\frac{|1-\omega(z)|^2}{1-|\omega(z)|^2}\cdot\frac{1-|z|^2}{|1-z|^2} : \ z \in \mathbf{D}\right\}.$$

Recall now the Julia-Carathéodory-Wolff theorem (see [6], [12, p. 82]), [2, p. 57], [5, p. 44]).

Theorem 1.2. *If $\omega \in \mathcal{B}_1$, then $\omega'_\angle(1)$ exists and*

$$0 < \omega'_\angle(1) = \lim_{r\to1^-}\frac{1-|\omega(r)|}{1-r} = \Lambda(\omega) \le \infty, \tag{1.2}$$

i.e., $\omega \in \mathcal{B}_1(\lambda)$, where $\lambda = \omega'_\angle(1)$ satisfies (1.2). *Moreover*

$$\mathcal{B}_1 = \bigcup_{\lambda\in(0,\infty]}\mathcal{B}_1(\lambda).$$

Define now the following classes of functions.

Definition 1.3. Let

2010 Mathematics Subject Classification. Primary 30C45.

doi: 10.3792/pjaa.89.133

$$\mathcal{B}_0 := \{\omega \in \mathcal{B} : \omega(0) = 0\}$$

be the class of *Schwarz functions*. Let

$$\mathcal{B}_{0,1} := \mathcal{B}_0 \cap \mathcal{B}_1.$$

For each $n \in \mathbf{N}$ let $\mathcal{B}^{(n)}$ be the class of all functions $\omega \in \mathcal{B}$ of the form

$$\omega(z) = b_0 + \sum_{k=n}^{\infty} b_k z^k, \quad z \in \mathbf{D}, \tag{1.3}$$

with $b_n \neq 0$, and let

$$\mathcal{B}_0^{(n)} := \mathcal{B}^{(n)} \cap \mathcal{B}_0,\ \mathcal{B}_1^{(n)} := \mathcal{B}^{(n)} \cap \mathcal{B}_1,$$

$$\mathcal{B}_{0,1}^{(n)} := \mathcal{B}_0^{(n)} \cap \mathcal{B}_1.$$

Observe that for $\omega \in \mathcal{B}_1$,

$$\Lambda(\omega) \geq \frac{|1-\omega(0)|^2}{1-|\omega(0)|^2}.$$

Hence and from Theorem 1.2 we have at once the following assertion.

Corollary 1.4. *If $\omega \in \mathcal{B}_1$ and $b_0 = \omega(0)$, then*

$$\omega'_{\angle}(1) \geq \frac{|1-b_0|^2}{1-|b_0|^2}.$$

The assertion of the next theorem can be found in Remark 3 of [11]. In the form below the theorem was reproved in [9].

Theorem 1.5. *Let $n \in \mathbf{N}$. If $\omega \in \mathcal{B}_{0,1}^{(n)}$, then*

$$\omega'_{\angle}(1) \geq n.$$

2. Main result. In this section we improve Theorem 1.5 and Corollary 1.4 by showing the sharp lower bound of the angular derivative at 1 of functions in the class $\mathcal{B}_1^{(n)}$ (Theorem 2.5) dependent on their initial coefficients b_0 and b_n.

The version of the Schwarz lemma formulated below is a simple consequence of Lemma 1 of [4]. The inequality (2.2) can be found in Remark 3 of [11]. Lemma 1 of [4] was reproved and called Interior Schwarz Lemma in [11].

Lemma 2.1. *Let $n \in \mathbf{N}$. If $\omega \in \mathcal{B}_0^{(n)}$ is of the form*

$$\omega(z) = \sum_{k=n}^{\infty} b_k z^k, \quad z \in \mathbf{D}, \tag{2.1}$$

then

$$|\omega(z)| \leq |z|^n \frac{|z| + |b_n|}{1 + |b_n||z|}, \quad z \in \mathbf{D}. \tag{2.2}$$

Equality in (2.2) *for some $z \neq 0$ can occur only for*

$$\omega(z) := z^n \varphi_{-\xi}(\kappa z), \quad z \in \mathbf{D}, \tag{2.3}$$

where $\kappa \in \mathbf{T}$ and $\xi \in \mathbf{D}$.

Notation 2.2. Let $n \in \mathbf{N}$. For $\omega \in \mathcal{B}_{0,1}^{(n)}$ of the form (2.1) let

$$\lambda_n(\omega) := n + \frac{1-|b_n|}{1+|b_n|}.$$

The assertion of the theorem below can be found in Remark 3 of [11] (see also [3, p. 3624]). The case $n = 1$ was proved by Unkelbach [13, p. 741]. For more information on the classical versions of the Schwarz lemma at the boundary of $\mathbf{D}$ including commentary on the history and the applications see the survey paper by Boas [1].

Theorem 2.3. *Let $n \in \mathbf{N}$. If $\omega \in \mathcal{B}_{0,1}^{(n)}$ is of the form* (2.1)*, then*

$$\omega'_{\angle}(1) \geq \lambda_n(\omega), \tag{2.4}$$

i.e., $\omega \in \mathcal{B}_1(\lambda)$ for some $\lambda \geq \lambda_n(\omega)$.

In particular, when $n = 1$, then

$$\omega'_{\angle}(1) \geq \lambda_1(\omega) = \frac{2}{1+|b_1|}.$$

Equality in (2.4) *holds for*

$$\omega(z) := z^n \varphi_{-\xi}(z), \quad z \in \overline{\mathbf{D}}, \tag{2.5}$$

where $\xi \in (0,1)$.

Notation 2.4. Let $n \in \mathbf{N}$. For $\omega \in \mathcal{B}_1^{(n)}$ of the form (1.3) denote

$$\lambda_{0,n}(\omega) := \frac{|1-b_0|^2}{1-|b_0|^2}\left(n + \frac{1-|b_0|^2-|b_n|}{1-|b_0|^2+|b_n|}\right).$$

Note that when $b_0 \in (-1,1)$, then

$$\lambda_{0,n}(\omega) = \frac{1-b_0}{1+b_0}\left(n + \frac{1-b_0^2-|b_n|}{1-b_0^2+|b_n|}\right).$$

Applying Theorem 2.3 we generalize now the so-called General Boundary Lemma proved by Osserman [11, p. 3515] for the case $n = 1$.

Theorem 2.5. *Let $n \in \mathbf{N}$. If $\omega \in \mathcal{B}_1^{(n)}$ is of the form* (1.3)*, then*

$$\omega'_{\angle}(1) \geq \lambda_{0,n}(\omega), \tag{2.6}$$

i.e., $\omega \in \mathcal{B}_1(\lambda)$ for some $\lambda \geq \lambda_{0,n}(\omega)$.

In particular, when $n = 1$, then

$$\omega'_{\angle}(1) \geq \lambda_{0,1}(\omega) = \frac{2|1-b_0|^2}{1-|b_0|^2+|b_1|}.$$

Equality in (2.6) *holds for*

$$\omega(z) := \kappa\varphi_\xi(z^n), \quad z \in \overline{\mathbf{D}}, \tag{2.7}$$

where $\xi \in \mathbf{D}$ *and* $\kappa := (1-\overline{\xi})/(1-\xi)$.

Proof. Since $b_0 = \omega(0) \in \mathbf{D}$, we can write

$$\mathrm{e}^{\mathrm{i}\theta_0} = \frac{1-b_0}{1-\overline{b_0}}$$

for some $\theta_0 \in \mathbf{R}$, and define

$$\psi(z) := \mathrm{e}^{-\mathrm{i}\theta_0}\varphi_{b_0}(\omega(z)), \quad z \in \mathbf{D}. \tag{2.8}$$

We see that $\psi(\mathbf{D}) \subset \mathbf{D}$, $\psi(0) = 0$ and $\psi_{\angle}(1) = 1$, so $\psi \in \mathcal{B}_{0,1}$. Since ω is of the form (1.3), one has

$$\begin{aligned}\varphi_{b_0}(\omega(z)) &= \frac{\omega(z)-b_0}{1-\overline{b_0}\omega(z)}\\ &= \frac{b_n z^n + O(z^{n+1})}{1-|b_0|^2 + O(z^n)}\\ &= \frac{b_n}{1-|b_0|^2} z^n + O(z^{n+1}), \quad z \in \mathbf{D}.\end{aligned}$$

Thus by (2.8),

$$\psi(z) = \sum_{k=n}^{\infty} c_k z^k, \quad z \in \mathbf{D},$$

with

$$c_n = \frac{\mathrm{e}^{-\mathrm{i}\theta_0} b_n}{1-|b_0|^2}.$$

So $\psi \in \mathcal{B}_{0,1}^{(n)}$, and applying now Theorem 2.3 for ψ we obtain

$$\begin{aligned}\psi'_{\angle}(1) \geq n + \frac{1-|c_n|}{1+|c_n|} &= n + \frac{1-\dfrac{|b_n|}{1-|b_0|^2}}{1+\dfrac{|b_n|}{1-|b_0|^2}}\\ &= n + \frac{1-|b_0|^2-|b_n|}{1-|b_0|^2+|b_n|}.\end{aligned} \tag{2.9}$$

Since in view of (2.8),

$$\omega(z) = \varphi_{-b_0}(\mathrm{e}^{\mathrm{i}\theta_0}\psi(z)), \quad z \in \mathbf{D},$$

so

$$\omega'(z) = \mathrm{e}^{\mathrm{i}\theta_0}\frac{1-|b_0|^2}{(1+\overline{b_0}\mathrm{e}^{\mathrm{i}\theta_0}\psi(z))^2}\psi'(z), \quad z \in \mathbf{D}.$$

Thus

$$\begin{aligned}\omega'_{\angle}(1) &= \frac{1-b_0}{1-\overline{b_0}}\frac{1-|b_0|^2}{\left(1+\overline{b_0}\dfrac{1-b_0}{1-\overline{b_0}}\right)^2}\psi'_{\angle}(1)\\ &= \frac{1-b_0}{1-\overline{b_0}}\frac{(1-|b_0|^2)(1-\overline{b_0})^2}{(1-|b_0|^2)^2}\psi'_{\angle}(1)\\ &= \frac{|1-b_0|^2}{1-|b_0|^2}\psi'_{\angle}(1).\end{aligned}$$

This and (2.9) give (2.6).

Consider now the function (2.7). In view of (1.1) and by the fact that $\omega(1) = 1$, we see that $\omega \in \mathcal{B}_1^{(n)}$ with $b_0 = -\kappa\xi$ and $b_n = \kappa(1-|\xi|^2)$. Observe that

$$|b_n| = 1-|b_0|^2 \tag{2.10}$$

and

$$\begin{aligned}\frac{|1-b_0|^2}{1-|b_0|^2} &= \frac{|1+\kappa\xi|^2}{1-|\xi|^2}\\ &= \frac{\left|1+\dfrac{\xi-|\xi|^2}{1-\xi}\right|^2}{1-|\xi|^2} = \frac{1-|\xi|^2}{|1-\xi|^2}.\end{aligned} \tag{2.11}$$

On the other hand we have

$$\omega'(1) = n\kappa\varphi'_\xi(1) = n\frac{1-\overline{\xi}}{1-\xi}\frac{1-|\xi|^2}{(1-\overline{\xi})^2} = n\frac{1-|\xi|^2}{|1-\xi|^2}.$$

This with (2.10) and (2.11) yields the equality in (2.6). □

Note that the following result as a consequence of the Schwarz-Pick lemma (see e.g. [5, Lemma 1.2]) holds.

Lemma 2.6. *Let* $n \in \mathbf{N}$. *If* $\omega \in \mathcal{B}^{(n)}$ *is of the form* (1.3), *then*

$$|b_n| \leq 1-|b_0|^2. \tag{2.12}$$

Equality in (2.12) *can occur only for*

$$\omega(z) := \kappa\varphi_\xi(z^n), \quad z \in \mathbf{D},$$

where $\xi \in \mathbf{D}$ *and* $\kappa \in \mathbf{T}$.

The last lemma yields

Corollary 2.7. *Let* $n \in \mathbf{N}$. *If* $\omega \in \mathcal{B}_1^{(n)}$ *is of the form* (1.3), *then*

$$\lambda_{0,n}(\omega) \geq n\frac{|1-b_0|^2}{1-|b_0|^2}. \tag{2.13}$$

Equality in (2.13) *can occur only for functions of the form* (2.7).

Example 2.8. 1. Fix $n \in \mathbf{N}$. For each $a \in (0,1]$ let

$$\omega(z) := 1 - a + az^n, \quad z \in \overline{\mathbf{D}}.$$

Clearly, $\omega \in \mathcal{B}_1^{(n)}$ with $b_0 = 1-a$ and $b_n = a$. We have $\omega'(1) = na$ and $\omega \in \mathcal{B}_1(na)$. On the other hand we have

$$\lambda_{0,n}(\omega) = \frac{|1-(1-a)|^2}{1-(1-a)^2}\left(n + \frac{1-(1-a)^2-a}{1-(1-a)^2+a}\right) = \frac{a}{2-a}\left(n + \frac{1-a}{3-a}\right).$$

In particular, when $n = 1$ we have

$$\lambda_{0,1}(\omega) = \frac{2a}{3-a}.$$

An easy calculation shows that $\omega'(1) \ge \lambda_{0,n}(\omega)$ with equality only for $a = 1$.

2. For each $p > 0$ let

$$\omega_p(z) := \frac{1+z}{2} + \frac{(1-z)^p}{2^{p+1}}, \quad z \in \overline{\mathbf{D}}.$$

As proved in [10, Proposition 3.1(ii)], the function ω_p is a self-map of $\overline{\mathbf{D}}$ for every $1 \le p \le 3$. Let $p \in [1,3]$. Since $\omega_p(1) = 1$, so $\omega_p \in \mathcal{B}_1$. We have

$$\omega_p'(z) = \frac{1}{2} - \frac{p}{2^{p+1}}(1-z)^{p-1}, \quad z \in \overline{\mathbf{D}}.$$

By the fact that

$$b_0 = \omega_p(0) = \frac{1}{2} + \frac{1}{2^{p+1}}, \quad b_1 = \omega_p'(0) = \frac{1}{2} - \frac{p}{2^{p+1}},$$

we see that $\omega_p \in \mathcal{B}_1^{(1)}$. Moreover

$$\omega_1'(1) = \frac{1}{4}, \quad \omega_p'(1) = \frac{1}{2}, \quad p \in (1,3].$$

Thus $\omega_1 \in \mathcal{B}_1(1/4)$ and $\omega_p \in \mathcal{B}_1(1/2)$ for $p \in (1,3]$. On the other hand, since $b_1 > 0$ for $p \in [1,3]$, we get

$$\lambda_{0,1}(\omega_p) = \frac{2|1-b_0|^2}{1-|b_0|^2+|b_1|} = \frac{2\left(\frac{1}{2} - \frac{1}{2^{p+1}}\right)^2}{1 - \left(\frac{1}{2} + \frac{1}{2^{p+1}}\right)^2 + \left(\frac{1}{2} - \frac{p}{2^{p+1}}\right)} = \frac{2(2^p-1)^2}{5\cdot 4^p - 2(1+p)2^p - 1}.$$

Note that

$$\omega_1'(1) = \frac{1}{4} > \lambda_{0,1}(\omega_1) = \frac{2}{11}$$

and for every $p \in (1,3]$,

$$\omega_p'(1) = \frac{1}{2} > \lambda_{0,1}(\omega_p).$$

Observe that ω_1 coincides with the function ω considered in the previous example for $n = 1$ and $a = 1/4$.

3. For $b \in (-1,1)$ and $a \in [-1,b)$ let

$$\omega(z) := \frac{(ab+b-2a)z - ab + b}{(2b-a-1)z - a + 1}, \quad z \in \overline{\mathbf{D}}.$$

We see that $\omega(0) = b_0 = b$, $\omega(1) = 1$ and $\omega(-1) = a$, so $\omega(\mathbf{D}) \subset \mathbf{D}$ and $\omega \in \mathcal{B}_1$. Moreover

$$\omega'(z) = \frac{2(b-a)(1-a)(1-b)}{((2b-a-1)z-a+1)^2}, \quad z \in \overline{\mathbf{D}}.$$

Hence

$$b_1 = \omega'(0) = \frac{2(b-a)(1-b)}{1-a}$$

and $b_1 > 0$. Moreover

$$\omega'(1) = \frac{(1-a)(1-b)}{2(b-a)} = \frac{(1-b)^2}{\omega'(0)} = \frac{(1-b_0)^2}{b_1}.$$

But

$$\lambda_{0,1}(\omega) = \frac{2(1-b_0)^2}{1-b_0^2+b_1}.$$

We see that

$$\omega'(1) = \frac{(1-b_0)^2}{b_1} \ge \frac{2(1-b_0)^2}{1-b_0^2+b_1} = \lambda_{0,1}(\omega).$$

Observe that when $b_0 = b \to -1^+$, then $a \to -1^+$ and, consequently, $\omega'(1) \to \infty$.

Theorem 2.5 can be slightly generalized.

Corollary 2.9. *Let $n \in \mathbf{N}$. Assume that $\omega \in \mathcal{B}^{(n)}$ is of the form (1.3) and at $z_0 \in \mathbf{T}$ a limit $\omega_{\angle}(z_0)$ exists with*

$$|\omega_{\angle}(z_0)| = \sup\{|\omega(z)| : z \in \mathbf{D}\}.$$

Then $\omega_{\angle}'(z_0)$ exists and

$$\frac{z_0\omega_{\angle}'(z_0)}{\omega_{\angle}(z_0)} \ge \frac{|\omega_{\angle}(z_0) - b_0|^2}{|\omega_{\angle}(z_0)|^2 - |b_0|^2}$$

$$\times\left(n+\frac{\left|\omega_{\angle}(z_0)\right|^2-|b_0|^2-\left|\omega_{\angle}(z_0)b_n\right|}{\left|\omega_{\angle}(z_0)\right|^2-|b_0|^2+\left|\omega_{\angle}(z_0)b_n\right|}\right).$$

Proof. Since $\omega\not\equiv 0$, so $\omega_{\angle}(z_0)\neq 0$. Thus the function

$$\mathbf{D}\ni z\mapsto \psi(z):=\frac{\omega(z_0z)}{\omega_{\angle}(z_0)}$$

is well defined and analytic in $\mathbf{D}$. Obviously, $\psi\in\mathcal{B}_1^{(n)}$. Moreover from (1.3) it follows that

$$\psi(z)=c_0+\sum_{k=n}^{\infty}c_kz^k,\quad z\in\mathbf{D},$$

where

$$c_0=\frac{b_0}{\omega_{\angle}(z_0)},\quad c_n=\frac{z_0^nb_n}{\omega_{\angle}(z_0)}.$$

Since, by Theorem 1.2, $\psi'_{\angle}(1)$ exists, so $\omega'_{\angle}(z_0)$ exists and

$$\psi'_{\angle}(1)=\frac{z_0\omega'_{\angle}(z_0)}{\omega_{\angle}(z_0)}.$$

Applying now Theorem 2.5 we have finally

$$\begin{aligned}\frac{z_0\omega'_{\angle}(z_0)}{\omega_{\angle}(z_0)}&\geq\frac{\left|1-\dfrac{b_0}{\omega_{\angle}(z_0)}\right|^2}{1-\left|\dfrac{b_0}{\omega_{\angle}(z_0)}\right|^2}\\&\quad\times\left(n+\frac{1-\left|\dfrac{b_0}{\omega_{\angle}(z_0)}\right|^2-\left|\dfrac{z_0^nb_n}{\omega_{\angle}(z_0)}\right|}{1-\left|\dfrac{b_0}{\omega_{\angle}(z_0)}\right|^2+\left|\dfrac{z_0^nb_n}{\omega_{\angle}(z_0)}\right|}\right)\\&=\frac{\left|\omega_{\angle}(z_0)-b_0\right|^2}{\left|\omega_{\angle}(z_0)\right|^2-|b_0|^2}\\&\quad\times\left(n+\frac{\left|\omega_{\angle}(z_0)\right|^2-|b_0|^2-\left|\omega_{\angle}(z_0)b_n\right|}{\left|\omega_{\angle}(z_0)\right|^2-|b_0|^2+\left|\omega_{\angle}(z_0)b_n\right|}\right).\end{aligned}$$

□

When $\omega_{\angle}(z_0)\in\mathbf{T}$, we have the following corollary.

Corollary 2.10. *Let $n\in\mathbf{N}$. Assume that $\omega\in\mathcal{B}^{(n)}$ is of the form* (1.3) *and at $z_0\in\mathbf{T}$ a limit $\omega_{\angle}(z_0)$ exists and $\omega_{\angle}(z_0)\in\mathbf{T}$. Then $\omega'_{\angle}(z_0)$ exists and*

$$\frac{z_0\omega'_{\angle}(z_0)}{\omega_{\angle}(z_0)}\geq\frac{\left|\omega_{\angle}(z_0)-b_0\right|^2}{1-|b_0|^2}\left(n+\frac{1-|b_0|^2-|b_n|}{1-|b_0|^2+|b_n|}\right).$$

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