Explicit t-expansions for the elliptic curve $y^2 = 4(x^3 + Ax + B)$

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Abstract: For an elliptic curve $E : y^2 = 4(x^3 + Ax + B)$ over a field of characteristic $\neq 2$, we explicitly compute the pullback to the formal completion of E at the origin of some important objects on E including the functions x, y and the invariant differential $\omega = dx/y$ in terms of the formal parameter $t = -2x/y$.

Key words: Elliptic curves; invariant differential; sigma function.

1. Introduction and the main result. Let R be a commutative ring with unit on which 2 is invertible. Let E be an elliptic curve over $\operatorname{Spec} R$ whose affine form is given by the equation $y^2 = 4(x^3 + Ax + B)$ for some $A, B \in R$ satisfying $4A^3 + 27B^2 \in R^\times$. Let $\widehat{E}$ be the completion of E at the origin. We set $t = -2x/y$. Then $\widehat{E}$ is canonically isomorphic to the formal spectrum of $R[[t]]$.

In this paper, we give an explicit description of the pullbacks to $\widehat{E}$ of some important functions and 1-forms on E. Our main result is the following:

Theorem 1. *Let $\widehat{\omega} \in R[[t]]dt$ denote the pull back of the invariant differential $\omega = dx/y$ to $\widehat{E}$. Then for any integer k, the formal power series $\frac{x^k \widehat{\omega}}{dt} \in R((t))$ is equal to the sum*

$$(1) \quad \sum_{m,n=0}^{\infty} \frac{(m + 2n - k + 1)_{m+n}}{m!n!} A^m B^n t^{4m+6n-2k}.$$

Here $(m + 2n - k + 1)_{m+n}$ denotes the Pochhammer symbol

$$(m + 2n - k + 1)_{m+n} = \prod_{i=1}^{m+n} (m + 2n - k + i),$$

and we understand $(m + 2n - k + 1)_{m+n} = 1$ when $m = n = 0$.

The proof of Theorem 1 will be given in Section 2. We give some other results in Section 3.

Remark 2. According to [6, p. 924, Remark], the formula (1) for $k = 0$ was already obtained by Beukers [3]. According to [7, p. 273], a generalization of the formula for $k = 0$ to the case of an elliptic curve given by a more general Weierstraß equation was also obtained by Beukers [3], and recently perhaps independently by Sadek [5].

Remark 3. When A and B vary, the sum (1) is a formal power series of three variables A, B, and t with coefficients in $\mathbf{Z}$. If we set $A' = At^4$ and $B' = Bt^6$, then Theorem 1 for $k \leq 0$ is rewritten as

$$t^{2k} x^k \widehat{\omega}/dt = F((1 - k, k), -A', B'),$$

where the right hand side is the hypergeometric series of two variables in the sense of [2, Definition 3.1], associated to the set $\{2\nu_1 + 3\nu_2, -\nu_1 - 2\nu_2\}$ of linear forms. Similarly we have

$$t^{2k} x^k \widehat{\omega}/dt = \lim_{\epsilon \to 0} F((1 - k + \epsilon, k + \epsilon), -A', B')$$

for $k \geq 1$. Here the right hand side is the coefficient-wise limit in $\mathbf{R}$.

Now we give several consequences of Theorem 1. All of them follow immediately from Theorem 1 or from the argument of its proof, and the proofs are omitted.

Corollary 4. *Let the notation and the assumption be as in Theorem 1.*

(a) *We have*

$$\frac{\widehat{\omega}}{dt} = \sum_{m,n=0}^{\infty} \frac{(2m + 3n)!}{(m + 2n)!m!n!} A^m B^n t^{4m+6n}.$$

(b) *We have the equalities*

$$x = -\sum_{m,n=0}^{\infty} \frac{(2m + 3n - 2)!}{(m + 2n - 1)!m!n!} A^m B^n t^{4m+6n-2}$$

and

$$y = 2 \sum_{m,n=0}^{\infty} \frac{(2m + 3n - 2)!}{(m + 2n - 1)!m!n!} A^m B^n t^{4m+6n-3}$$

in $R((t))$. Here we understand $\frac{(2m+3n-2)!}{(m+2n-1)!m!n!} =$

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-1 when $m = n = 0$. (Observe that $\frac{(2m+3n-2)!}{(m+2n-1)!m!n!}$ is an integer for any integers $m, n \geq 0$.)

(c) *For any integer $p, q \in \mathbf{Z}$ satisfying $k := p + q \neq 0$, the monomial $x^p y^q$ is equal to $-(-2)^q/t^{2p+3q}$ times*

$$\sum_{m,n\geq 0} \frac{k(m+2n-k+1)_{m+n-1}}{m!n!} A^m B^m t^{4m+6n}$$

in $R((t))$. (Observe that $\frac{k(m+2n-k+1)_{m+n-1}}{m!n!}$ is an integer for any integers $m, n \geq 0$.)

□

Remark 5. To be precise, the formulae for x, y, and $x^p y^q$ in Corollary 4 are not consequences of Theorem 1 but immediate consequences of the proof of Theorem 1 given in Section 2.

Corollary 6. *Suppose that R is a $\mathbf{Q}$-algebra.*

(a) *Let $\log_{\widehat{E}} \in R[[t]]$ denote the formal logarithm associated to $\widehat{E}$ with respect to the formal parameter t. By definition $\log_{\widehat{E}}$ is the unique formal power series satisfying $d\log_{\widehat{E}} = \widehat{\omega}$ and $\log_{\widehat{E}}(0) = 0$. We then have*

$$\log_{\widehat{E}} = \sum_{m,n=0}^{\infty} \frac{(2m+3n)!}{(m+2n)!m!n!} A^m B^n \frac{t^{4m+6n+1}}{4m+6n+1}.$$

(b) *Let $\widehat{\zeta} \in R((t))$ be a formal Laurent power series satisfying $d\widehat{\zeta} = -x\widehat{\omega}$. Then $\widehat{\zeta}$ is equal to*

$$c - \sum_{m,n=0}^{\infty} \frac{(2m+3n-1)!}{(m+2n-1)!m!n!} A^m B^n \frac{t^{4m+6n-1}}{4m+6n-1}$$

for some constant $c \in R$. Here we understand $\frac{(2m+3n-1)!}{(m+2n-1)!m!n!} = 1$ when $m = n = 0$.

□

Remark 7. Corollary 6 (a) was announced (with the author's name) without proof in p. 289 of [4].

Corollary 8. *Let the notation and assumption be as in Theorem 1.*

(a) *Suppose that $B = 0$. We then have*

$$\frac{\widehat{\omega}}{dt} = \frac{1}{\sqrt{1-4At^4}}$$

and

$$\frac{x\widehat{\omega}}{dt} = \frac{1}{2t^2\sqrt{1-4At^4}} + \frac{1}{2t^2}.$$

(b) *Suppose that $A = 0$. (Observe that 6 is invertible in R in this case.) We then have*

$$\frac{\widehat{\omega}}{dt} = {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{1}{2}; \frac{27}{4} Bt^6\right)$$

and

$$\frac{x\widehat{\omega}}{dt} = \frac{2}{3t^2}\, {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{1}{2}; \frac{27}{4} Bt^6\right) + \frac{1}{3t^2},$$

where ${}_2F_1(\alpha, \beta; \gamma; z)$ is Gauß hypergeometric series

$${}_2F_1(\alpha, \beta; \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n n!} z^n.$$

□

Remark 9. The claim (a) in Corollary 8 can be proved directly without using Theorem 1. We include this for completeness.

Suppose that R is a field which is complete with respect to an absolute value $|\ |$. When the absolute value $|\ |$ is archimedean, let α be the unique real root of

$$4|A|^3(T^2-1)(T-4) - 27|B|^2T^3$$

satisfying $0 \leq \alpha \leq 1$ and set

$$r = \frac{1}{\sqrt{6}} \left((4-\alpha)^{(4-\alpha)} \left(\frac{3\alpha}{|A|}\right)^{3\alpha} \left(\frac{2-2\alpha}{|B|}\right)^{2-2\alpha} \right)^{\frac{1}{12}}.$$

When $|\ |$ is non-archimedean, we set $r = 1/\max(|A|^{1/4}, |B|^{1/6})$. We use the terminology "analytic" to stand for real analytic, complex analytic, and rigid analytic in the case when $|\ |$ is real archimedean, complex archimedean, and non-archimedian, respectively. When $|\ |$ is archimedean (resp. non-archimedean), we let E^{an} denote $E(R)$ regarded as an analytic manifold (resp. an analytic space over R associated to E). It then can be checked easily that there exists a unique open neighborhood (resp. a unique admissible open neighborhood with respect to the strong G-topology) of the origin O in E^{an} such that the rational function t on E gives an isomorphism from U to the open disk $\{t \mid |t| < r\}$.

Corollary 10. *Let the notation and assumption be as above.*

(a) *The formulae in Theorem 1 and Corollary 4, with $\widehat{\omega}$ replaced with ω, are valid on U.*

(b) *Suppose that R is of characteristic zero. Then there exists a unique analytic function $\log_E$ on U and an analytic function ζ on $U \setminus \{O\}$ such that $d\log_E = \omega$, $d\zeta = -x\omega$ and that the value of $\log_E$ at the origin is equal to zero. The formulae (a) and (b) in Corollary 6, with $\log_{\widehat{E}}$ and $\widehat{\zeta}$ replaced with $\log_E$ and ζ, are valid on U and*

$U \setminus \{O\}$, respectively. $\square$

Almost all the material of this manuscript is a translation into English of my handwritten notes and my emails to Shinichi Kobayashi, all of which were written in Japanese on January 2004.

2. Proof of Theorem 1. In this section we give a proof of Theorem 1. Let the notation and assumption be as in Theorem 1. If suffices to prove the claim for the universal case when R is the localization $R = \mathbf{Z}[1/2, A, B, 1/(4A^3 + 27B^2)]$ of the polynomial ring over $\mathbf{Z}[1/2]$ of the two variables A and B. By choosing an injective ring homomorphism $\iota : \mathbf{Z}[1/2, A, B, 1/(4A^3 + 27B^2)] \hookrightarrow \mathbf{C}$ such that $\iota(A)$ and $\iota(B)$ are real numbers, we can reduce the proof to that in the case when $R = \mathbf{C}$ and both A and B are real numbers.

Let us assume that $R = \mathbf{C}$ and both A and B are real numbers. For $k \in \mathbf{Z}$, we let $F_k(t)$ denote the formal power series (1) with coefficients in $\mathbf{R}$. Observe that the formal power series $t^{2k}F_k(t)$ is absolutely convergent on $|t| < c_k$ for a sufficiently small $c_k > 0$. Hence it suffices to prove that for each integer k, the value of $x^k\omega/dt$ at $t = a$ is equal to $F_k(a)$ for infinitely many complex numbers a with $0 < |a| < c_k$.

Observe that, if $(x, y) \in \mathbf{C}^\times \times \mathbf{C}^\times$ satisfies $y^2 = 4(x^3 + Ax + B)$, then $(t, u) = (-2x/y, 1/x)$ satisfies the equality

$$1 = \frac{t^2}{u} + Aut^2 + Bu^2t^2.$$

Let us fix $t \in \mathbf{C}^\times$ and set

$$f(u) = u\left(1 - \frac{t^2}{u} - Aut^2 - Bu^2t^2\right),$$

which we regard as a holomorphic function of u. For $|t|$ sufficiently small, the function $f(u)$ have a unique zero on $|u| < 1$, which we denote by u_0. Then $(x, y) = (1/u_0, -2/(tu_0))$ is a point of $E(\mathbf{C})$ with $-2x/y = t$.

We prove the claim for $k = 0$. By Jensen's formula (cf. [1, p. 208])

$$\log|f(0)| = -\log\left|\frac{1}{u_0}\right| + \frac{1}{2\pi}\int_0^{2\pi} \log|f(e^{i\theta})|d\theta$$

we have

$$\log|-t^2| = \log|u_0| + \frac{1}{2\pi}\mathrm{Re}\int_0^{2\pi} \log(1 - g(e^{i\theta}))d\theta$$

where

$$g(u) = \frac{t^2}{u} + At^2u + Bt^2u^2.$$

Since we have assumed that A and B are real numbers, u_0 is a positive real number if t is a sufficiently small real number. Since

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} \log(1 - g(e^{i\theta}))d\theta \\
&\quad = -\sum_{n\geq 1} \frac{1}{2\pi n}\int_0^{2\pi} g(e^{i\theta})^n d\theta \\
&\quad = -\sum_{\substack{m,n\geq 0\\(m,n)\neq(0,0)}} \frac{(2m+3n)!}{(m+2n)!m!n!}\frac{A^mB^n(t^2)^{2m+3n}}{2m+3n},
\end{aligned}$$

we have

$$\begin{aligned}
&\log u_0 \\
&\quad = \log t^2 + \sum_{\substack{m,n\geq 0\\(m,n)\neq(0,0)}} \frac{(2m+3n)!}{(m+2n)!m!n!}\frac{A^mB^n(t^2)^{2m+3n}}{2m+3n}
\end{aligned}$$

if t is a sufficiently small real number. By differentiating with respect to t and by using $\omega = tdu_0/(2u_0)$, we obtain the desired equality $\omega/dt = F_0(t)$ for any sufficiently small real number t, which proves the claim for $k = 0$.

Next we consider the case when $k \neq 0$. Let r_k denote the residue of $u^{-k}(uf(u))'/(uf(u))$ at $u = 0$. By the residue theorem we have

$$x^k + r_k = \frac{1}{2\pi i}\int_{|u|=1} \frac{u^{-k}(uf(u))'}{uf(u)}du \tag{2}$$

for $|t|$ sufficiently small. We set

$$h(u) = \frac{u}{t^2} - Au^2 - Bu^3.$$

Since

$$\begin{aligned}
\frac{(uf(u))'}{uf(u)} &= \frac{(-\frac{1}{t^2} - 2Au - 3Bu^2)}{1 - h(u)} \\
&= \left(-\frac{1}{t^2} - 2Au - 3Bu^2\right)\sum_{n\geq 0} h(u)^n
\end{aligned}$$

where the last infinite sum is absolutely convergent if $|u|$ is much smaller than $|t|^2$, we have $r_k = 0$ for $k < 0$ and

$$r_k = -\frac{1}{t^2}C_{k,1} + 2AC_{k,2} + 3BC_{k,3} \tag{3}$$

for $k > 0$. Here $C_{k,j}$ is the finite sum

$$\sum_{\substack{m,n\geq 0\\ 2m+3n\leq k-j}} \frac{(m+n+\ell_{k,j}(m,n))!}{\ell_{k,j}(m,n)!m!n!} \frac{(-1)^{m+n}A^m B^n}{t^{\ell_{k,j}(m,n)}}$$

for $j = 1, 2, 3$, where $\ell_{k,j}(m,n) = (k-j) - (2m + 3n)$. On the other hands, since

$$\begin{aligned}\frac{(uf(u))'}{uf(u)} &= \frac{u^{-1} - 2At^2 - 3Bt^2u}{1-g(u)}\\ &= (u^{-1} - 2At^2 - 3Bt^2u)\sum_{n\geq 0} g(u)^n\end{aligned}$$

where the last infinite sum is absolutely convergent if $|u| = 1$ and $|t|$ is sufficiently small, the right hand side of (2) is equal to

$$D_{k,0} - 2At^2 D_{k,1} - 3Bt^2 D_{k,2}. \tag{4}$$

Here $D_{k,j}$ is the infinite sum

$$\sum_{\substack{m,n\geq 0\\ m+2n\geq k-j}} \frac{(m+n+\ell'_{k,j}(m,n))!}{\ell'_{k,j}(m,n)!m!n!} \frac{A^m B^n}{t^{\ell'_{k,j}(m,n)}}$$

for $j = 0, 1, 2$, where $\ell'_{k,j}(m,n) = m + 2n - (k-j)$. By (2), (3), and (4), the value of $-x^k/k$ is equal to the sum

$$\sum_{m,n\geq 0} \frac{(m+2n-k+1)_{m+n-1}}{m!n!} A^m B^n t^{2(2m+3n-k)}$$

for $|t|$ sufficiently small. Since $x^k\omega = t/2 \cdot d(-x^k/k)$, we have the equality $x^k\omega/dt = F_k(t)$ for $|t|$ sufficiently small, which proves the claim for $k \neq 0$. $\square$

3. Some other formulae. The method of the proof, given in Section 2, of Theorem 1 can be applied to a more general situation. Especially we can obtain in many cases explicit expansions of the pullbacks of functions or 1-forms on a plane curve over a field with respect to a local parameter at some closed point.

In this section we give several examples of such formulae. We omit the proofs of these formulae, since the main idea of the proofs is essentially the same as that of Theorem 1.

Theorem 11. *Let $m = 2g+1$ be a positive odd integer. Let C be a hyperelliptic curve over $\mathbf{Q}$ whose affine form is given by $y^2 = x^m - 1$. We set $t = -x^g/y$, which is a local parameter of C at the infinity. Let $\widehat{C}$ denote the completion of C at the infinity, which is canonically isomorphic to the formal spectrum of $\mathbf{Q}[[t]]$. Then we have*

$$x = \frac{1}{t^2} + \sum_{n\geq 1}(-1)^{n-1}\binom{mn-1}{n}\frac{t^{2(mn-1)}}{mn-1}$$

and

$$y = -\frac{1}{t^m} + g\sum_{n\geq 1}(-1)^n\binom{mn-g}{n}\frac{t^{m(2n-1)}}{mn-g}$$

in $\mathbf{Q}((t))$, and the pullback $\widehat{\omega}$ of $\omega = x^{g-1}dx/(2y)$ to $\widehat{C}$ has the following explicit description:

$$\frac{\widehat{\omega}}{dt} = \sum_{n\geq 0}(-1)^n\binom{mn}{n}t^{2mn}.$$

$\square$

Let us go back to the situation in Theorem 1 and suppose that R is a subring of the field $\mathbf{C}$ of complex numbers. Let r and U be as in the paragraph just before Corollary 10. Let $\log_E$ be the complex analytic function on U introduced in (b) of Corollary 10. We regard $\log_E$ as a complex analytic function of t on the open disk $\{t \mid |t| < r\}$. Let $\Lambda \subset \mathbf{C}$ denote the lattice generated by the periods of $E(\mathbf{C})$ with respect to ω. Let σ be the Weierstraß σ-function on $\mathbf{C}$ with respect to the lattice Λ. We end this paper with two formulae on the t-expansions of some functions related to σ. The author expect that they are useful for explicit computation related to the formal group law or the canonical height.

Theorem 12. *Let the notation and assumption be as above. Let S be the set of quadruples (a,b,c,d) of integers $a,b,c,d \geq 0$ satisfying $(a,b,c,d) \neq (0,0,0,0)$. For $(a,b,c,d) \in S$, we set*

$$V_{a,b,c,d} = \frac{(2a+3b-1)!\ (2c+3d)!}{(a+2b-1)!\ (c+2d)!\ a!b!c!d!}.$$

Here we understand $\frac{(2a+3b-1)!}{(a+2b-1)!} = 1$ when $(a,b) = (0,0)$. (Observe that $V_{a,b,c,d}$ is an integer for any $a,b,c,d \geq 0$.) Then $-\log(\sigma(\log_E(t))/t)$ is equal to the sum

$$\sum_{(a,b,c,d)\in S} V_{a,b,c,d}\frac{A^{a+c}B^{b+d}t^{4a+6b+4c+6d}}{(4a+6b-1)(4a+6b+4c+6d)}$$

for any complex number t with $|t| < r$.

In order to state the last formula in this paper, we need to introduce some more notation. For non-negative integers $m,n,a,b \geq 0$ satisfying the condition

$$2m + 3n = a + b, \tag{$*$}$$

let us introduce two integers $E_{m,n,a,b}, F_{m,n,a,b} \in \mathbf{Z}$.

Let $m, n, a, b \geq 0$ be integers satisfying the condition $(*)$. Let $\Xi(m, n, a, b)$ denote the set of pairs $(m_1, n_1) \in \mathbf{Z} \times \mathbf{Z}$ satisfying the following conditions:

$$0 \leq m_1 \leq m,\ 0 \leq n_1 \leq n,$$
$$m_1 + n_1 \leq a \leq 2m_1 + 3n_1 - 1.$$

For $(m_1, n_1) \in \Xi(m, n, a, b)$, we let $e_{m,n,a,b}(m_1, n_1)$ denote the integer

$$\frac{(2m_1 + 3n_1 - a)a!b!}{(a - (m_1 + n_1))!(b - (m_2 + n_2))!m_1!n_1!m_2!n_2!},$$

where $m_2 = m - m_1$ and $n_2 = n - n_1$. We set

$$E_{m,n,a,b} = \sum_{(m_1,n_1) \in \Xi(m,n,a,b)} e_{m,n,a,b}(m_1, n_1).$$

If either $a = 0$ or $b = 0$, then we have $E_{m,n,a,b} = 0$ since the set $\Xi(m, n, a, b)$ is an empty set. Let $\Theta(m, n, a, b)$ denote the set of integers m_1 satisfying the conditions

$$\max\{0, a - 3n\} \leq 2m_1 \leq \min\{2m, a, 2m + b - 1\},$$
$$2m_1 \equiv a \mod 3.$$

For $m_1 \in \Theta(m, n, a, b)$, we let $f_{m,n,a,b}(m_1)$ denote the integer

$$\frac{a!(b-1)!}{(a - (m_1 + n_1))!((b-1) - (m_2 + n_2))!m_1!n_1!m_2!n_2!},$$

where $m_2 = m - m_1$, $n_1 = \frac{a - 2m_1}{3}$, and $n_2 = n - n_1$. We set

$$F_{m,n,a,b} = \sum_{m_1 \in \Theta(m,n,a,b)} f_{m,n,a,b}(m_1).$$

If $b = 0$, then we have $F_{m,n,a,b} = 0$ since there exists no integer m_1 satisfying the condition above. When $b \geq 1$, we also set $F'_{m,n,a,b} = (2 - 1/b)E_{m,n,a,b} + F_{m,n,a,b}$.

Theorem 13. *Let the notation be as above. We then have*

$$\begin{aligned}
&\log \frac{\sigma(\log_E(s) + \log_E(t))}{s + t} \\
&\quad - \log \frac{\sigma(\log_E(s))}{s} - \log \frac{\sigma(\log_E(t))}{t} \\
&\quad = 2 \sum_{\substack{m,n \geq 0 \\ a,b \geq 1 \\ satisfying(*)}} E_{m,n,a,b} A^m B^n \frac{s^{2a}}{2a} \frac{t^{2b}}{2b} \\
&\qquad - \sum_{\substack{m,n \geq 0 \\ a \geq 0, b \geq 1 \\ satisfying(*)}} F'_{m,n,a,b} A^m B^n \frac{s^{2a+1}}{2a+1} \frac{t^{2b-1}}{2b-1}
\end{aligned}$$

for $(s, t) \in \mathbf{C} \times \mathbf{C}$ satisfying $|s|, |t| < r$. □

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