A generalization of Gu's normality criterion

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Abstract: Let $\mathcal{F}$ be a family of meromorphic functions on a domain D, $k \in \mathbf{N}$ and $\mathcal{H}$ be a normal family of meromorphic functions on D such that 0 is not in $\mathcal{H}$ and $\mathcal{H}$ has no sequence that converges to 0 or ∞ spherically locally uniformly on D. If for every $f \in \mathcal{F}$, $f(z) \neq 0$, and there exists an $h_f \in \mathcal{H}$ such that $f^{(k)}(z) \neq h_f(z)$ at every $z \in D$, then the family $\mathcal{F}$ is normal on D. This generalizes Gu's well-known normality criterion. It is interesting that the condition $f(z) \neq 0$ cannot be replaced by that all zeros of f have large multiplicities, at least $k+3$ for instance.

Key words: Meromorphic functions; normality; exceptional functions.

1. Introduction. Let $\mathcal{F}$ be a family of meromorphic functions on a domain $D \subset \mathbf{C}$. Then $\mathcal{F}$ is said to be normal on D in the sense of Montel, if each sequence of $\mathcal{F}$ contains a subsequence which converges spherically uniformly on each compact subset of D to a meromorphic function which may be ∞ identically. See [2,5,8]. We denote by $\mathcal{F}'$ the family of these limit functions, and let $\overline{\mathcal{F}} = \mathcal{F} \cup \mathcal{F}'$. For two functions f and g defined in D, we write $f \neq g$ on D if $f(z) \neq g(z)$ for every $z \in D$; write $f \not\equiv g$ if $f(z_0) \neq g(z_0)$ for some $z_0 \in D$; and write $f \equiv g$ if $f(z) = g(z)$ for every $z \in D$.

The well-known Gu's normality criterion [1] says that a family $\mathcal{F} = \{f\}$ of functions meromorphic on D is normal if $f \neq 0$ and $f^{(k)} \neq 1$ on D for each $f \in \mathcal{F}$. Our starting point is the following generalization of Gu's theorem proved by L. Yang [7].

Theorem A ([7, Theorem 1]). *Let $\mathcal{F}$ be a family of meromorphic functions on D, $k \in \mathbf{N}$ and h $(\not\equiv 0)$ be a holomorphic function on D. If for every $f \in \mathcal{F}$, $f \neq 0$ and $f^{(k)} \neq h$ on D, then $\mathcal{F}$ is normal on D.*

In Theorem A, the derivatives of all functions in $\mathcal{F}$ omit the same function h. Thus, it is interesting to consider the case that for different functions in $\mathcal{F}$, their k-th derivatives omit different functions. In this direction, S. Nevo, X. C. Pang and L. Zalcman [3] have proved the following result. We state their result by the following form.

Theorem B ([3, Lemma 3]). *Let $\mathcal{F}$ be a family of meromorphic functions on a domain D, $k \in \mathbf{N}$ and $\mathcal{H}$ be a normal family of holomorphic functions on D such that $h \neq 0, \infty$ on D for each function $h \in \overline{\mathcal{H}}$. If for every $f \in \mathcal{F}$, $f \neq 0$ on D, and there exists an $h_f \in \mathcal{H}$ such that $f^{(k)} \neq h_f$ on D, then the family $\mathcal{F}$ is normal on D.*

Here, we generalize this result by allowing that $\mathcal{H}$ consists of meromorphic functions and that $h \not\equiv 0, \infty$ for each $h \in \overline{\mathcal{H}}$.

Theorem 1. *Let $\mathcal{F}$ be a family of meromorphic functions on a domain D, $k \in \mathbf{N}$ and $\mathcal{H}$ be a normal family of meromorphic functions on D such that $0, \infty \notin \overline{\mathcal{H}}$. If for every $f \in \mathcal{F}$, $f \neq 0$ on D, and there exists an $h_f \in \mathcal{H}$ such that $f^{(k)} \neq h_f$ on D, then the family $\mathcal{F}$ is normal on D.*

There are many further studies [4,6] about Gu's criterion and Yang's theorem. For example, we have the following normality criteria.

Theorem C ([4, Theorem 1]). *Let $k \in \mathbf{N}$ and $\mathcal{F}$ be a family of meromorphic functions on D, all of whose zeros have multiplicity at least $k+3$, and $h(\not\equiv 0)$ be a holomorphic function on D. If for every $f \in \mathcal{F}$, $f^{(k)} \neq h$ on D, then the family $\mathcal{F}$ is normal on D.*

Theorem D ([4, Theorem 3]). *Let $k \in \mathbf{N}$ and $\mathcal{F}$ be a family of meromorphic functions on D, all of whose zeros have multiplicity at least $k+2$ and all of whose poles are multiple, and $h(\not\equiv 0)$ be a holomorphic function on D. If for every $f \in \mathcal{F}$, $f^{(k)} \neq h$ on D, then the family $\mathcal{F}$ is normal on D.*

Hence, it is natural to ask whether there are similar results for Theorems C and D. The answer is

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no. Indeed, for given positive integers M, N and k, we can construct a non-normal family $\mathcal{F}$ of meromorphic functions such that for each $f \in \mathcal{F}$, all poles of f have multiplicity at least M, all zeros of f have multiplicity at least N, and $f^{(k)}$ omits a function contained in a normal family $\mathcal{H}$ satisfying $0, \infty \notin \overline{\mathcal{H}}$.

Example 1. Let M, N, k be positive integers with $N > k$, and let for every $n \in \mathbf{N}$

$$f_n(z) = \frac{\left(z^M - \frac{1}{n^M}\right)^N}{z^M},$$

$$h_n(z) = \left(\sum_{j=0}^{N-1} \frac{(-1)^j}{n^{Mj}} \binom{N}{j} z^{(N-j-1)M}\right)^{(k)}.$$

Then we have

$$f_n(z) = \frac{\sum_{j=0}^{N} \binom{N}{j} \frac{(-1)^j}{n^{Mj}} z^{(N-j)M}}{z^M} = \frac{(-1)^N}{n^{MN} z^M} + \sum_{j=0}^{N-1} \frac{(-1)^j}{n^{Mj}} \binom{N}{j} z^{(N-j-1)M},$$

and hence

$$f_n^{(k)}(z) = \frac{(-1)^{N+k}(M+k-1)!}{(M-1)! n^{MN} z^{M+k}} + h_n(z) \neq h_n(z).$$

We also see that $\mathcal{H} = \{h_n\}$ is normal on $\mathbf{C}$ and $0, \infty \notin \overline{\mathcal{H}}$, since $h_n \not\equiv 0, \infty$ on D and

$$h_n(z) \to h(z) := (z^{(N-1)M})^{(k)} \not\equiv 0, \infty.$$

However, the sequence $\{f_n\}$ is not normal at 0, since $f_n(0) = \infty$ and $f_n(1/n) = 0$.

This example shows that the condition $f \neq 0$ on D for every $f \in \mathcal{F}$ in Theorem 1 cannot be relaxed in general. The other conditions are also essential.

Example 2. The condition that $\mathcal{H}$ is normal is necessary. Let $f_n(z) = e^{nz}$ for every $n \in \mathbf{N}$, and $h_n(z) = n^k e^{nz} + 1$. Then on $\mathbf{C}$, f_n is zero-free and $f_n^{(k)} \neq h_n$. We see that both $\{f_n\}$ and $\{h_n\}$ are not normal at 0.

Example 3. The condition that $0, \infty \notin \overline{\mathcal{H}}$ is necessary. Let $f_n(z) = e^{nz}$ for every $n \in \mathbf{N}$. Then on the unit disk $\Delta(0, 1)$, $f_n \neq 0$, $f_n^{(k)}(z) = n^k e^{nz} \neq n^k e^n$ and $f_n^{(k)}(z) \neq n^k e^{-n}$. Obviously, we have $n^k e^n \to \infty$ and $n^k e^{-n} \to 0$. However, $\{f_n\}$ is not normal at 0.

Example 4. The condition $f^{(k)} \neq h_f$ cannot be replaced by $f^{(k)} - h_f \neq 0$, even for all h_f are the same. Let $f_n(z) = 1/(nz)$ for every $n \in \mathbf{N}$, and $h(z) = (-1)^k k!/z^{k+1}$. Then we have $f_n \neq 0$ and $f_n^{(k)} - h \neq 0$ on $\mathbf{C}$ for $n > 1$. However, $\{f_n\}$ is not normal at 0.

2. Proof of Theorem 1. Let $\{f_n\} \subset \mathcal{F}$ be a sequence. We are required to prove that $\{f_n\}$ contains a subsequence which converges spherically locally uniformly on D.

By the condition, there exists a corresponding sequence $\{h_n\} \subset \mathcal{H}$ such that $f_n^{(k)} \neq h_n$ on D. If $\{h_n\}$ contains a subsequence in which all functions are the same, then the conclusion follows from Theorem A. So we can assume that the functions h_n are distinct.

Since $\mathcal{H}$ is normal, $\{h_n\}$ contains a subsequence, which we continue to call $\{h_n\}$, such that $\{h_n\}$ converges spherically locally uniformly on D to a meromorphic function h_0, which may be ∞ identically. Since $0, \infty \notin \overline{\mathcal{H}}$, we have $h_0 \not\equiv 0, \infty$. Set $E = h_0^{-1}(0) \cup h_0^{-1}(\infty)$, where $h_0^{-1}(0)$ and $h_0^{-1}(\infty)$ stand respectively for the set of zeros and the set of poles of h_0 in D. Since $h_0 \not\equiv 0, \infty$, the set E has no accumulation point in D.

We claim that $\{f_n\}$ is normal on $D \setminus E$. It suffices to show that $\{f_n\}$ is normal at every point $z_0 \in D \setminus E$. Let $U = U(z_0)$ be a neighborhood of z_0 such that $\overline{U} \subset D \setminus E$. Then $h_0 \neq 0, \infty$ on $\overline{U}$. Since $h_n \to h_0$ on D, by Hurwitz's theorem, $h_n \neq 0, \infty$ on U (for sufficiently large n). So the conditions of Theorem B are satisfied on U, and hence the normality of $\{f_n\}$ on U (and hence at z_0) follows.

It follows from the claim that we can say $f_n \to f_0$ on $D \setminus E$, where f_0 is meromorphic on $D \setminus E$ or $f_0 \equiv \infty$.

Suppose first that $f_0 \not\equiv 0$. Then we have $1/f_n \to 1/f_0 \not\equiv \infty$ on $D \setminus E$. Since $f_n \neq 0$ on D, $1/f_n$ is holomorphic on D. Hence, by the maximum modulus principle, $1/f_n \to 1/f_0$ on whole D. It follows that $f_n \to f_0$ on D.

Suppose now that $f_0 \equiv 0$. Then f_n is locally uniformly holomorphic on $D \setminus E$, i.e., for each bounded and closed sub-domain of $D \setminus E$, there exists an $N \in \mathbf{N}$ such that for $n > N$, f_n is holomorphic on this sub-domain.

Now let F be a bounded and closed subset of D. Since D is a domain, there exists a bounded and closed sub-domain of D with smooth boundary that contains F. So we can assume that F is a bounded and closed sub-domain of D with smooth boundary ∂F. Also, as E has no accumulation point in D, we can assume that no point in E lies on the boundary ∂F. We denote by F° the interior of F.

Thus by $f_n \to f_0 \equiv 0$ on $D \setminus E$, we have $f_n^{(k)} \to 0$ and $f_n^{(k+1)} \to 0$ on $D \setminus E$, and hence $f_n^{(k)} - h_n \to -h_0$ and $f_n^{(k+1)} - h_n' \to -h_0'$ on ∂F. Now we apply the argument principle to the functions $f_n^{(k)} - h_n$. We have

$$(1)\qquad n\left(F^\circ, \frac{1}{f_n^{(k)} - h_n}\right) - n(F^\circ, f_n^{(k)} - h_n) = \frac{1}{2\pi i}\int_{\partial F} \frac{f_n^{(k+1)} - h_n'}{f_n^{(k)} - h_n}\,dz \to \frac{1}{2\pi i}\int_{\partial F} \frac{h_0'}{h_0}\,dz = n\left(F^\circ, \frac{1}{h_0}\right) - n(F^\circ, h_0),$$

where $n(F^\circ, g)$ and $n(F^\circ, 1/g)$ are respectively the number of poles of g and the number of zeros of g in F°, counting multiplicity. Since both sides of (1) are integers, it follows that for sufficiently large n,

$$(2)\qquad n\left(F^\circ, \frac{1}{f_n^{(k)} - h_n}\right) - n(F^\circ, f_n^{(k)} - h_n) = n\left(F^\circ, \frac{1}{h_0}\right) - n(F^\circ, h_0).$$

From $f_n^{(k)} \neq h_n$, we get $n(F^\circ, \frac{1}{f_n^{(k)} - h_n}) = 0$ and know that $f_n^{(k)}$ and h_n have no common poles. It follows that

$$(3)\qquad n(F^\circ, f_n^{(k)} - h_n) = n(F^\circ, f_n^{(k)}) + n(F^\circ, h_n).$$

Further as $h_n \to h_0$ spherically uniformly on F and $h_0 \neq 0, \infty$ on ∂F, by Hurwitz's theorem, we have $n(F^\circ, h_n) = n(F^\circ, h_0)$ for sufficiently large n. Thus, by (2) and (3), we have

$$(4)\qquad n(F^\circ, f_n^{(k)}) + n\left(F^\circ, \frac{1}{h_0}\right) = 0.$$

It follows that each f_n has no pole on F and hence is holomorphic for sufficiently large n. Thus by $f_n \to 0$ on $D \setminus E$ and the maximum modulus principle, $f_n \to 0$ uniformly on F. This shows that $f_n \to 0$ locally uniformly on D. □

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