The Shi arrangement of the type D_ℓ

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Abstract: In this paper, we give a basis for the derivation module of the cone over the Shi arrangement of the type D_ℓ explicitly.

Key words: Hyperplane arrangement; Shi arrangement; free arrangement.

1. Introduction. Let V be an ℓ-dimensional vector space. An *affine arrangement of hyperplanes* $\mathcal{A}$ is a finite collection of affine hyperplanes in V. If every hyperplane $H \in \mathcal{A}$ goes through the origin, then $\mathcal{A}$ is called to be *central*. When $\mathcal{A}$ is central, for each $H \in \mathcal{A}$, choose $\alpha_H \in V^*$ with $\ker(\alpha_H) = H$. Let S be the algebra of polynomial functions on V and let Der_S be the module of derivations

$$\mathrm{Der}_S := \{\theta : S \to S \mid \theta(fg) = f\theta(g) + g\theta(f), f, g \in S, \\ \theta \text{ is } \mathbf{R}\text{-linear}\}.$$

For a central arrangement $\mathcal{A}$, recall

$$D(\mathcal{A}) := \{\theta \in \mathrm{Der}_S \mid \theta(\alpha_H) \in \alpha_H S \text{ for all } H \in \mathcal{A}\}.$$

We say that $\mathcal{A}$ is a *free arrangement* if $D(\mathcal{A})$ is a free S-module. The freeness was defined in [8]. The Factorization Theorem [9] states that, for any free arrangement $\mathcal{A}$, the characteristic polynomial of $\mathcal{A}$ factors completely over the integers.

Let $E = \mathbf{R}^\ell$ be an ℓ-dimensional Euclidean space with a coodinate system $x_1, \dots, x_\ell$, and Φ be a crystallographic irreducible root system. Fix a positive root system $\Phi^+ \subset \Phi$. For each positive root $\alpha \in \Phi^+$ and $k \in \mathbf{Z}$, we define an affine hyperplane

$$H_{\alpha,k} := \{v \in V \mid (\alpha, v) = k\}.$$

In [5], J.-Y. Shi introduced the *Shi arrangement*

$$\mathcal{S}(A_\ell) := \{H_{\alpha,k} \mid \alpha \in \Phi^+,\ 0 \le k \le 1\}$$

when the root system is of the type A_ℓ. This definition was later extended to the *generalized Shi arrangement* (e.g., [1])

$$\mathcal{S}(\Phi) := \{H_{\alpha,k} \mid \alpha \in \Phi^+,\ 0 \le k \le 1\}.$$

Embed E into $V = \mathbf{R}^{\ell+1}$ by adding a new coordinate z such that E is defined by the equation $z = 1$ in V. Then, as in [3], we have the cone $\mathbf{c}\mathcal{S}(\Phi)$ of $\mathcal{S}(\Phi)$

$$\mathbf{c}\mathcal{S}(\Phi) := \{\mathbf{c}H_{\alpha,k} \mid \alpha \in \Phi^+,\ 0 \le k \le 1\} \cup \{\{z = 0\}\}.$$

In [10], M. Yoshinaga proved that the cone $\mathbf{c}\mathcal{S}(\Phi)$ is a free arrangement with exponenets $(1, h, \dots, h)$ (h appears ℓ times), where h is the Coxeter number of Φ. (He actually verified the conjecture by P. H. Edelman and V. Reiner in [1], which is far more general.) He proved the freeness without finding a basis.

In [6], for the first time, the authors gave an explicit construction of a basis for $D(\mathbf{c}\mathcal{S}(A_\ell))$. Then D. Suyama constructed bases for $D(\mathbf{c}\mathcal{S}(B_\ell))$ and $D(\mathbf{c}\mathcal{S}(C_\ell))$ in [7]. In this paper, we will give an explicit construction of a basis for $D(\mathbf{c}\mathcal{S}(D_\ell))$. A defining polynomial of the cone over the Shi arrangement of the type D_ℓ is given by

$$Q := z \prod_{1 \le s < t \le \ell} \prod_{\epsilon \in \{-1,1\}} (x_s + \epsilon x_t - z)(x_s + \epsilon x_t).$$

Note that the number of hyperplanes in $\mathbf{c}\mathcal{S}(D_\ell)$ is equal to $2\ell(\ell-1)+1$. Our construction is similar to the construction in the case of the type B_ℓ. The essential ingredients of the recipe are the Bernoulli polynomials and their relatives.

2. The basis construction.

Proposition 2.1. *For* $(p, q) \in \mathbf{Z}_{\ge -1} \times \mathbf{Z}_{\ge 0}$, *consider the following two conditions for a rational function* $B_{p,q}(x)$:

1. $$B_{p,q}(x+1) - B_{p,q}(x) = \frac{(x+1)^p - (-x)^p}{(x+1) - (-x)}(x+1)^q(-x)^q,$$
2. $B_{p,q}(-x) = -B_{p,q}(x)$.

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Then such a rational function $B_{p,q}(x)$ uniquely exists. Morever, the $B_{p,q}(x)$ is a polynomial unless $(p,q) = (-1,0)$ and $B_{-1,0}(x) = -(1/x)$.

Proof. Suppose $(p,q) \neq (-1,0)$. Since the right hand side of the first condition is a polynomial in x, there exists a polynomial $B_{p,q}(x)$ satisfying the first condition. Note that $B_{p,q}(x)$ is unique up to a constant term. Define a polynomial $F(x) = B_{p,q}(x) + B_{p,q}(-x)$. Since

$$\begin{aligned} &B_{p,q}(-x) - B_{p,q}(-x-1) \\ =&\frac{(-x)^p - (x+1)^p}{(-x)-(x+1)}(-x)^q(x+1)^q \\ =&\frac{(x+1)^p - (-x)^p}{(x+1)-(-x)}(x+1)^q(-x)^q \\ =&B_{p,q}(x+1) - B_{p,q}(x), \end{aligned}$$

we have $F(x+1) = F(x)$ for any x. Therefore $F(x)$ is a constant function. Then the polynomial $B_{p,q}(x) - (F(0)/2)$ is the unique solution satisfying the both conditions. Next we suppose $(p,q) = (-1,0)$. Then we compute

$$\begin{aligned} &B_{-1,0}(x+1) - B_{-1,0}(x) \\ =&\frac{(x+1)^{-1} - (-x)^{-1}}{(x+1)-(-x)} = -\frac{1}{x+1} + \frac{1}{x}. \end{aligned}$$

Thus $B_{-1,0}(x) = -(1/x)$ is the unique solution satisfying the both conditions. □

Definition 2.2. Define a rational function $\overline{B}_{p,q}(x,z)$ in x and z by

$$\overline{B}_{p,q}(x,z) := z^{p+2q}B_{p,q}(x/z).$$

Then $\overline{B}_{p,q}(x,z)$ is a homogeneous polynomial of degree $p+2q$ except the two cases: $\overline{B}_{-1,0}(x,z) = -(1/x)$ and $\overline{B}_{0,q}(x,z) = 0$.

For a set $I := \{y_1,\dots,y_m\}$ of variables, let

$$\sigma_n^I := \sigma_n(y_1,\dots,y_m),\ \tau_{2n}^I := \sigma_n(y_1^2,\dots,y_m^2),$$

where σ_n stands for the elementary symmetric function of degree n.

Definition 2.3. Define derivations

$$\begin{aligned} \varphi_j := (x_j - x_{j+1} - z)\sum_{i=1}^{\ell}\sum_{\substack{K_1\cup K_2\subseteq J\\ K_1\cap K_2=\emptyset}}\Bigl(\prod K_1\Bigr)\Bigl(\prod K_2\Bigr)^2 \\ (-z)^{|K_1|}\sum_{\substack{0\le n_1\le|J_1|\\ 0\le n_2\le|J_2|}}(-1)^{n_1+n_2}\sigma_{n_1}^{J_1}\tau_{2n_2}^{J_2}\overline{B}_{k,k_0}(x_i,z)\frac{\partial}{\partial x_i} \end{aligned}$$

for $j = 1,\dots,\ell-1$ and

$$\begin{aligned} \varphi_\ell := \sum_{i=1}^{\ell}\sum_{\substack{K_1\cup K_2\subseteq J\\ K_1\cap K_2=\emptyset}}\Bigl(\prod K_1\Bigr)\Bigl(\prod K_2\Bigr)^2(-z)^{|K_1|} \\ (-x_\ell)\overline{B}_{-1,k_0}(x_i,z)\frac{\partial}{\partial x_i} \end{aligned}$$

for $j = \ell$, where

$$\begin{aligned} J &:= \{x_1,\dots,x_{j-1}\},\ J_1 := \{x_j,x_{j+1}\}, \\ J_2 &:= \{x_{j+2},\dots,x_\ell\}, \\ \prod K_p &:= \prod_{x_i\in K_p} x_i\ (p=1,2), \\ k_0 &:= |J\setminus(K_1\cup K_2)| \ge 0, \\ k &:= (|J_1|-n_1) + 2(|J_2|-n_2) - 1 \ge -1. \end{aligned}$$

Note that $\varphi_j(z) = 0$ $(1\le j\le\ell)$. In the rest of the paper, we will give a proof of the following theorem:

Theorem 2.4. *The derivations $\varphi_1,\dots,\varphi_\ell$, together with the Euler derivation*

$$\theta_E := z\frac{\partial}{\partial z} + \sum_{i=1}^{\ell}x_i\frac{\partial}{\partial x_i},$$

form a basis for $D(\mathbf{c}\mathcal{S}(D_\ell))$.

Note that $\theta_E(x_i) = x_i (1\le i\le\ell)$ and $\theta_E(z) = z$.

Lemma 2.5. *Let $1\le i\le\ell$ and $1\le j\le\ell$. Suppose $\varphi_j(x_i)$ is nonzero. Then $\varphi_j(x_i)$ is a homogeneous polynomial of degree $2(\ell-1)$.*

Proof. Define

$$\begin{aligned} F_{ij} &:= (x_j - x_{j+1} - z)\Bigl(\prod K_1\Bigr)\Bigl(\prod K_2\Bigr)^2 z^{|K_1|} \\ &\quad \sigma_{n_1}^{J_1}\tau_{2n_2}^{J_2}\overline{B}_{k,k_0}(x_i,z) \qquad (1\le j\le\ell-1), \\ F_{i\ell} &:= \Bigl(\prod K_1\Bigr)\Bigl(\prod K_2\Bigr)^2 z^{|K_1|}x_\ell\overline{B}_{-1,k_0}(x_i,z) \end{aligned}$$

when K_1, K_2, n_1, n_2 are fixed. Then $\varphi_j(x_i)$ is a linear combination of the F_{ij}'s over $\mathbf{R}$.

Note that $\overline{B}_{k,k_0}(x_i,z)$ is a polynomial unless $(k,k_0) = (-1,0)$.

Assume that $1\le j\le\ell-1$ and $(k,k_0) = (-1,0)$. Then $J = K_1\cup K_2$, $n_1 = |J_1|$, $n_2 = |J_2|$, and $\overline{B}_{-1,0}(x_i,z) = -1/x_i$. Therefore each F_{ij} is a polynomial. Thus $\varphi_j(x_i)$ is a nonzero polynomial and there exists a nonzero polynomial F_{ij}. Compute

$$\begin{aligned} \deg\varphi_j(x_i) &= \deg F_{ij} \\ &= 1 + |K_1| + 2|K_2| + |K_1| + n_1 + 2n_2 \\ &\quad + \deg\overline{B}_{k,k_0}(x_i,z) \end{aligned}$$

$$
\begin{aligned}
&= 1 + 2|K_1| + 2|K_2| + n_1 + 2n_2 + (2k_0 + k) \\
&= 1 + 2|K_1| + 2|K_2| + n_1 + 2n_2 \\
&\quad + 2(|J| - |K_1| - |K_2|) + |J_1| - n_1 \\
&\quad + 2(|J_2| - n_2) - 1 \\
&= 2(|J| + |J_1| + |J_2|) - |J_1| = 2\ell - 2.
\end{aligned}
$$

Next consider $\varphi_\ell(x_i)$. If $k_0 = 0$, then $J = K_1 \cup K_2$. Therefore each $F_{i\ell}$ is a polynomial. Thus so is $\varphi_\ell(x_i)$. Compute

$$
\begin{aligned}
&\deg \varphi_\ell(x_i) \\
&= |K_1| + 2|K_2| + |K_1| + 1 + \deg \overline{B}_{-1,k_0}(x_i, z) \\
&= 2(|K_1| + |K_2|) + 1 + (2k_0 - 1) \\
&= 2(|K_1| + |K_2| + k_0) = 2(\ell - 1).
\end{aligned}
$$

□

Let $<$ denote the *pure lexicographic order* of monomials with respect to the total order

$$x_1 > x_2 > \cdots > x_\ell > z.$$

When $f \in S = \mathbf{C}[x_1, x_2, \ldots, x_\ell, z]$ is a nonzero polynomial, let $\operatorname{in}(f)$ denote the *initial monomial* (e.g., see [2]) of f with respect to the order $<$.

Proposition 2.6. *Suppose $\varphi_j(x_i)$ is nonzero. Then*

(1) $\operatorname{in}(\varphi_j(x_i)) \le x_1^2 \cdots x_{i-1}^2 x_i^{2\ell-2i}$,

(2) $\operatorname{in}(\varphi_j(x_i)) < x_1^2 \cdots x_{i-1}^2 x_i^{2\ell-2i}$ *for $i < j$,*

(3) $\operatorname{in}(\varphi_i(x_i)) = x_1^2 \cdots x_{i-1}^2 x_i^{2\ell-2i}$ *for $1 \le i \le \ell$.*

Proof. Recall F_{ij} $(1 \le j \le \ell - 1)$ and $F_{i\ell}$ from the proof of Lemma 2.5 when K_1, K_2, n_1, n_2 are fixed. Let $\deg^{(x_i)} f$ denote the degree of f with respect to x_i when $f \neq 0$.

(1) Since, for every nonzero F_{ij}, we obtain

$$\deg^{(x_p)} F_{ij} \le 2 \ (1 \le p < i), \quad \deg(F_{ij}) = 2\ell - 2.$$

Hence we may conclude

$$\operatorname{in}(F_{ij}) \le x_1^2 \cdots x_{i-1}^2 x_i^{2\ell-2i}$$

and thus

$$\operatorname{in}(\varphi_j(x_i)) \le \max\{\operatorname{in}(F_{ij})\} \le x_1^2 \cdots x_{i-1}^2 x_i^{2\ell-2i}.$$

(2) Suppose $i < j < \ell$. Since $x_i > x_j > z$, one has

$$
\begin{aligned}
&\operatorname{in}(\sigma_{n_1}^{J_1} \tau_{2n_2}^{J_2} \overline{B}_{k,k_0}(x_i, z)) \\
&\qquad \le x_i^{n_1+2n_2+2k_0+k} = x_i^{2\ell-2j+2k_0-1}
\end{aligned}
$$

when $\overline{B}_{k,k_0}(x_i, z)$ is nonzero. The equality holds if and only if $n_1 = n_2 = 0$.

Suppose that F_{ij} is nonzero. For $1 \le i < j \le \ell - 1$, we have

$$
\begin{aligned}
&\operatorname{in}(F_{ij}) \\
&= \operatorname{in}(x_j - x_{j+1} - z) \operatorname{in}\Big(\Big(\prod K_1\Big) \Big(\prod K_2\Big)^2 (-z)^{|K_1|} \Big) \\
&\quad \operatorname{in}(\sigma_{n_1}^{J_1} \tau_{2n_2}^{J_2} \overline{B}_{k,k_0}(x_i, z)) \\
&\le x_j \operatorname{in}\Big(\Big(\prod K_1\Big) \Big(\prod K_2\Big)^2 (-z)^{|K_1|} \Big) x_i^{2\ell-2j+2k_0-1} \\
&= x_j \operatorname{in}\Big(\Big(\prod K_1\Big) \Big(\prod K_2\Big)^2 (-z)^{|K_1|} x_i^{2k_0} \Big) x_i^{2\ell-2j-1} \\
&\le x_j (x_1^2 \cdots x_{i-1}^2 x_i^{2j-2i}) x_i^{2\ell-2j-1} \ (*) \\
&= x_1^2 \cdots x_{i-1}^2 x_i^{2\ell-2i-1} x_j < x_1^2 \cdots x_{i-1}^2 x_i^{2\ell-2i}.
\end{aligned}
$$

Thus

$$\operatorname{in}(\varphi_j(x_i)) < x_1^2 \cdots x_{i-1}^2 x_i^{2\ell-2i}.$$

For $1 \le i < j = \ell$,

$$
\begin{aligned}
&\operatorname{in}(F_{i\ell}) \\
&= x_\ell \operatorname{in}\Big(\Big(\prod K_1\Big) \Big(\prod K_2\Big)^2 (-z)^{|K_1|} \Big) \operatorname{in}(\overline{B}_{-1,k_0}(x_i, z)) \\
&= x_\ell \operatorname{in}\Big(\Big(\prod K_1\Big) \Big(\prod K_2\Big)^2 (-z)^{|K_1|} \Big) x_i^{2k_0-1} \\
&= x_\ell \operatorname{in}\Big(\Big(\prod K_1\Big) \Big(\prod K_2\Big)^2 (-z)^{|K_1|} x_i^{2k_0} \Big) x_i^{-1} \\
&\le x_\ell (x_1^2 \cdots x_{i-1}^2 x_i^{2\ell-2i}) x_i^{-1} \quad (**) \\
&= x_1^2 \cdots x_{i-1}^2 x_i^{2\ell-2i-1} x_\ell < x_1^2 \cdots x_{i-1}^2 x_i^{2\ell-2i}.
\end{aligned}
$$

This proves (2).

Now we only need to prove (3). Let $i = j < \ell$ in $(*)$. Then the equality

$$\operatorname{in}(F_{ii}) = x_1^2 \cdots x_{i-1}^2 x_i^{2\ell-2i}$$

holds if and only if

$$K_1 = \emptyset,\ K_2 = J,\ n_1 = n_2 = k_0 = 0,\ k = 2\ell - 2i - 1$$

because the leading term of $\overline{B}_{2\ell-2i-1,0}(x_i, z)$ is equal to

$$\frac{x_i^{2\ell-2i-1}}{2\ell - 2i - 1}.$$

Next let $i = \ell$ in $(**)$. Then the equality

$$\operatorname{in}(F_{\ell\ell}) = x_1^2 \cdots x_{\ell-1}^2$$

holds if and only if

$$K_1 = \emptyset,\ K_2 = J = \{x_1, \ldots, x_{\ell-1}\},\ k_0 = 0.$$

Therefore, for $1 \le i \le \ell$,

$$\operatorname{in}(\varphi_i(x_i)) = x_1^2 \cdots x_{i-1}^2 x_i^{2\ell-2i}.$$

□

From Proposition 2.6, we immediately obtain the following Corollary:

Corollary 2.7.

(1) $\mathrm{in}(\det[\varphi_j(x_i)]) = \prod_{i=1}^{\ell} \mathrm{in}(\varphi_i(x_i)) = \prod_{i=1}^{\ell-1} x_i^{4(\ell-i)}$.

(2) *Moreover, the leading term of* $\det[\varphi_j(x_i)]$ *is equal to*

$$\frac{1}{(2\ell-3)!!}\prod_{i=1}^{\ell-1} x_i^{4(\ell-i)}.$$

(3) *In particular,* $\det[\varphi_j(x_i)]$ *does not vanish.*

Next, we will prove $\varphi_j \in D(\mathbf{c}\mathcal{S}(D_\ell))$ for $1 \le j \le \ell$. We denote $\mathbf{c}\mathcal{S}(D_\ell)$ simply by $\mathcal{S}_\ell$ from now on. Before the proof, we need the following two lemmas:

Lemma 2.8. *Fix* $1 \le j \le \ell - 1$ *and* $\epsilon \in \{-1, 1\}$*. Then*

(1)

$$\prod_{x_i \in J}(x_i - x_s)(x_i - \epsilon x_t) = \sum_{\substack{K_1 \cup K_2 \subseteq J \\ K_1 \cap K_2 = \emptyset}} \Big(\prod K_1\Big) \times \Big(\prod K_2\Big)^2 [-(x_s + \epsilon x_t)]^{|K_1|}(\epsilon x_s x_t)^{k_0}.$$

(2)

$$\sum_{\substack{0 \le n_1 \le |J_1| \\ 0 \le n_2 \le |J_2|}} (-1)^{|J_1|+|J_2|-n_1-n_2} \sigma_{n_1}^{J_1} \tau_{2n_2}^{J_2} (\epsilon x_s)^{k+1} = \prod_{x_i \in J_1}(x_i - \epsilon x_s) \prod_{x_i \in J_2}(x_i^2 - x_s^2).$$

Proof. (1) is easy because the left handside is equal to

$$\prod_{x_i \in J}(x_i^2 - (x_s + \epsilon x_t)x_i + \epsilon x_s x_t).$$

(2) The left handside is equal to

$$\sum_{0 \le n_1 \le |J_1|} (-\epsilon x_s)^{|J_1|-n_1} \sigma_{n_1}^{J_1} \sum_{0 \le n_2 \le |J_2|} (-x_s^2)^{|J_2|-n_2} \tau_{2n_2}^{J_2}$$

which is equal to the right handside. □

Lemma 2.9.

(1) *The polynomial*

$$x_s \overline{B}_{k,k_0}(x_s, z) - x_t \overline{B}_{k,k_0}(x_t, z)$$

is divisible by $x_s^2 - x_t^2$,

(2) *For* $\epsilon \in \{-1, 1\}$, *the polynomial*

$$(x_s - \epsilon x_t)\epsilon x_s x_t[\overline{B}_{k,k_0}(x_s, z) + \epsilon \overline{B}_{k,k_0}(x_t, z)] - (x_s + \epsilon x_t)(\epsilon x_s x_t)^{k_0}[\epsilon x_t x_s^{k+1} - x_s(\epsilon x_t)^{k+1}]$$

is divisible by $x_s + \epsilon x_t - z$.

Proof. (1) follows from the fact that $-\overline{B}_{k,k_0}(x, z) = \overline{B}_{k,k_0}(-x, z)$ in Proposition 2.1.

(2) follows from the following congruence relation of polynomials modulo the ideal $(x_s + \epsilon x_t - z)$:

$$\begin{aligned}
&(x_s - \epsilon x_t)\epsilon x_s x_t[\overline{B}_{k,k_0}(x_s, z) + \epsilon \overline{B}_{k,k_0}(x_t, z)] \\
&= (x_s - \epsilon x_t)\epsilon x_s x_t z^{k+2k_0}\left[B_{k,k_0}\left(\frac{x_s}{z}\right) - B_{k,k_0}\left(\frac{-\epsilon x_t}{z}\right)\right] \\
&\equiv (x_s - \epsilon x_t)\epsilon x_s x_t (x_s + \epsilon x_t)^{k+2k_0} \\
&\quad \left[B_{k,k_0}\left(\frac{x_s}{x_s + \epsilon x_t}\right) - B_{k,k_0}\left(\frac{-\epsilon x_t}{x_s + \epsilon x_t}\right)\right] \\
&= (x_s - \epsilon x_t)\epsilon x_s x_t (x_s + \epsilon x_t)^{k+2k_0} \\
&\quad \frac{\left(\frac{x_s}{x_s + \epsilon x_t}\right)^k - \left(\frac{\epsilon x_t}{x_s + \epsilon x_t}\right)^k}{\left(\frac{x_s}{x_s + \epsilon x_t}\right) - \left(\frac{\epsilon x_t}{x_s + \epsilon x_t}\right)} \left(\frac{\epsilon x_t}{x_s + \epsilon x_t}\right)^{k_0}\left(\frac{x_s}{x_s + \epsilon x_t}\right)^{k_0} \\
&= (x_s + \epsilon x_t)(\epsilon x_s x_t)^{k_0}[\epsilon x_t x_s^{k+1} - x_s(\epsilon x_t)^{k+1}].
\end{aligned}$$

□

Proposition 2.10. *Every* φ_j *lies in* $D(\mathcal{S}_\ell)$.

Proof. For $1 \le j \le \ell - 1, 1 \le s < t \le \ell$, and $\epsilon \in \{-1, 1\}$, by Lemma 2.9 and Lemma 2.8, we have the following congruence relation of polynomials modulo the ideal $(x_s + \epsilon x_t - z)$:

$$\begin{aligned}
&(x_s - \epsilon x_t)\epsilon x_s x_t[\varphi_j(x_s + \epsilon x_t - z)] \\
&= (x_j - x_{j+1} - z)\sum_{\substack{K_1 \cup K_2 \subseteq J \\ K_1 \cap K_2 = \emptyset}} \Big(\prod K_1\Big)\Big(\prod K_2\Big)^2 \\
&\quad \times (-z)^{|K_1|} \sum_{\substack{0 \le n_1 \le |J_1| \\ 0 \le n_2 \le |J_2|}} (-1)^{n_1+n_2}\sigma_{n_1}^{J_1}\tau_{2n_2}^{J_2} \\
&\quad \times (x_s - \epsilon x_t)\epsilon x_s x_t[\overline{B}_{k,k_0}(x_s, z) + \epsilon \overline{B}_{k,k_0}(x_t, z)] \\
&\equiv (x_j - x_{j+1} - z)(x_s + \epsilon x_t) \\
&\quad \times \sum_{K_1, K_2}\Big(\prod K_1\Big)\Big(\prod K_2\Big)^2[-(x_s + \epsilon x_t)]^{|K_1|}(\epsilon x_s x_t)^{k_0} \\
&\quad \times \sum_{n_1, n_2}(-1)^{n_1+n_2}\sigma_{n_1}^{J_1}\tau_{2n_2}^{J_2}[\epsilon x_t x_s^{k+1} - x_s(\epsilon x_t)^{k+1}] \\
&= (x_j - x_{j+1} - z)(x_s + \epsilon x_t)\prod_{x_i \in J}(x_i - x_s)(x_i - \epsilon x_t) \\
&\quad \times (-1)^{|J_2|}\left[\epsilon x_t \prod_{x_i \in J_1}(x_i - x_s)\prod_{x_i \in J_2}(x_i^2 - x_s^2)\right. \\
&\quad \left. - x_s \prod_{x_i \in J_1}(x_i - \epsilon x_t)\prod_{x_i \in J_2}(x_i^2 - x_t^2)\right] \quad (\dagger).
\end{aligned}$$

Case 1. When $x_s \in J$, $(\dagger) = 0$.
Case 2. When $x_s \in J_2$ and $x_t \in J_2$, $(\dagger) = 0$.

Case 3. When $x_s \in J_1$ and $x_t \in J_2$, $(\dagger) = 0$.
Case 4. When $x_s \in J_1$, $x_t \in J_1$ and $\epsilon = 1$, $(\dagger) = 0$.
Case 5. If $x_s \in J_1$, $x_t \in J_1$ and $\epsilon = -1$, then $s = j < t = j+1$. So $(\dagger)$ is divisible by $x_s + \epsilon x_t - z$.

We also have the following congruence relation of polynomials modulo the ideal $(x_s + \epsilon x_t - z)$:

$$(x_s - \epsilon x_t)\epsilon x_s x_t[\varphi_\ell(x_s + \epsilon x_t - z)] = \sum_{\substack{K_1 \cup K_2 \subseteq J \\ K_1 \cap K_2 = \emptyset}} \Big(\prod K_1\Big)\Big(\prod K_2\Big)^2 (-z)^{|K_1|}(-x_\ell)$$

$$(x_s - \epsilon x_t)\epsilon x_s x_t[\overline{B}_{-1,k_0}(x_s, z) + \epsilon \overline{B}_{-1,k_0}(x_t, z)] \equiv (x_s + \epsilon x_t)(-x_\ell)(\epsilon x_t - x_s)$$

$$\sum_{K_1, K_2} \Big(\prod K_1\Big)\Big(\prod K_2\Big)^2 [-(x_s + \epsilon x_t)]^{|K_1|}(\epsilon x_s x_t)^{k_0} = (x_s^2 - x_t^2)x_\ell \prod_{x_i \in J}(x_i - x_s)(x_i - \epsilon x_t) \quad (\dagger\dagger).$$

Since $s < t \le \ell$, we have $x_s \in J = \{x_1, \ldots, x_{\ell-1}\}$. Thus $(\dagger\dagger) = 0$. Therefore $\varphi_j(x_s + \epsilon x_t - z)$ is divisible by $x_s + \epsilon x_t - z$ for $1 \le j \le \ell, 1 \le s < t \le \ell$.

For $1 \le j \le \ell$,

$$\varphi_j(x_s^2 - x_t^2) = 2x_s\varphi_j(x_s) - 2x_t\varphi_j(x_t)$$

is divisible either by $x_s\overline{B}_{k,k_0}(x_s, z) - x_t\overline{B}_{k,k_0}(x_t, z)$ or by $x_s\overline{B}_{-1,k_0}(x_s, z) - x_t\overline{B}_{-1,k_0}(x_t, z)$, we have

$$\varphi_j(x_s^2 - x_t^2) \equiv 0 \mod (x_s^2 - x_t^2)$$

by Lemma 2.9 (1). This implies $\varphi_j \in D(\mathcal{S}_\ell)$. □

Applying Saito's lemma [4] [3, Theorem 4.19], we complete our proof of Theorem 2.4 thanks to Lemma 2.5, Corollay 2.7 (3) and Proposition 2.10. Theorem 2.4 implies that $\det[\varphi_j(x_i)]$ is a nonzero multiple of (Q/z). By Corollary 2.7 (2) one obtains

Corollary 2.11.

$$\det[\varphi_j(x_i)] = \frac{1}{(2\ell-3)!!} \prod_{1 \le s < t \le \ell} \prod_{\epsilon \in \{-1,1\}} (x_s + \epsilon x_t - z)(x_s + \epsilon x_t).$$

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