On Brauer's height zero conjecture

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Abstract: We give a reduction theorem for Brauer's height zero conjecture.

Key word: Brauer's height zero conjecture.

1. Introduction. Let G be a finite group and p a prime. Let (K, R, k) be a p-modular system ([NaTs, p.230]); that is, R is a complete discrete valuation ring with quotient field K of characteristic 0 and k is the residue field of R of characteristic p. We assume that K contains a primitive $|G|$-th root of unity and that k is algebraically closed. In this paper a block of G means a block ideal of RG. Let B be a block of G with a defect group D. We say B is an HZ-block if every irreducible character in B has height 0. Brauer's height zero conjecture states

(HZ) B is an HZ-block if and only if D is abelian.

This conjecture is naturally divided into two parts.

(AHZ) If D is abelian, then B is an HZ-block.

(HZA) If B is an HZ-block, then D is abelian.

Berger and Knörr [BK] have reduced the conjecture (AHZ) to the case of quasi-simple groups. (See also [Mu1].) In [Mu2] the conjecture (HZA) for principal blocks has been reduced to the case of simple groups under the Alperin-McKay conjecture. In the present paper we consider reduction of the conjecture (HZA) for general blocks. This time we need Dade's projective conjecture [Da], a conjecture much stronger than the Alperin-McKay conjecture. We also need the following statement (see (A) of Section 7 of [NaTi]):

(GW) Suppose that N is a normal subgroup of a group G and let $\theta \in \mathrm{Irr}(N)$ be G-invariant. If p does not divide $\chi(1)/\theta(1)$ for all $\chi \in \mathrm{Irr}(G)$ lying over θ, then G/N has an abelian Sylow p-subgroup.

Gluck and Wolf [GW] have proved (GW) is true when G/N is p-solvable, which constitutes the central part of their proof of (HZA) for p-blocks of p-solvable groups. Navarro and Tiep [NaTi, Section 5] remark that (GW) is a consequence of (HZA). We prove the following

Theorem. *Assume that Dade's projective conjecture is true. Assume that (GW) is always true. Assume that (HZA) is true for any block of any quasi-simple group G with $Z(G)$ a cyclic p'-group. Then (HZA) is true for any block of any group.*

It is well known that Dade's projective conjecture implies the conjecture (AHZ), see [Da, p.97]. However, the relevance of Dade's projective conjecture to the conjecture (HZA) seems to be new.

We obtain the following corollary.

Corollary. *Let $p = 2$. Assume that Dade's projective conjecture is true. Assume that (HZA) is true for any 2-block of any quasi-simple group G with $Z(G)$ a cyclic group of odd order. Then (HZA) is true for any 2-block of any group.*

2. Proof of Theorem. For an integer $n \neq 0$, let $p^{\nu(n)}$ be the highest power of p dividing n. For an irreducible character χ, let $\mathrm{ht}(\chi)$ be the height of χ. For a group X, let X' be the commutator subgroup of X.

Proof of Theorem. Let B be an HZ-block of a group G. Let D be a defect group of B. We prove D is abelian by induction firstly on $|G : Z(G)|$ and secondly on $|G|$. First we note the following:

($*$) Let M be a normal subgroup of G. Let β be a block of M covered by B. Then we may assume β is G-invariant.

Indeed, let T_β be the inertial group of β in G. Then $T_\beta \geq Z(G)$. So, if $T_\beta \neq G$, then D is abelian by the Fong-Reynolds theorem and induction. So we may assume β is G-invariant.

If G is abelian, there is nothing to prove. So we assume G is non-abelian. Let $N/Z(G)$ be a maximal

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proper normal subgroup of $G/Z(G)$. Let b be a block of N covered by B. By $(*)$, we may assume b is G-invariant. Put $Q = D \cap N$. Then Q is a defect group of b. Since B is an HZ-block, b is also an HZ-block by [Mu3, Lemma 2.2]. Hence Q is abelian by induction. Let B' be the Harris-Knörr correspondent of B in $N_G(Q)$. Then D is a defect group of B'. Since we are assuming Dade's projective conjecture is true, we can apply Theorem 3 of [Ro] to conclude that B' is also an HZ-block. If $N_G(Q) \neq G$, then D is abelian by induction. Therefore we may assume $N_G(Q) = G$.

We may assume $Z(G)$ is a cyclic p'-group. Indeed, let Z be a Sylow p-subgroup of $Z(G)$ and assume that $Z \neq 1$. Put $\bar{G} = G/Z$. Let $\bar{B}$ be a unique block of $\bar{G}$ which is dominated by B. Then D/Z is a defect group of $\bar{B}$. So $\bar{B}$ is an HZ-block. Since $|\bar{G} : Z(\bar{G})| \leq |G : Z(G)|$ and $|\bar{G}| < |G|$, D/Z is abelian by induction. So $D' \leq Z$. For any linear character μ of Z, there is an irreducible character χ in B such that χ lies over μ. Since $\mathrm{ht}\,\chi = 0$, μ extends to D by [Mu3, Theorem 2.3]. Let $\hat{\mu}$ be an extension. Then $D' \leq \mathrm{Ker}\,\hat{\mu} \cap Z = \mathrm{Ker}\,\mu$. Since μ is arbitrary, we obtain $D' = 1$, so D is abelian. Therefore we may assume $Z = 1$ and $Z(G)$ is a p'-group. Let $\mu \in \mathrm{Irr}(Z(G))$ be covered by B. Assume $\mathrm{Ker}\,\mu \neq 1$ and let $\bar{B}$ be a unique block of $G/\mathrm{Ker}\,\mu$ dominated by B. Then $D\mathrm{Ker}\,\mu/\mathrm{Ker}\,\mu$ $(\simeq D)$ is a defect group of $\bar{B}$. Clearly $\bar{B}$ is an HZ-block. So D is abelian by induction. Thus we may assume $\mathrm{Ker}\,\mu = 1$ and $Z(G)$ is cyclic.

Since $C_G(Q) \lhd G$ and G/N is simple, either of the following holds.

(i) $G = C_G(Q)N$; (ii) $C_G(Q) \leq N$.

Case (i). We may assume that G is quasi-simple. Indeed, let b_0 be a block of $C_N(Q)$ covered by b. By $(*)$, we may assume b_0 is G-invariant. Thus $|N : C_N(Q)|$ is a p'-integer. Therefore $|G : C_G(Q)| = |N : C_N(Q)|$ is a p'-integer. Thus $C_G(Q) \geq D$. Let b_1 be a block of $C_G(Q)$ covered by B. By $(*)$, we may assume b_1 is G-invariant. Then D is a defect group of b_1. By [Mu3, Lemma 2.2], b_1 is an HZ-block. If $C_G(Q) \neq G$, then D is abelian by induction. So we may assume $Q \leq Z(G)$. Since $Z(G)$ is a p'-group, we obtain $Q = 1$. Let ξ be a unique irreducible character in b. Since b is G-invariant, ξ is G-invariant. Let P be a Sylow p-subgroup of G and let b' be a unique block of PN covering b. Choose an irreducible character ζ of height 0 in b'. Then ζ lies over ξ, and we obtain $\zeta_N = \xi$. Thus ξ extends to PN. Therefore there exists a central extension of G,

$$1 \longrightarrow Z \longrightarrow \hat{G} \xrightarrow{f} G \longrightarrow 1$$

with the following properties: For some $N_1 \lhd \hat{G}$ we have $f^{-1}(N) = N_1 \times Z$, so that we can identify N_1 with N via f. Then ξ extends to $\hat{G}$; Z is a p'-group.

Let $\hat{\xi}$ be an extension of ξ to $\hat{G}$. Let λ be an irreducible constituent of $\hat{\xi}_Z$. Let $\hat{B}$ be the inflation of B to $\hat{G}$. Let $\hat{D}$ be a defect group of $\hat{B}$ such that $\hat{D}Z/Z = D$. Choose an irreducible character χ in B. Let $\hat{\chi}$ be the inflation of χ to $\hat{G}$. Then $\hat{\chi} = \hat{\xi} \otimes \varphi$ for a unique irreducible character φ of $\tilde{G} := \hat{G}/N$. Let $\tilde{B}$ be the block of $\tilde{G}$ containing φ. Then $\tilde{B}$ is $\hat{\xi}$-dominated by $\hat{B}$. (For $\hat{\xi}$-domination, see [Mu3, p.35].) Put $\tilde{Z} = ZN/N$. We denote again by λ the character of $\tilde{Z}$ which is identified with λ via the natural isomorphism: $\tilde{Z} \simeq Z$. Then φ lies over λ^{-1}. So $\tilde{B}$ covers a block of $\tilde{Z}$ containing λ^{-1}. Let θ be any irreducible character in $\tilde{B}$. Since $\tilde{Z}$ is a p'-group, θ lies over λ^{-1}. So $Z \leq \mathrm{Ker}(\hat{\xi} \otimes \theta)$ and, since $\tilde{B}$ is $\hat{\xi}$-dominated by $\hat{B}$, $\hat{\xi} \otimes \theta$ belongs to $\hat{B}$. Thus $\hat{\xi} \otimes \theta$ is the inflation of an irreducible character in B, say η. Then

$$0 = \mathrm{ht}(\eta) = \mathrm{ht}(\xi) + \mathrm{ht}(\theta) + \nu(|\hat{D}N/N|) - \nu(|\tilde{D}|),$$

where $\tilde{D}$ is a defect group of $\tilde{B}$. Since $\hat{D}N/N \geq_{\tilde{G}} \tilde{D}$ by [Mu3, Corollary 1.5], we obtain $\mathrm{ht}(\theta) = 0$ and $\hat{D}N/N =_{\tilde{G}} \tilde{D}$. Thus $\tilde{B}$ is an HZ-block. Now assume $N \neq Z(G)$. Then, $|\tilde{G} : Z(\tilde{G})| \leq |\tilde{G} : \tilde{Z}| = |G : N| < |G : Z(G)|$, so $\tilde{D}$ is abelian by induction. Therefore $\hat{D}' \leq N$. So $D' \leq N \cap D = Q = 1$. Thus D is abelian. Hence we may assume $N = Z(G)$. If G/N is of prime order, then G is abelian, a contradiction. Hence G/N is non-abelian simple. Then $G = G'Z(G)$ with G' quasi-simple. Let B' be a block of G' covered by B. Since G/G' is a p'-group, D is a defect group of B'. Further B' is an HZ-block by [Mu3, Lemma 2.2]. We have $|G' : Z(G')| \leq |G : Z(G)|$. Thus, if $G' \neq G$, then D is abelian by induction. Thus we may assume $G = G'$. So G is quasi-simple.

Then we conclude that D is abelian by the assumption of Theorem.

Case (ii). Let β be a block of $C_G(Q)$ covered by b. Since B covers β, we may assume β is G-invariant by $(*)$. So β is N-invariant and $Q \cap C_G(Q) = Q$ is a defect group of β. Thus B is a unique block of G covering β and β is nilpotent. Therefore B is isomorphic to the full matrix algebra of some degree over a twisted group algebra $R^\alpha[L]$ by [KP, 1.20.3], where the group L is an extension of $G/C_G(Q)$ by Q ([KP, 1.8.1]), the 2-cocycle α takes p'-th roots of unity as values ([KP, 2.4]) and D is isomorphic to a Sylow p-subgroup of L ([KP, Remark 1.9]). There is a p'-central extension H of L

$$1 \longrightarrow Z \longrightarrow H \longrightarrow L \longrightarrow 1$$

such that for a linear character μ of Z, $R^\alpha[L] \simeq RHe_\mu$, where e_μ is the central idempotent of KZ corresponding to μ. Put $B' = RHe_\mu$. Then we see B' is a block of H, which is necessarily a unique block of H covering the block of Z containing μ. So, if P is a defect group of B', then $|H : PZ|$ is a p'-integer by Fong's theorem. Thus P is a Sylow p-subgroup of H. In particular, B' contains an irreducible character of p'-degree. On the other hand, since B is an HZ-block, the Morita equivalence between B and B' yields that all character degrees in B' have the same p-part. Thus all characters in B', that is, all characters of H lying over μ, have p'-degree. Hence by (GW), a Sylow p-subgroup of $H/Z \simeq L$ is abelian. So D is abelian. This completes the proof. $\square$

Remark. In the proof of Theorem, we do not use the full strength of Dade's projective conjecture. We have assumed only that the conclusion of Theorem 3 of [Ro] is true in general.

Proof of Corollary. If $p = 2$, then (GW) is true, see [NaTi, Theorem 5.2]. $\square$

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(1) Professor G. Navarro has kindly informed the author that he and P. H. Tiep have recently proved that (GW) is always true, see "Characters of relative p'-degree over normal subgroups", preprint. Therefore Corollary is true for any prime p.

(2) Recently (AHZ) has been proved, see "R. Kessar and G. Malle, Quasi-isolated blocks and Brauer's height zero conjecture", arXiv:1112.2642v1 [math.GR].

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