Optimal Korn's inequality for solenoidal vector fields on a periodic slab

By Yoshiaki TERAMOTO*) and Kyoko TOMOEDA**),†)

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Abstract: We obtain the best constant in Korn's inequality for solenoidal vector fields on a periodic slab which vanish on a part of its boundary. To do this we consider the Stokes equations with Dirichlet boundary conditions, following H. Ito [6], [7].

Key words: Korn inequality; best constant.

1. Introduction and result. Let Ω be a periodic slab $\mathbf{I}\times\mathbf{T}$, where $\mathbf{I}$ denotes the interval $(-1,0)$ and $\mathbf{T}$ the torus $\mathbf{R}/(2\pi/a)\mathbf{Z}$ with period $2\pi/a$ for a given constant $a>0$. We set

$$ {}_0H^1_\sigma(\Omega)=\{\boldsymbol{u}=(u_1,u_2)\in\{H^1(\Omega)\}^2 : \operatorname{div}\boldsymbol{u}=0,\quad \boldsymbol{u}=0 \text{ on } x_1=-1\}. $$

Korn's inequality on ${}_0H^1_\sigma(\Omega)$ states that there exists a constant $K>0$ such that

$$ \|\varepsilon(\boldsymbol{u})\|^2_{L^2(\Omega)}\geq K\|\nabla\boldsymbol{u}\|^2_{L^2(\Omega)} \tag{1.1} $$

for any $\boldsymbol{u}\in{}_0H^1_\sigma(\Omega)$, where $\varepsilon(\boldsymbol{u})=(\varepsilon_{ij}(\boldsymbol{u}))$ is the rate-of-strain tensor whose elements are given by

$$ \varepsilon_{ij}(\boldsymbol{u})=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right),\quad i,j=1,2. $$

Our problem is to find the best constant $K_{\max}$ of (1.1), i.e., the largest number K such that (1.1) holds, and we obtain the following result.

Theorem 1.1. *The best constant of* (1.1) *is given by*

$$ K_{\max}=\frac{1}{3}. $$

Remark 1. Note that the value 1/3 of the best constant coincides with that for the case of half-space obtained by H. Ito [7]. For the results in other situations, see, for example, [1–9] and their references.

The plan of this paper is as follows. In section 2, we obtain the solution to the Stokes equations with Dirichlet boundary conditions. By using this, we determine $K_{\max}$ in section 3. We show the lemma used in the proof of Theorem 1.1 in appendix.

2. Preliminary. We begin with writing down explicitly, a solution $\{\boldsymbol{u},p\}$ of the Stokes equations with Dirichlet boundary conditions:

$$ -(1-\kappa)\Delta\boldsymbol{u}+\nabla p=0,\quad \operatorname{div}\boldsymbol{u}=0\quad\text{in }\Omega, \tag{2.1} $$

$$ \int_\Omega p(x_1,x_2)dx=0, \tag{2.2} $$

$$ \boldsymbol{u}(-1,x_2)=0,\quad \boldsymbol{u}(0,x_2)=\phi(x_2), \tag{2.3} $$

where $\kappa<1$ is a constant and $\phi=(\phi_1,\phi_2)$ is a given function.

We expand u_j, p and ϕ_j $(j=1,2)$ into Fourier series in $x_2\in\mathbf{T}$ as follows:

$$ \begin{aligned} u_j(x_1,x_2)&=\sum_{\ell\in\mathbf{Z}}u_j^{(\ell)}(x_1)\exp(ia\ell x_2),\\ p(x_1,x_2)&=\sum_{\ell\in\mathbf{Z}}p^{(\ell)}(x_1)\exp(ia\ell x_2),\\ \phi_j(x_2)&=\sum_{\ell\in\mathbf{Z}}\phi_j^{(\ell)}\exp(ia\ell x_2),\quad j=1,2. \end{aligned} $$

Then, for each $\ell\in\mathbf{Z}$, we obtain the boundary value problem on the interval $\{x_1:-1<x_1<0\}$ for the system of the ordinary differential equations

$$ -(1-\kappa)\left\{\left(\frac{d}{dx_1}\right)^2u_1^{(\ell)}-(a\ell)^2u_1^{(\ell)}\right\}+\frac{d}{dx_1}p^{(\ell)}=0, \tag{2.4} $$

$$ -(1-\kappa)\left\{\left(\frac{d}{dx_1}\right)^2u_2^{(\ell)}-(a\ell)^2u_2^{(\ell)}\right\}+(ia\ell)p^{(\ell)}=0, \tag{2.5} $$

$$ \frac{d}{dx_1}u_1^{(\ell)}+(ia\ell)u_2^{(\ell)}=0,\quad\text{in }-1<x_1<0, \tag{2.6} $$

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*) Institute for Fundamental Sciences, Faculty of Science and Engineering, Setsunan University, Neyagawa, Osaka 572-8508, Japan.

**) Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan.

†) Current Address: Institute for Fundamental Sciences, Faculty of Science and Engineering, Setsunan University, Neyagawa, Osaka 572-8508, Japan.

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$$(2.7)\qquad u_j^{(\ell)}(-1)=0,\quad u_j^{(\ell)}(0)=\phi_j^{(\ell)},\quad j=1,2.$$

To solve this, we must assume

$$(2.8)\qquad \phi_1^{(0)}=0.$$

Then, for $\ell=0$, the solution to the system (2.4)–(2.7) is given by

$$(2.9)\qquad u_1^{(0)}(x_1)=0,\quad u_2^{(0)}(x_1)=(x_1+1)\phi_2^{(0)},\quad p^{(0)}=0.$$

In the case $\ell\neq 0$, the solution to (2.4)–(2.6) are written as follows:

$$(2.10)\qquad u_1^{(\ell)}(x_1)=(C_1+C_2(x_1+1))e^{|a\ell|(x_1+1)}+(C_3+C_4(x_1+1))e^{-|a\ell|(x_1+1)},$$

$$u_2^{(\ell)}(x_1)=\frac{-1}{ia\ell}\{(C_2+|a\ell|(C_1+C_2(x_1+1)))e^{|a\ell|(x_1+1)}+(C_4-|a\ell|(C_3+C_4(x_1+1)))e^{-|a\ell|(x_1+1)}\},$$

$$p^{(\ell)}(x_1)=2(1-\kappa)(C_2e^{|a\ell|(x_1+1)}+C_4e^{-|a\ell|(x_1+1)}).$$

Substituting (2.10) into the boundary conditions (2.7), we have

$$(2.11)\qquad \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14}\\ b_{21} & b_{22} & b_{23} & b_{24}\\ b_{31} & b_{32} & b_{33} & b_{34}\\ b_{41} & b_{42} & b_{43} & b_{44}\end{pmatrix}\begin{pmatrix} C_1\\ C_2\\ C_3\\ C_4\end{pmatrix}=\begin{pmatrix} 0\\ 0\\ \phi_1^{(\ell)}\\ -ia\ell\phi_2^{(\ell)}\end{pmatrix},$$

where

$$b_{11}=b_{13}=b_{22}=b_{24}=1,\ b_{12}=b_{14}=0,\quad b_{21}=|a\ell|,$$
$$b_{23}=-|a\ell|,\ b_{31}=b_{32}=e^{|a\ell|},\ b_{33}=b_{34}=e^{-|a\ell|},$$
$$b_{41}=|a\ell|e^{|a\ell|},\ b_{42}=(1+|a\ell|)e^{|a\ell|},$$
$$b_{43}=-|a\ell|e^{-|a\ell|},\ b_{44}=(1-|a\ell|)e^{-|a\ell|}.$$

To guarantee the unique solvability of (2.11), we give the following lemma.

Lemma 2.1. *If $\ell\neq 0$, then* $\det(b_{ij})>0$.

Proof. Calculating determinant, we have

$$\det(b_{ij})=4(\sinh^2|a\ell|-|a\ell|^2)>0.$$

□

Set $\mathcal{D}=\det(b_{ij})$. By (2.11) and Lemma 2.1, we have

$$(2.12)\qquad \begin{pmatrix} C_1\\ C_2\\ C_3\\ C_4\end{pmatrix}=(b_{ij})^{-1}\begin{pmatrix} 0\\ 0\\ \phi_1^{(\ell)}\\ -ia\ell\phi_2^{(\ell)}\end{pmatrix}=\frac{1}{\mathcal{D}}\begin{pmatrix} f_{11}(a\ell) & f_{12}(a\ell)\\ f_{21}(a\ell) & f_{22}(a\ell)\\ f_{31}(a\ell) & f_{32}(a\ell)\\ f_{41}(a\ell) & f_{42}(a\ell)\end{pmatrix}\begin{pmatrix}\phi_1^{(\ell)}\\ \phi_2^{(\ell)}\end{pmatrix},$$

where

$$f_{11}(a\ell)=2(|a\ell|\cosh|a\ell|+\sinh|a\ell|),$$
$$f_{12}(a\ell)=2ia\ell(\sinh|a\ell|),$$
$$f_{41}(a\ell)=-2|a\ell|(\sinh|a\ell|+|a\ell|e^{|a\ell|}),$$
$$f_{42}(a\ell)=2ia\ell(\sinh|a\ell|-|a\ell|e^{|a\ell|}),$$
$$f_{2j}(a\ell)=-2|a\ell|f_{1j}(a\ell)-f_{4j}(a\ell),$$
$$f_{3j}(a\ell)=-f_{1j}(a\ell),\quad j=1,2.$$

We next show the regularity of the formal solution $\{\boldsymbol{u},p\}$ of (2.1)–(2.3) in the form of Fourier series in x_2 with coefficients (2.8)–(2.10) specified by (2.12).

Lemma 2.2. *For a given $\phi\in\{H^{3/2}(\mathbf{T})\}^2$ with* (2.8), *it holds that*

$$(2.13)\qquad \{\boldsymbol{u},p\}\in\{H^2(\Omega)\}^2\times H^1(\Omega).$$

Proof. We begin with showing $u_1\in H^2(\Omega)$. Since each term $u_1^{(\ell)}(x_1)\exp(ia\ell x_2)$ of the Fourier series of u_1 in x_2 is a smooth function on $\bar{\Omega}$, it is sufficient to show

$$\sum_{|a\ell|\geq 1}u_1^{(\ell)}(x_1)\exp(ia\ell x_2)\in H^2(\Omega).$$

So let $|a\ell|\geq 1$. Since

$$\mathcal{D}=(e^{|a\ell|}-e^{-|a\ell|}-2|a\ell|)(e^{|a\ell|}-e^{-|a\ell|}+2|a\ell|)\geq\frac{e^{2|a\ell|}}{5},$$

we have

$$(2.14)\qquad |C_1+C_2| =\frac{2}{\mathcal{D}}\Big|(\sinh|a\ell|-|a\ell|(|a\ell|-1)e^{-|a\ell|})\phi_1^{(\ell)}+ia\ell|a\ell|e^{-|a\ell|}\phi_2^{(\ell)}\Big|$$
$$\leq 5e^{-2|a\ell|}(e^{|a\ell|}|\phi_1^{(\ell)}|+2|a\ell|^2e^{-|a\ell|}|\phi_2^{(\ell)}|)$$
$$\leq 5e^{-|a\ell|}(|\phi_1^{(\ell)}|+|\phi_2^{(\ell)}|),$$

and

$$(2.15)\qquad |C_2|\leq\frac{2|a\ell|}{\mathcal{D}}\times(|(|a\ell|e^{-|a\ell|}+\sinh|a\ell|)\phi_1^{(\ell)}|+|(-|a\ell|e^{-|a\ell|}+\sinh|a\ell|)\phi_2^{(\ell)}|)$$
$$\leq 10|a\ell|e^{-|a\ell|}(|\phi_1^{(\ell)}|+|\phi_2^{(\ell)}|).$$

Similarly, we have

$$(2.16)\qquad |C_3+C_4| \le 14(|\phi_1^{(\ell)}|+|\phi_2^{(\ell)}|),$$
$$|C_4| \le 9(|\phi_1^{(\ell)}|+|\phi_2^{(\ell)}|).$$

Therefore we have by (2.10) and (2.14)–(2.16)

$$(2.17)\qquad \begin{aligned} &\int_{-1}^0 |u_1^{(\ell)}(x_1)|^2 dx_1 \\ &\le 4\biggl(e^{2|a\ell|}|C_1+C_2|^2\int_{-1}^0 e^{2|a\ell|x_1}dx_1 \\ &\quad + e^{2|a\ell|}|C_2|^2\int_{-1}^0 |x_1|^2 e^{2|a\ell|x_1}dx_1\biggr) \\ &\quad + 4\biggl(|C_3+C_4|^2\int_{-1}^0 e^{-2|a\ell|(x_1+1)}dx_1 \\ &\quad + |C_4|^2\int_{-1}^0 |x_1|^2 e^{-2|a\ell|(x_1+1)}dx_1\biggr) \\ &\le 2|a\ell|^{-1}e^{2|a\ell|}(|C_1+C_2|^2+|C_2|^2) \\ &\quad + 2|a\ell|^{-1}(|C_3+C_4|^2+|C_4|^2) \\ &\le C_5|a\ell|^{-1}(|\phi_1^{(\ell)}|^2+|\phi_2^{(\ell)}|^2) \end{aligned}$$

for some constant C_5. In the same way as (2.17), we have for $\alpha_1+\alpha_2 \le 2$

$$(2.18)\qquad \begin{aligned} &|a\ell|^{2\alpha_1}\int_{-1}^0 \left|\left(\frac{d}{dx_1}\right)^{\alpha_2} u_1^{(\ell)}\right|^2 dx_1 \\ &\le C_6|a\ell|^{2(\alpha_1+\alpha_2)-1}(|\phi_1^{(\ell)}|^2+|\phi_2^{(\ell)}|^2), \end{aligned}$$

where C_6 is independent of ℓ. Since $\phi \in \{H^{3/2}(\mathbf{T})\}^2$, we see from Parseval's identity that (2.18) implies

$$\|u_1\|_{H^2(\Omega)} \le C_7\|\phi\|_{\{H^{3/2}(\mathbf{T})\}^2},$$

which yields $u_1 \in H^2(\Omega)$. Using a similar argument to (2.17)–(2.18), we can prove (2.13) for u_2 and p. This completes the proof of Lemma 2.2. □

3. Proof of Theorem 1.1. For $\boldsymbol{v}, \boldsymbol{w} \in \{H^1(\Omega)\}^2$ and $\kappa \in \mathbf{R}$ we set

$$E_\kappa(\boldsymbol{v},\boldsymbol{w}) = 2(\varepsilon(\boldsymbol{v}),\varepsilon(\boldsymbol{w}))_{L^2(\Omega)} - \kappa(\nabla\boldsymbol{v},\nabla\boldsymbol{w})_{L^2(\Omega)},$$
$$E_\kappa(\boldsymbol{v}) = E_\kappa(\boldsymbol{v},\boldsymbol{v}) = 2\|\varepsilon(\boldsymbol{v})\|^2_{L^2(\Omega)} - \kappa\|\nabla\boldsymbol{v}\|^2_{L^2(\Omega)}.$$

By integration by parts we have the following

Lemma 3.1. *For* $\boldsymbol{v} \in {}_0H^1_\sigma(\Omega) \cap \{H^2(\Omega)\}^2$, $\boldsymbol{w} \in {}_0H^1_\sigma(\Omega)$ *and* $q \in H^1(\Omega)$, *it holds*

$$\begin{aligned} &(-(1-\kappa)\Delta\boldsymbol{v} + \nabla q,\ \boldsymbol{w})_{L^2(\Omega)} = E_\kappa(\boldsymbol{v},\boldsymbol{w}) \\ &\quad - \sum_{i,j=1}^2\int_{\partial\Omega}\left(2\varepsilon_{ij}(\boldsymbol{v}) - \kappa\frac{\partial v_j}{\partial x_i} - q\delta_{ij}\right)n_i\overline{w_j}\,ds, \end{aligned}$$

where δ_{ij} *is the* (i,j)*-element of the* 2×2 *unit matrix and* (n_1,n_2) *is the outward unit normal vector to the boundary* $\partial\Omega$.

Now we determine the best constant $K_{\max}$. By definition $2K_{\max}$ is equal to the supremum of $\kappa \in \mathbf{R}$ such that $E_\kappa(\boldsymbol{v}) \ge 0$ for all $\boldsymbol{v} \in {}_0H^1_\sigma(\Omega)$. Since each $\boldsymbol{v} \in {}_0H^1_\sigma(\Omega) \cap \{H^1_0(\Omega)\}^2$ satisfies $E_\kappa(\boldsymbol{v}) = (1-\kappa)\|\nabla\boldsymbol{v}\|^2_{L^2(\Omega)}$, we see that $2K_{\max} \le 1$, and hence

$$(3.1)\qquad 2K_{\max} = \sup\{\kappa < 1 : E_\kappa(\boldsymbol{v}) \ge 0\ \forall\ \boldsymbol{v} \in {}_0H^1_\sigma(\Omega)\}.$$

Let $\kappa < 1$. For an arbitrarily given $\boldsymbol{v} \in {}_0H^1_\sigma(\Omega) \cap \{H^2(\Omega)\}^2$ let $\{\boldsymbol{u}, p\}$ be the solution to (2.1)–(2.3) with $\phi(x_2) = \boldsymbol{u}(0,x_2)$; by Lemma 2.2 $\boldsymbol{u} \in {}_0H^1_\sigma(\Omega) \cap \{H^2(\Omega)\}^2$ and $p \in H^1(\Omega)$. Setting $\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{u} \in {}_0H^1_\sigma(\Omega) \cap \{H^1_0(\Omega)\}^2$ and applying Lemma 3.1 to $\{\boldsymbol{u}, p\}$ and $\boldsymbol{w}$, we have $E_\kappa(\boldsymbol{u},\boldsymbol{w}) = 0$, so that

$$\begin{aligned} E_\kappa(\boldsymbol{v}) &= E_\kappa(\boldsymbol{u}+\boldsymbol{w}) \\ &= E_\kappa(\boldsymbol{u},\boldsymbol{u}) + 2\,\mathrm{Re}\,E_\kappa(\boldsymbol{u},\boldsymbol{w}) + E_\kappa(\boldsymbol{w},\boldsymbol{w}) \\ &= E_\kappa(\boldsymbol{u}) + (1-\kappa)\|\nabla\boldsymbol{w}\|^2_{L^2(\Omega)} \\ &\ge E_\kappa(\boldsymbol{u}), \end{aligned}$$

where the last equality holds if and only if $\boldsymbol{v} = \boldsymbol{u}$. Since ${}_0H^1_\sigma(\Omega) \cap \{H^2(\Omega)\}^2$ is dense in ${}_0H^1_\sigma(\Omega)$, we see from the argument above that $2K_{\max}$ is equal to the supremum of $\kappa < 1$ such that $E_\kappa(\boldsymbol{u}) \ge 0$ for all $\boldsymbol{u} \in {}_0H^1_\sigma(\Omega) \cap \{H^2(\Omega)\}^2$ such that $\{\boldsymbol{u}, p\}$ is the solution to (2.1)–(2.3) for some $\phi \in \{H^{3/2}(\Omega)\}^2$ satisfying (2.8). For such $\{\boldsymbol{u}, p\}$, by Lemma 3.1 and the orthogonality of $\{\exp(ia\ell x_2) : \ell \in \mathbf{Z}\}$ in $L^2(\mathbf{T})$ we have

$$\begin{aligned} &E_\kappa(\boldsymbol{u}) \\ &= \frac{2\pi}{a}\Biggl\{\sum_{\ell\in\mathbf{Z}}\left((2-\kappa)\frac{du_1^{(\ell)}}{dx_1}(0) - p^{(\ell)}(0)\right)\overline{u_1^{(\ell)}(0)} \\ &\quad + \sum_{\ell\in\mathbf{Z}}\left((1-\kappa)\frac{du_2^{(\ell)}}{dx_1}(0) + ia\ell u_1^{(\ell)}(0)\right)\overline{u_2^{(\ell)}(0)}\Biggr\}. \end{aligned}$$

By (2.9), (2.10) and (2.12), this is rewritten in the form of inner product in $\mathbf{C}^2$:

$$E_\kappa(\boldsymbol{u}) = \sum_{\ell\in\mathbf{Z}}(M_\kappa(\ell)\phi^{(\ell)}, \phi^{(\ell)})_{\mathbf{C}^2}$$
$$\text{with } \phi^{(\ell)} = \begin{pmatrix}\phi_1^{(\ell)} \\ \phi_2^{(\ell)}\end{pmatrix},\ \phi_1^{(0)} = 0,$$

where

$$M_\kappa(0) = \frac{2\pi}{a}\begin{pmatrix}0 & 0 \\ 0 & 1-\kappa\end{pmatrix}$$

and for $\ell \ne 0$

$$M_\kappa(\ell) = \frac{2\pi}{a}\frac{1}{\mathcal{D}}\begin{pmatrix}f_5(a\ell,\kappa) & f_6(a\ell,\kappa) \\ \overline{f_6(a\ell,\kappa)} & f_7(a\ell,\kappa)\end{pmatrix}$$

with components

$$f_5(a\ell,\kappa) = 4(1-\kappa)|a\ell|(2|a\ell| + \sinh 2|a\ell|),$$
$$f_6(a\ell,\kappa) = 4(ia\ell)\{(2-\kappa)|a\ell|^2 - \kappa\sinh^2|a\ell|\},$$
$$f_7(a\ell,\kappa) = -4(1-\kappa)|a\ell|(2|a\ell| - \sinh 2|a\ell|).$$

Since $(M_\kappa(0)\phi^{(0)},\phi^{(0)})_{\mathbf{C}^2} = (1-\kappa)|\phi_2^{(0)}|^2 \geq 0$ for $\kappa < 1$, it follows from (3.1) that

$$2K_{\max} = \sup\{\kappa < 1 : M_\kappa(\ell) \geq O\ \forall\ell \neq 0\},$$

where $M_\kappa(\ell) \geq O$ signifies that $M_\kappa(\ell)$ is nonnegative definite.

Let $\kappa < 1$. For each $\ell \neq 0$ fixed, since the trace of $M_\kappa(\ell)$ is positive as easily verified, $M_\kappa(\ell) \geq O$ if and only if $\det M_\kappa(\ell) \geq 0$, or equivalently $\kappa \leq \kappa_0(a\ell)$ with

$$\kappa_0(a\ell) = \frac{2|a\ell|^2 + 4\sinh^2|a\ell| + 4}{|a\ell|^2 + 3\sinh^2|a\ell| + 4} - \frac{2\sqrt{\sinh^4|a\ell| + \sinh^2|a\ell| - |a\ell|^2}}{|a\ell|^2 + 3\sinh^2|a\ell| + 4}.$$

Thus it follows that

$$2K_{\max} = \sup\left(\bigcap_{\ell\neq 0}(-\infty,\kappa_0(a\ell)]\right) = \inf_{\ell\neq 0}\kappa_0(a\ell).$$

The graph of $\kappa_0(a\ell)$ drawn by Mathematica is shown in Fig. 1.

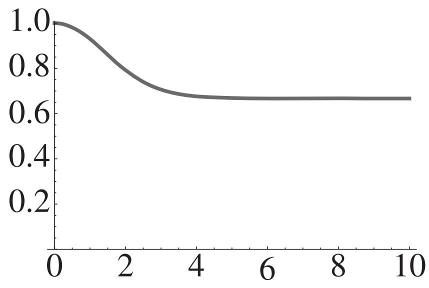

Fig. 1. The horizontal and vertical axes indicate $|a\ell|$ and $\kappa_0(a\ell)$, respectively.

We obtain from Lemma A.1 in appendix that

$$K_{\max} = \frac{1}{2}\inf_{\ell\neq 0}\kappa_0(a\ell) = \frac{1}{3}.$$

This completes the proof of Theorem 1.1. □

Appendix. In this section we prove the lemma below.

Lemma A.1. *The function*

$$g(x) = \frac{2x^2 + 4\sinh^2 x + 4 - 2\sqrt{\sinh^4 x + \sinh^2 x - x^2}}{x^2 + 3\sinh^2 x + 4}$$

satisfies

$$g(x) \geq \frac{2}{3} \quad \text{for any} \quad x \geq 0 \quad \text{and} \quad \lim_{x\to\infty} g(x) = \frac{2}{3}.$$

Moreover $g(x)$ is monotone decreasing for $x \geq 2$.

Proof. Since

$$\sqrt{\sinh^4 x + \sinh^2 x - x^2} \leq \sinh^2 x + \frac{2}{3}x^2 + \frac{2}{3},$$

we have

$$g(x) \geq \frac{\frac{2}{3}x^2 + 2\sinh^2 x + \frac{8}{3}}{x^2 + 3\sinh^2 x + 4} = \frac{2}{3}.$$

Differentiating $g(x)$, we have

$$\frac{d}{dx}g(x) = g_1(x)\{g_2(x) + g_3(x)g_4(x)\},$$

where

$$g_1(x) = (x^2 + 3\sinh^2 x + 4)^{-2},$$
$$g_2(x) = -2x(\sinh x)(x\cosh x - 2\sinh x) - 2(x^2-4)\sinh x\cosh x + 8x,$$
$$g_3(x) = (\sinh^4 x + \sinh^2 x - x^2)^{-1/2},$$
$$g_4(x) = -\{4x(\sinh^3 x)(x\cosh x - \sinh x) + (10\sinh^2 x + 8)(\sinh x\cosh x - x) + 14x^2\sinh x\cosh x + 2x^3\}.$$

By $x \geq 2$ and Taylor expansion, we have

$$x\cosh x - 2\sinh x \geq 0, \quad x^2 - 4 \geq 0,$$

and

$$\sinh x\cosh x - x \geq 0,$$

which yield

$$g_1(x) > 0,$$
$$g_2(x) \leq 8x,$$
$$g_3(x)g_4(x) \leq -14x^2\sinh x\cosh x(2\sinh^4 x)^{-1/2} \leq -14x.$$

Therefore,

$$g_1(x)\{g_2(x) + g_3(x)g_4(x)\} < 0$$

for $x \geq 2$. This completes the proof. □

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