On weakly transitive operators

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Abstract: In the present paper, transitivity of a linear operators acting on a reflexive Banach space with the weak topology is investigated. We show any bounded operator, transitive on an open bounded subset of X with the weak topology, is weakly hypercyclic.

Key words: Hypercyclic operator; weighted backward shift; weak topology.

1. Introduction. Let X be a separable, infinite-dimensional Banach space. If T is a bounded linear operator on X, the orbit of a vector $x \in X$ under T is defined by $orb(T, x) = \{T^n x : n \in \mathbf{N}\}$. The operator T is said to be hypercyclic if there exists a vector $x \in X$ whose orbit, is norm dense in X. The study of such operators on Banach space was initiated by Rolewicz [6], who showed that if B is the unilateral backward shift on $\ell^p(\mathbf{N})$, $1 \leq p < +\infty$, then the operator λB is hypercyclic for any scalar λ with $|\lambda| > 1$.

A generalization of hypercyclicity was proposed recently. It replaces the norm topology of X with the weak topology: T is said to be weakly hypercyclic if there is a vector $x \in X$ whose orbit is dense in the weak topology of X. Since the norm topology is strictly stronger than the weak topology, every hypercyclic operator is weakly hypercyclic but it is shown in [3] that weakly hypercyclic operators may fail to be hypercyclic. A weakly hypercyclic operator has many of the same properties as a hypercyclic operator. For example, its adjoint, has no eigenvalue [3] and every component of whose spectrum must intersect the unite circle [4]. Good sources of background information on weakly hypercyclic operators include [1–3] and [7].

On every separable Banach space, hypercyclicity is equivalent to transitivity, i.e., for every pair of nonempty, norm open sets (U, V), $T^n U \cap V \neq \emptyset$, for some integer $n \geq 0$. To prove it, Godefroy and Shapiro [5] used the Baire Category Theorem, which is not available in the weak topology. In the present paper, we consider the transitivity of linear operators on Banach spaces with separable dual with the weak topology. We prove a bounded linear operator, weakly transitive on an open bounded subset of X, must be weakly hypercyclic on X. However, we show there is some example of weakly hypercyclic operator on $\ell^2(\mathbf{Z})$ without being weakly transitive on any bounded open set.

2. Main result. In what follows, X will be a Banach space with separable dual, $\mathcal{L}(X)$ denotes the algebra of all bounded linear operators on X. For any integer $N > 1$, use $B_N(X)$ for denote the open N-ball of X, i.e., the set of all vectors $x \in X$ with $\|x\| < N$ and $\bar{B}_N(X)$ for the close N-ball of X as well.

Definition 2.1. Let $T \in \mathcal{L}(X)$ and D be a nonempty subset of X. The operator T is said to be weakly transitive on D whenever for each nonempty norm open set U and nonempty weakly open set V in X that $U \cap D \neq \emptyset$ and $V \cap D \neq \emptyset$, there exists some integer $n \geq 0$ such that

$$T^n(U \cap D) \cap V \cap D \neq \emptyset. \tag{1}$$

Note that norm transitivity implies weak transitivity on any open subset of X. Also in Lemma 2.4 below, we prove that if D is norm open, T is surjective and condition (1) is true for some positive integer n, then it must be true for infinitely many positive integers n.

Remember that if $T \in \mathcal{L}(X)$, the orbit of a set $D \subseteq X$ under T is defined by

$$\mathrm{orb}(T, D) = \bigcup_{n \in \mathbf{N}} T^n D.$$

Theorem 2.2. *Let $T \in \mathcal{L}(X)$ and it is weakly transitive on a bounded open subset of X whose orbit under T is weakly dense in X, then T is weakly hypercyclic on X.*

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Proof. Let D be a bounded open subset of X for which $\mathrm{orb}(T, D)$ is weakly dense in X and let $D \subseteq B_N$ for some positive integer N. Since X has separable dual and every bounded subset of X is weakly metrizable, every closed balls in X are first countable in the weak topology. Let E be a countable dense subset of X. For each $x \in E$, let

$$\{W_{p,q}(x) \cap \bar{B}_p\}_{q \in \mathbf{N}}$$

be a countable basis of weakly neighborhoods of x in $\bar{B}_p$; where $W_{p,q}(x)$ is a weakly open set in X containing x. Consider the index set

$$I_{p,q}(x) = \{r \in \mathbf{N} : T^{-r}W_{p,q}(x) \cap D \neq \emptyset\} \quad (x \in E, p, q \in \mathbf{N})$$

which is nonempty by the weak density of $\mathrm{orb}(T, D)$. Then each of the sets

$$A(x, p, q, r) = \bigcup_{n \in \mathbf{N}} T^{-n}(T^{-r}W_{p,q}(x) \cap D)$$

is norm open in X and norm dense in $\bar{D}$ for $r \in I_{p,q}(x)$. To prove the norm density, let $x \in E$, $p, q \in \mathbf{N}$, $r \in I_{p,q}(x)$ and let $U \cap \bar{D}$ be a nonempty norm open set in $\bar{D}$. Then

$$U \cap D \neq \emptyset, \quad T^{-r}W_{p,q}(x) \cap D \neq \emptyset \text{ (because } r \in I_{p,q}(x))$$

and $T^{-r}W_{p,q}(x)$ is weakly open set in X. By weak transitivity of T on D, we find that

$$T^{-n}(T^{-r}W_{p,q}(x) \cap D) \cap U \cap D \neq \emptyset$$

for some integer $n \geq 1$ which follows that $A(x, p, q, r)$ is norm dense in $\bar{D}$. Now the Baire Category Theorem for $\bar{D}$ implies that the set

$$W_{hy}(T) = \bigcap_{x \in E} \bigcap_{p,q \in \mathbf{N}} \bigcap_{r \in I_{p,q}(x)} A(x, p, q, r)$$

is norm dense in $\bar{D}$. We claim every element of $W_{hy}(T)$ is a weakly hypercyclic vector for T. For this, let $z \in W_{hy}(T)$ and W be an arbitrary weakly open set in X. Then there is a weakly neighborhood $W(x)$ of some $x \in E$ contained in W. Using the weak density of $\mathrm{orb}(T, D)$ to get, $T^rD \cap W(x) \neq \emptyset$ for some integer $r \geq 1$. Choose a positive integer $p > N\|T\|^r$ in such a way that the set $T^rD \cap W(x) \cap B_p$ is nonempty. Thus

$$T^{-r}(W(x) \cap \bar{B}_p) \cap D \neq \emptyset.$$

The first countability of $\bar{B}_p$ implies that for some $q \in \mathbf{N}$, $T^{-r}(W_{p,q}(x) \cap \bar{B}_p) \cap D$ is nonempty. This yields that $T^{-r}W_{p,q}(x) \cap D$ is also nonempty and hence $r \in I_{p,q}(x)$. Since $z \in A(x, p, q, r)$, there is some integer $n \geq 1$, such that $T^{n+r}z \in W_{p,q}(x)$ and $T^nz \in D \subseteq B_N$. But $\|T^{r+n}z\| \leq N\|T\|^r < p$ and $T^{r+n}z \in W_{p,q}(x) \cap \bar{B}_p$ which is a subset of $W(x) \cap \bar{B}_p$, deducing $T^{r+n}z \in W$. Therefore, z is a weakly hypercyclic vector for T and T is weakly hypercyclic. □

Assume that T is weakly transitive on B_1. If U is a norm open and W is a weakly open set in X then for some integer $m > 1$ both $m^{-1}U \cap B_1$ and $m^{-1}W \cap B_1$ are nonempty. Hence,

$$T^n(m^{-1}U \cap B_1) \cap m^{-1}W \cap B_1 \neq \emptyset$$

and so $T^n(U) \cap W \neq \emptyset$ for some $n \in \mathbf{N}$. Therefore, $\mathrm{orb}(T, U)$ is weakly dense in X for any norm open set U and in particular for $U = B_1$. This led to the following corollary

Corollary 2.3. *If $T \in \mathcal{L}(X)$ is weakly transitive on B_1 then it is weakly hypercyclic on X.*

Since the set of weakly hypercyclic vectors for T is norm dense in X, a necessary condition for weakly hypercyclic operators in terms of open sets can be given. In fact, if $T \in \mathcal{L}(X)$ is a weakly hypercyclic operator, then for every nonempty norm open set U and weakly open set W, there is a positive integer r such that $T^r(U) \cap W \neq \emptyset$. This means that the converse of the above theorem is true for unbounded set $D = X$ while the following example shows it can not be true for any bounded open set. Before state it we need the following simple lemma:

Lemma 2.4. *If $T \in \mathcal{L}(X)$ is a surjective operator satisfying the hypotheses of Definition* 2.1 *for a norm open set D, i.e, weakly transitive on norm open set D, then condition* (1) *holds for infinitely many positive integers n.*

Proof. Let U and V be as in Definition 2.1 then by weak transitivity of T on D, there is a positive integer n_1 such that

$$T^{n_1}(U \cap D) \cap V \cap D \neq \emptyset. \tag{2}$$

By the open mapping Theorem $U_1 := T^{n_1}(U \cap D)$, is norm open and by (2), $U_1 \cap D \neq \emptyset$. Again by weak transitivity,

$$T^{n_2}(U_1 \cap D) \cap V \cap D \neq \emptyset$$

which implies that

$$T^{n_2}(U_1) \cap V \cap D \neq \emptyset$$

and so

$$T^{n_1+n_2}(U \cap D) \cap V \cap D \neq \emptyset.$$

Repeating this method, we construct a sequence of positive integers $\{n_k\}$ such that

$$T^{n_1+n_2+\dots n_k}(U\cap D)\cap V\cap D\neq\emptyset$$

and this completes the proof. □

Example 2.5. In [3] Chan and Sanders have shown that the bilateral weighted shift T with the weight sequence $w_n=2$ if $n\geq 0$, $w_n=1$ if $n<0$ acting on $X:=\ell^2(\mathbf{Z})$ is weakly hypercyclic. We prove the operator T is not weakly transitive on any bounded open subset D of X. Let $\{e_n\}_{n\in\mathbf{Z}}$ be the standard base for X.

Fix a bounded open subset D and let $R>0$ be such that $D\subseteq B_R$. Also, let $\varepsilon>0$ be such that D contains a ball of radius 2ε centered at a point $a\in D$. Choose a positive integer q such that $2^{q-1}\varepsilon>R$, and let $U=\{x:|<x,e_q>|>\varepsilon/2\}$. Then, U is norm open (weakly open) in X and we claim $D\cap U\neq\emptyset$ and $T^nU\cap B_R=\emptyset$ for all $n\geq q$. Indeed, if $D\cap U=\emptyset$, then consider two vectors $x:=a+\varepsilon e_q$ and $y:=a-\varepsilon e_q$ in D. Hence, x,y do not belong to U and so $|<x-y,e_q>|\leq\varepsilon$ while by definition $|<x-y,e_q>|=2\varepsilon$. This contradiction says that $D\cap U\neq\emptyset$.

To see the next part, fix an integer $n\geq q$, it is enough to prove

$$T^n(U)\cap B_R=\emptyset.$$

Assume that $x=\sum_{j\in\mathbf{Z}}\lambda_j e_j$ is any vector in U. Then

$$\|T^nx\|^2=\sum_{j\in\mathbf{Z}}|\lambda_j|^2(w_jw_{j-1}\dots w_{j-n+1})^2$$
$$\geq|\lambda_q|^2(w_qw_{q-1}\dots w_{q-n+1})^2$$
$$=|\lambda_q|^2 4^q>\varepsilon^2 4^{q-1}>R^2.$$

It follows that $T^nx\notin B_R$ for all $n\geq q$ and by the preceding lemma the surjective operator T is not weakly hypercyclic on D. □

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