On the linearity of some sets of sequences defined by L_p-functions and L_1-functions determining ℓ_1

Dedicated to Prof. Shinnosuke Oharu on his 70th birthday

By Gen NAKAMURA*) and Kazuo HASHIMOTO**)

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Abstract: In this paper, we discuss the linearity of a sequence space $\Lambda_p(f)$, and the conditions such that $\ell_1 = \Lambda_1(f)$ holds are characterized in term of the essential bounded variation of $f \in L_1(\mathbf{R})$, i.e. $\ell_1 = \Lambda_1(f)$ if and only if $f \in BV(\mathbf{R})$.

Key words: Sequence space; linearity; essential bounded variation; Sobolev space.

1. Introduction. Let $f(\neq 0)$ be an L_p-function defined on the real line $\mathbf{R}$ and assume $1 \leq p < +\infty$. For a sequence of real numbers $\boldsymbol{a} = (a_n) \in \mathbf{R}^\infty$, define

$$\Psi_p(\boldsymbol{a}; f) := \left(\sum_k \int_{\mathbf{R}} |f(x - a_k) - f(x)|^p dx\right)^{1/p}$$

and

$$\Lambda_p(f) := \{\boldsymbol{a} \in \mathbf{R}^\infty : \Psi_p(\boldsymbol{a}; f) < +\infty\}.$$

The following results are known (cf. [1]):

- For every $\boldsymbol{a} = (a_n) \in \mathbf{R}^\infty$, $\Psi_p(|\boldsymbol{a}|; f) = \Psi_p(\boldsymbol{a}; f)$, where $|\boldsymbol{a}| = (|a_n|)$;
- $\Psi_p(\boldsymbol{a} - \boldsymbol{b}; f) \leq \Psi_p(\boldsymbol{a}; f) + \Psi_p(\boldsymbol{b}; f)$ for every $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{R}^\infty$, i.e, the sets $\Lambda_p(f)$ are additive subgroups of $\mathbf{R}^\infty$.

Let $W^{1,p}(\mathbf{R})$ be a Sobolev space, i.e, $f \in W^{1,p}(\mathbf{R})$ if and only if $f \in L_p(\mathbf{R})$ and the derivative Df of f in the sense of distribution belongs to $L_p(\mathbf{R})$. In particular, if $f \in L^1(\mathbf{R})$ and Df is a Radon measure of bounded variation on $\mathbf{R}$, f called a function of bounded variation. The class of all such functions will be denoted by $BV(\mathbf{R})$. Thus, $f \in BV(\mathbf{R})$ if and only if there is a Radon measure μ defined in $\mathbf{R}$ such that $|\mu|(\mathbf{R}) < +\infty$ and

$$\int_{\mathbf{R}} f\varphi' dx = -\int \varphi d\mu, \ \ \varphi \in C_0^\infty(\mathbf{R}),$$

where, $|Df|(\mathbf{R}) = |\mu|(\mathbf{R})$ means the total variation of μ.

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*) Matsue College of Technology, 14-4 Nishi-ikuma, Matsue, Shimane 690-8518, Japan.
**) Hiroshima Jogakuin University, 4-13-1 Ushita Higashi Higashi-ku, Hiroshima 732-0063, Japan.

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It is obvious that a function f on $\mathbf{R}$ is absolutely continuous and the derivative f' is in $L_1(\mathbf{R})$, then f is of bounded variation. In particular, $W^{1,1}(\mathbf{R}) \subset BV(\mathbf{R})$ (see [3]).

In [1], A. Honda, Y. Okazaki and H. Sato provided the following results:

(i) ([1, Theorem 1, Theorem 2]) If $1 \leq p < +\infty$ and $f(\neq 0) \in L_p(\mathbf{R})$, then $\Lambda_p(f) \subset \ell_p$. In particular, $f \in W^{1,p}(\mathbf{R})$ implies $\ell_p = \Lambda_p(f)$.

(ii) ([1, Corollary 4]) If $1 < p < +\infty$ and $f(\neq 0) \in L_p(\mathbf{R})$, then $\ell_p = \Lambda_p(f)$ if and only if $f \in W^{1,p}(\mathbf{R})$.

In (ii), we should note that the case of $p = 1$ is excluded. In this paper, we discuss the linearity of the space $\Lambda_p(f)$, and the conditions such that $\ell_1 = \Lambda_1(f)$ holds are characterized in term of the essential bounded variation of $f \in L_1(\mathbf{R})$, i.e. $\ell_1 = \Lambda_1(f)$ if and only if $f \in BV(\mathbf{R})$ (Theorem 3.5).

2. The linearity of $\Lambda_p(f)$. We first give necessary and sufficient conditions for the linearity of $\Lambda_p(f)$.

Theorem 2.1. *Let $1 \leq p < +\infty$ and $f(\neq 0) \in L_p(\mathbf{R})$. Then the following are equivalent*:

(i) *$\Lambda_p(f)$ is a linear subspace of $\mathbf{R}^\infty$*;

(ii) *For any $0 \leq k \leq 1$, there exists a constant $C(k) > 0$ such that*

$$\int_{\mathbf{R}} |f(x - ka) - f(x)|^p dx \leq C(k) \int_{\mathbf{R}} |f(x - a) - f(x)|^p dx, \ \forall a > 0;$$

(iii) *There exits a constant $C > 0$ such that*

$$\int_{\mathbf{R}} |f(x - ka) - f(x)|^p dx \leq C \int_{\mathbf{R}} |f(x - a) - f(x)|^p dx, \ 0 \leq \forall k \leq 1, \ \forall a > 0.$$

Proof. Since $\Lambda_p(f)$ is additive as mentioned in the introduction, it suffices to show that $\alpha \in \mathbf{R}$ and $\boldsymbol{a} \in \Lambda_p(f)$ implies $\alpha\boldsymbol{a} \in \Lambda_p(f)$. Condition (ii) means that $\boldsymbol{a} \in \Lambda_p(f)$ implies $\alpha\boldsymbol{a} \in \Lambda_p(f)$ for all $0 \leq \alpha \leq 1$. Since $\Lambda_p(f)$ is an additive group, we see that $\alpha \in \mathbf{R}$ and $\boldsymbol{a} \in \Lambda_p(f)$ implies $\alpha\boldsymbol{a} \in \Lambda_p(f)$. Thus we see that $\Lambda_p(f)$ is linear.

Conversely, suppose that (ii) does not hold. Then there exists $0 < k_0 \leq 1$ such that for any natural number n, we can take $a_n > 0$ such that

$$\int_{\mathbf{R}} |f(x - k_0 a_n) - f(x)|^p dx > 3^n \int_{\mathbf{R}} |f(x - a_n) - f(x)|^p dx. \tag{2.1}$$

On the other hand, we have

$$\begin{aligned}\int_{\mathbf{R}} |f(x - k_0 a_n) - f(x)|^p dx &\leq \int_{\mathbf{R}} (|f(x - k_0 a_n)| + |f(x)|)^p dx \\ &\leq 2^{p-1}\left\{\int_{\mathbf{R}} |f(x - k_0 a_n)|^p dx + \int_{\mathbf{R}} |f(x)|^p dx\right\} \\ &= 2^p \|f\|_{L_p}^p.\end{aligned} \tag{2.2}$$

Since $f(\neq 0) \in L_p$, we have

$$\|f(\cdot - a_n) - f(\cdot)\|_{L_p} \neq 0.$$

We have from (2.1) and (2.2) that

$$0 < \int_{\mathbf{R}} |f(x - a_n) - f(x)|^p dx < \frac{2^p}{3^n} \|f\|_{L_p}^p < 2^{p-n} \|f\|_{L_p}^p.$$

Also, for each n, let $N(n)$ be the maximum of a natural number N such that the following inequality holds

$$N \int_{\mathbf{R}} |f(x - a_n) - f(x)|^p dx \leq 2^{p-n} \|f\|_{L_p}^p. \tag{2.3}$$

Form the maximality of $N(n)$ we have

$$\begin{aligned}2^{p-n}\|f\|_{L_p}^p &< (N(n) + 1) \int_{\mathbf{R}} |f(x - a_n) - f(x)|^p dx \\ &\leq 2N(n) \int_{\mathbf{R}} |f(x - a_n) - f(x)|^p dx,\end{aligned}$$

and hence form this equality and (2.1), we have

$$\begin{aligned}2^{p-n-1}\|f\|_{L_p}^p / N(n) &< \int_{\mathbf{R}} |f(x - a_n) - f(x)|^p dx \\ &< \frac{1}{3^n} \int_{\mathbf{R}} |f(x - k_0 a_n) - f(x)|^p dx.\end{aligned}$$

Thus we have

$$(3/2)^n 2^{p-1} \|f\|_{L_p}^p < N(n) \int_{\mathbf{R}} |f(x - k_0 a_n) - f(x)|^p dx. \tag{2.4}$$

Let $N(0) = 0$, and define a sequence $\boldsymbol{b} = (b_n)$ in the following way

$$b_j = a_k,\ 1 + \sum_{i=0}^{k-1} N(i) \leq j \leq \sum_{i=0}^{k} N(i),$$

where $j, k = 1, 2, 3, \cdots$. Then, from (2.3) we have

$$\begin{aligned}\sum_{n=1}^{\infty} \int_{\mathbf{R}} |f(x - b_n) - f(x)|^p dx &= \sum_{n=1}^{\infty} N(n) \int_{\mathbf{R}} |f(x - a_n) - f(x)|^p dx. \\ &\leq \|f\|_{L_p}^p \sum_{n=1}^{\infty} 2^{p-n} < +\infty.\end{aligned}$$

Hence $\boldsymbol{b} \in \Lambda_p(f)$.

On the other hand, using (2.4) we have

$$\begin{aligned}\sum_{n=1}^{\infty} \int_{\mathbf{R}} |f(x - k_0 b_n) - f(x)|^p dx &= \sum_{n=1}^{\infty} N(n) \int_{\mathbf{R}} |f(x - k_0 a_n) - f(x)|^p dx \\ &\geq \sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n 2^{p-1} \|f\|_{L_p}^p = +\infty.\end{aligned}$$

This means $k_0 \boldsymbol{b} \notin \Lambda_p(f)$. Hence we have (i) $\Leftrightarrow$ (ii).

Next, we show that (ii) $\Leftrightarrow$ (iii). Since it is obvious that (iii) $\Rightarrow$ (ii), it is sufficient to prove that (ii) $\Rightarrow$ (iii). Put

$$M(k) = \sup_{a>0} \frac{\|f(\cdot - ka) - f(\cdot)\|_{L_p}}{\|f(\cdot - a) - f(\cdot)\|_{L_p}}$$

for $k \in \mathbf{R}$. Then for $k_1,\ k_2 \in \mathbf{R}$, we have the following inequality

$$\begin{aligned}M(k_1 + k_2) &= \sup_{a>0} \frac{\|f(\cdot - (k_1 + k_2)a) - f(\cdot)\|_{L_p}}{\|f(\cdot - a) - f(\cdot)\|_{L_p}} \\ &\leq \sup_{a>0} \frac{\|f(\cdot - k_1 a) - f(\cdot)\|_{L_p} + \|f(\cdot - k_2 a) - f(\cdot)\|_{L_p}}{\|f(\cdot - a) - f(\cdot)\|_{L_p}} \\ &\leq M(k_1) + M(k_2).\end{aligned}$$

Now, suppose that (iii) does not hold, and thus $\sup_{0 \leq k \leq 1} M(k) = \infty$. Let (k_n) be a sequence in $[0, 1]$ such that $M(k_n) \to \infty$ and $k_n \to k_0$ for some $k_0 \in [0, 1]$.

For every $a \in [0, 1]$, put $a_n = k_n - k_0 + a$ ($n = 1, 2, 3, \cdots$), then

$$\begin{aligned} M(k_n) &= M(k_0 - a + a_n) \\ &\le M(k_0 - a) + M(a_n) \\ &= M(|k_0 - a|) + M(a_n). \end{aligned}$$

Since $|k_0 - a| \in [0,1]$ by (ii), $M(|k_0 - a|) < \infty$. Thus $M(a_n) \to \infty$ and $a_n \to a$ as $n \to \infty$. Consequently, for every $n \in \mathbf{N}$, put

$$L_n = \{x \in [0,1] : M(x) \le n\},$$

then it is easily verified that each L_n is nowhere dense and

$$\bigcup_{n=1}^{\infty} L_n = [0,1],$$

which contradicts the Baire category theorem. Thus, (ii) $\Rightarrow$ (iii) holds. □

The following theorem have already been proved by [2], but we give an alternative proof in this paper.

Theorem 2.2. *Let* $f \in L^p(\mathbf{R})$, $1 \le p < \infty$. *If there exists a countable partition* $(a_i)_{-\infty}^{\infty}$ *on* $\mathbf{R}$ *satisfying the following conditions*:

(1) $a_i < a_{i+1}$ *and* $\lim_{i\to\pm\infty} a_i = \pm\infty$;

(2) $\inf_i (a_{i+1} - a_i) > 0$;

(3) f *is monotone on* (a_i, a_{i+1}).

Then $\Lambda_p(f)$ *is linear.*

Proof. In what follows, let

$$\varepsilon = (\inf_{i\in\mathbf{Z}} |a_{i+1} - a_i|)/3 > 0.$$

Then, for every $0 < b < a < \varepsilon$, $x \in \mathbf{R}$, we have

$$\begin{aligned} (2.5) \qquad & |f(x-b) - f(x)|^p \\ &\le 2^{p-1}(|f(x-b) - f(x-a-b)|^p \\ &\quad + |f(x-a) - f(x)|^p \\ &\quad + |f(x-b) - f(x+a-b)|^p \\ &\quad + |f(x+a) - f(x)|^p). \end{aligned}$$

To show this, put

$$I_1 = [x-a-b, x-b],\ I_2 = [x, x+a],\ I_3 = [x-a-b, x+a].$$

Then it is obvious that I_1, $I_2 \subset I_3$ and $I_1 \cap I_2 = \emptyset$. Moreover, since the length of the interval I_3 is $2a + b$ and less than $3\varepsilon(\le \inf_{i\in\mathbf{Z}} |a_{i+1} - a_i|)$, the number of elements of $\{i : a_i \in I_3\}$ is at most single. Hence either of the following holds

(a) $\{i : a_i \in I_1\} = \emptyset$;

(b) $\{i : a_i \in I_2\} = \emptyset$.

CASE (a): By hypothesis, sinse f is monotone on $I_1 = [x-a-b, x-b]$, we see that $f(x-a-b) \le f(x-a) \le f(x-b)$ or $f(x-a-b) \ge f(x-a) \ge f(x-b)$, and so

$$\begin{aligned} & |f(x-b) - f(x)| \\ &\le |f(x-b) - f(x-a)| + |f(x-a) - f(x)| \\ &\le |f(x-b) - f(x-a-b)| + |f(x-a) - f(x)|. \end{aligned}$$

Hence

$$\begin{aligned} & |f(x-b) - f(x)|^p \\ &\le 2^{p-1}(|f(x-b) - f(x-a-b)|^p + \\ & |f(x-a) - f(x)|^p). \end{aligned}$$

CASE (b): By hypothesis, since f is monotone on $I_2 = [x, x+a]$, we have that either $f(x) \le f(x+a-b) \le f(x+a)$ or $f(x) \ge f(x+a-b) \ge f(x+a)$ holds, and so

$$\begin{aligned} & |f(x-b) - f(x)| \\ &\le |f(x-b) - f(x+a-b)| + |f(x+a-b) - f(x)| \\ &\le |f(x-b) - f(x+a-b)| + |f(x+a) - f(x)|. \end{aligned}$$

Consequently, we have

$$\begin{aligned} & |f(x-b) - f(x)|^p \\ &\le 2^{p-1}(|f(x-b) - f(x+a-b)|^p + \\ & |f(x+a) - f(x)|^p). \end{aligned}$$

Thus we see that (2.5) holds. Finally, to show that the statement (iii) of Theorem 2.1 holds, let $0 < k < 1$, $a > 0$, and so $0 < ka < a$.

Now we consider the two case of $a < \varepsilon$ or $a \ge \varepsilon$.

First, suppose that $a < \varepsilon$. Put $b = ka$ in (2.5), then by $0 < ka < a < \varepsilon$ we see

$$\begin{aligned} & |f(x-ka) - f(x)|^p \\ &\le 2^{p-1}(|f(x-ka) - f(x-a-ka)|^p \\ &\quad + |f(x-a) - f(x)|^p \\ &\quad + |f(x-ka) - f(x+a-ka)|^p \\ &\quad + |f(x+a) - f(x)|^p), \end{aligned}$$

and so

$$\begin{aligned} & \|f(\cdot - ka) - f(\cdot)\|_p^p \\ &\le 2^{p-1}(\|f(\cdot - ka) - f(\cdot - a - ka)\|_p^p \\ &\quad + \|f(\cdot - a) - f(\cdot)\|_p^p \\ &\quad + \|f(\cdot - ka) - f(\cdot + a - ka)\|_p^p \\ &\quad + \|f(\cdot + a) - f(\cdot)\|_p^p) \\ &= 2^{p+1}\|f(\cdot - a) - f(\cdot)\|_p^p. \end{aligned}$$

Next suppose that $a \ge \varepsilon$, put

$$c = \inf_{\alpha \geq \varepsilon} \|f(\cdot - \alpha) - f(\cdot)\|_p,$$

then $c > 0$ holds. In deed, a function $\|f(\cdot - \alpha) - f(\cdot)\|_p$ is positive and continuous with respect to $\alpha > 0$ and

$$\lim_{\alpha \to \infty} \|f(\cdot - \alpha) - f(\cdot)\|_p = 2\|f\|_p > 0.$$

Thus we see that $c > 0$.

We now observe that

$$\frac{\|f(\cdot - ka) - f(\cdot)\|_p}{\|f(\cdot - a) - f(\cdot)\|_p} \leq \frac{\|f(\cdot - ka)\|_p + \|f\|_p}{c} = \frac{2\|f\|_p}{c}.$$

Then we have

$$\|f(\cdot - ka) - f(\cdot)\|_p^p \leq \left(\frac{2\|f\|_p}{c}\right)^p \|f(\cdot - a) - f(\cdot)\|_p^p.$$

Put $C = \max\{2^{p+1}, (\frac{2\|f\|_p}{c})^p\} > 0$, we conclude that

$$\|f(\cdot - ka) - f(\cdot)\|_p^p \leq C\|f(\cdot - a) - f(\cdot)\|_p^p \quad \text{for } 0 \leq k \leq 1,\ a > 0.$$

Thus we see that Theorem 2.1(iii) holds, and that $\Lambda_p(f)$ is a linear subspace in $\mathbf{R}^\infty$. □

Here, we give examples without the proof such that each $\Lambda_p(f)$ is not a linear space.

Example 3. $f_0 \in C_0(\mathbf{R})(\neq 0)$, $\mathrm{supp} f_0 \subset [0, \pi]$. For m and $n \in \mathbf{N}$, we define $f_{m,n} \in C(\mathbf{R})$ by

$$f_{m,n}(x) = 1 + \frac{1}{m}\sin(nx).$$

Then there exist subsequences $\{m_i\}$ and $\{n_i\}$ satisfying the following conditions (i) and (ii):

(i) $f(x) = \lim_{j\to\infty} f_0(x) \prod_{i=1}^{j} f_{m_i,n_i}(x)$ (uniformly on $\mathbf{R}$).

(ii) $\lim_{i\to\infty} \frac{\int_{-\infty}^{\infty} |f(x - \frac{\pi}{n_i}) - f(x)|^p dx}{\int_{-\infty}^{\infty} |f(x - \frac{2\pi}{n_i}) - f(x)|^p dx} = \infty.$

We can show that (i) implies $f \in C_0(\mathbf{R}) \subset L_p(\mathbf{R})$ and (ii) implies that f does not satisfy Theorem 2.1(ii). Thus we see that $\Lambda_p(f)$ is not a linear subspace in $\mathbf{R}^\infty$.

Next we give an example of a more smooth function f such that $\Lambda_p(f)$ is not linear.

Example 4. Let $1 \leq p < \infty$. Then there exists an function $f \in L_p(\mathbf{R})$ such that:

(i) $f \in C^\infty(\mathbf{R}) \cap L_p(\mathbf{R})$ and $f(x) > 0$ $(x \in \mathbf{R})$;

(ii) the number of x satisfying $f'(x) = 0$ on every subinterval I of $\mathbf{R}$ is at most countable;

(iii) $\Lambda_p(f)$ is not a linear subspace of $\mathbf{R}^\infty$.

In fact, we can construct f as follows: Let

$$\rho(x) = \begin{cases} e^{-\frac{1}{1-x^2}} & (-1 < x < 1) \\ 0 & |x| \geq 1. \end{cases}$$

Then $\rho \in C_0^\infty(\mathbf{R})$ and $\mathrm{supp}\,\rho = [-1.1]$. Moreover, for all $n \in \mathbf{N}$, let $\rho_n(x) = \rho(6(x - n + 1/2))$, then we have $\mathrm{supp}\,\rho_n = [n - 2/3, n - 1/3]$ and $0 \leq \rho_n(x) \leq 1/e$. For every subsequence $(n_k)_{k=1}^\infty$ of the natural number, let

$$f(x) = \begin{cases} e^{-x^2} & (x < 0) \\ e^{-x^2}(1 + \rho_k(x)\sin n_k\pi x) & (k - 1 \leq x < k). \end{cases}$$

Then, the above conditions (i) and (ii) hold. On the other hand, choose a sequence (n_k) so that n_k is a multiple of n_{k-1} for each $k \in \mathbf{N}$ and

$$\lim_{k\to\infty} e^{k^2}\frac{n_k}{n_{k-1}} = \infty$$

holds (for example, $n_k = (k!)!$). Then we have

$$\lim_{k\to\infty} \frac{\|f(\cdot - 1/n_k) - f(\cdot)\|_p}{\|f(\cdot - 2/n_k) - f(\cdot)\|_p} = \infty.$$

Let $a/2 = 1/n_k$, then we can not take a constant C such that

$$\|f(\cdot - a/2) - f(\cdot)\|_p^p \leq C\|f(\cdot - a) - f(\cdot)\|_p^p$$

for all $a > 0$. Hence, we see from Theorem 2.1 that $\Lambda_p(f)$ is not a linear subspace of $\mathbf{R}^\infty$.

Remark. We should note that example 2 means that condition (2) of Theorem 2.2 is essential.

3. $\ell_1 = \Lambda_1(f)$.

Let $f \in L_1(\mathbf{R})$. We define a subset D_f of $\mathbf{R}$ by

$$D_f = \left\{x \in \mathbf{R} : \lim_{h\to 0}\tfrac{1}{h}\int_x^{x+h} |f(t) - f(x)|\,dt = 0\right\}.$$

It is well known that the Lebesgue measure of $\mathbf{R} \setminus D_f$ is zero.

Let $f : \mathbf{R} \to \mathbf{R}$. The *essential variation* $\mathrm{ess}\,V(f)$ is defined as

$$\mathrm{ess}\,V(f) = \sup\left\{\sum_{i=1}^{k} |f(x_i) - f(x_{i-1})|; x_0 < \cdots < x_k,\ x_i \in D_f\right\}.$$

Theorem 3.1. *Let* $f \in L_1(\mathbf{R})$. *Then we have*

$$\lim_{h\to 0}\int_{\mathbf{R}} \left|\frac{f(x-h) - f(x)}{h}\right| dx = \mathrm{ess}\,V(f)$$

Proof. Let $(x_k)_{k=1}^n$ be a finite sequence of elements of D_f such that $a_1 < a_2 < \cdots < a_n$. Then for $h \neq 0$,

$$\int_{\mathbf{R}} \left| \frac{f(x-h)-f(x)}{h} \right| dx \geq \frac{1}{|h|} \sum_{k=1}^{n-1} \left| \int_{a_k}^{a_{k+1}} f(x-h) - f(x)\, dx \right| = \sum_{k=1}^{n-1} \left| \frac{1}{h} \int_{a_k-h}^{a_k} f(x)\,dx - \frac{1}{h} \int_{a_{k+1}-h}^{a_{k+1}} f(x)\,dx \right|.$$

Hence

$$\liminf_{h\to 0} \int_{\mathbf{R}} \left| \frac{f(x-h)-f(x)}{h} \right| dx \geq \sum_{k=1}^{n-1} |f(a_k) - f(a_{k+1})|.$$

Since (x_k) is arbitrary, we have

$$\liminf_{h\to 0} \int_{\mathbf{R}} \left| \frac{f(x-h)-f(x)}{h} \right| dx \geq \operatorname{ess} V(f).$$

To show the converse inequality, It suffices to show that the statement holds for $h > 0$.

$$\int_{\mathbf{R}} \left| \frac{f(x-h)-f(x)}{h} \right| dx = \frac{1}{h} \sum_{k=-\infty}^{\infty} \int_0^h |f(x+(k+1)h) - f(x+kh)|\,dx = \frac{1}{h} \int_0^h \sum_{k=-\infty}^{\infty} |f(x+(k+1)h) - f(x+kh)|\,dx.$$

We should note that the Lebesgue measure of

$$\bigcup_{k=-\infty}^{\infty} \{(\mathbf{R} \setminus D_f) - kh\}$$

is zero. Let $x \notin \bigcup_{k=-\infty}^{\infty} \{(\mathbf{R} \setminus D_f) - kh\}$, then we have $x + kh \in D_f$ for every $k \in \mathbf{Z}$.

$$\sum_{k=-\infty}^{\infty} |f(x+(k+1)h) - f(x+kh)| \leq \operatorname{ess} V(f).$$

Thus we have

$$\int_{\mathbf{R}} \left| \frac{f(x-h)-f(x)}{h} \right| dx \leq \operatorname{ess} V(f), \ h > 0.$$

□

Corollary 3.2. *$f = g$ a.e. implies* $\operatorname{ess} V(f) = \operatorname{ess} V(g)$.

Lemma 3.3. *Let $f \in L_1(\mathbf{R})$ and $\operatorname{ess} V(f) < \infty$. Then*

(1) *For every $x \in \mathbf{R}$, $\lim_{\substack{h\downarrow 0 \\ x+h\in D_f}} f(x+h)$ converges.*

(2) *By (1), we can define*

$$g(x) = \lim_{\substack{h\downarrow 0 \\ x+h\in D_f}} f(x+h) \text{ for } x \in \mathbf{R}.$$

Then $g(x)$ is right continuous on $\mathbf{R}$ and $g(x) = f(x)$ for $x \in D_f$.

(3) *Let g be the function defined on $\mathbf{R}$ in (2). Then $V(g) = \operatorname{ess} V(f)$, where $V(g)$ is a total variation on $\mathbf{R}$ of g.*

Proof. (1) Let $x \in \mathbf{R}$. From the density of D_f in $\mathbf{R}$, we can take a sequence such that $t_1 > t_2 > \cdots \downarrow x$ and $t_n \in D_f$. Then,

$$\sum_{n=1}^{\infty} |f(t_{n+1}) - f(t_n)| \leq \operatorname{ess} V(f) < +\infty.$$

Hence $\lim_{n\to\infty} f(t_n)$ converges. Since the choice of $\{t_n\}$ is arbitrary, $\lim_{h\downarrow 0, x+h\in D_f} f(x+h)$ converges.

(2) It is clear from (1) that g is right continuous. Let $x \in D_f$,

$$f(x) = \lim_{h\downarrow 0} \frac{1}{h} \int_0^h f(x+t)\,dt = \lim_{\substack{h\downarrow 0 \\ x+h\in D_f}} f(x+h) = g(x).$$

(3) We see from (2) and $g|_{D_f} = f$ that $\operatorname{ess} V(f) \leq V(g)$. To show the converse inequality, take any sequence of $\mathbf{R}$ with $a_1 < a_2 < \cdots < a_n$. Since g is right continuous and D_f is dense in $\mathbf{R}$, for every $\varepsilon > 0$, there exists (b_k) such that $b_k \in [a_k, a_{k+1}) \cap D_f$ $(1 \leq k \leq n)$ and $|g(a_k) - g(b_k)| < \varepsilon/2(n-1)$. Then

$$\begin{aligned}\sum_{k=1}^{n-1} |g(a_{k+1}) - g(a_k)| &\leq \sum_{k=1}^{n-1} \{|g(a_{k+1}) - g(b_{k+1})| \\ &\quad + |g(b_{k+1}) - g(b_k)| + |g(b_k) - g(a_k)|\} \\ &\leq \sum_{k=1}^{n-1} |g(b_{k+1}) - g(b_k)| + \varepsilon \\ &\leq \operatorname{ess} V(f) + \varepsilon.\end{aligned}$$

Thus we have $V(g) \leq \operatorname{ess} V(f)$. □

Theorem 3.4. *For every $f \in L_1(\mathbf{R})$, the following statements are equivalent*:

(i) $\operatorname{ess} V(f) < \infty$.

(ii) $\{\|f(\cdot+h) - f(\cdot)\|_1/|h| : h \neq 0, h \in \mathbf{R}\}$ *is bounded.*

(iii) $f \in BV(\mathbf{R})$.

Moreover, $|Df|(\mathbf{R}) = \operatorname{ess} V(f)$.

Proof. The equivalence of statements (i) and (ii) is clear from Theorem 3.1. The equivalence of statements (i) and (iii) follows from [3, Theorem 7.8]. □

Theorem 3.5. *For every* $f \in L_1(\mathbf{R})$, $f \in BV(\mathbf{R})$ *if and only if* $\Lambda_1(f) = \ell_1$.

Proof. Let $f \in BV(\mathbf{R})$. We see from the previous theorem that $\ell_1 \subseteq \Lambda_1(f)$.

The converse inclusion $\ell_1 \supseteq \Lambda_1(f)$ follows from (i) ([1, Theorem 1]) appeared in the introduction. Thus $\ell_1 = \Lambda_1(f)$.

To show the converse, suppose that $f \notin BV(\mathbf{R})$. Then we see from Theorem 3.4 that

$$\left\{\|f(\cdot + h) - f(\cdot)\|_1/|h| : h \neq 0,\ h \in \mathbf{R}\right\}$$

is unbounded. Hence, for each $n \in \mathbf{N}$ there exists $h_n \neq 0$ such that

$$\int_{\mathbf{R}} \left|\frac{f(x-h_n) - f(x)}{h_n}\right| dx > 2^n.$$

Hence

$$|h_n| < \frac{1}{2^n}\int_{\mathbf{R}} |f(x-h_n) - f(x)|\, dx \leq 2^{1-n}\|f\|_1.$$

Now, let $N(n)$ be the maximum of natural numbers satisfying $N|h_n| \leq 2^{1-n}\|f\|_1$. We have from the maximality of $N(n)$ that

$$2^{1-n}\|f\|_1 < (N(n)+1)|h_n| < 2N(n)|h_n|.$$

and so

$$\|f\|_1 < N(n)2^n|h_n| < N(n)\int_{\mathbf{R}} |f(x-h_n) - f(x)|\, dx.$$

Using (h_n) and $(N(n))$, we can construct a sequence (a_n) as follows:

$$a_j = h_k,\ 1 + \sum_{i=0}^{k-1} N(i) \leq j \leq \sum_{i=0}^{k} N(i),$$

where $N(0) = 0$ and $j, k = 1, 2, 3, \cdots$. Consequently, we have

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} N(n)|h_n| \leq \sum_{n=1}^{\infty} 2^{1-n}\|f\|_1 < +\infty,$$

and so $\boldsymbol{a} \in \ell_1$.

On the other hand,

$$\Psi_1(\boldsymbol{a}; f) = \sum_{n=1}^{\infty} N(n)\int_{\mathbf{R}} |f(x-h_n) - f(x)|\, dx \geq \sum_{n=1}^{\infty} \|f\|_1 = \infty.$$

Hence $\boldsymbol{a} \notin \Lambda_1(f)$, which contradicts $\ell_1 = \Lambda_1(f)$. □

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