Finite order meromorphic solutions of linear difference equations

By Sheng LI*) and Zong-Sheng GAO*)

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Abstract: In this paper, we mainly investigate the growth and the value distribution of meromorphic solutions of the linear difference equation

$$a_n(z)f(z+n)+\cdots+a_1(z)f(z+1)+a_0(z)f(z)=b(z),$$

where $a_0(z), a_1(z), \cdots, a_n(z), b(z)$ are entire functions such that $a_0(z)a_n(z)\not\equiv 0$. For a finite order meromorphic solution $f(z)$, some interesting results on the relation between $\rho=\rho(f)$ and $\lambda_f=\max\{\lambda(f),\lambda(1/f)\}$, are proved. And examples are provided for our results.

Key words: Difference equations; value distribution; finite order.

1. Introduction. In this paper, a meromorphic function means meromorphic in the complex plane. We will use the basic notions in Nevanlinna theory of meromorphic functions (see e.g., [8,10,17]).

Recently, there has been an increasing renewed interest in complex difference equations and difference analogues of Nevanlinna theory (see e.g., [1,3–7,9,11,12]). We firstly recall some existence results for meromorphic solutions of difference equations. The following two results have been proved by Shimomura [14] and Yanagihara [16], respectively.

Theorem A. *For any nonconstant polynomial $P(y)$, the difference equation*

$$y(z+1)=P(y(z))$$

has a nontrivial entire solution.

Theorem B. *For any nonconstant rational function $R(y)$, the difference equation*

$$y(z+1)=R(y(z))$$

has a nontrivial meromorphic solution in the complex plane.

The following two results concerning both existence and growth restriction for meromorphic solutions of linear difference equations have been proved by Bank and Kaufman [2] and Whittaker [15], respectively.

Theorem C. *For any nonconstant rational function $R(z)$, the difference equation*

$$y(z+1)-y(z)=R(z)$$

has a nontrivial meromorphic solution $y(z)$ such that $T(r,y)=O(r)$.

Theorem D. *Let ρ be a real number, and let $\Psi(z)$ be a given entire function with order $\rho(\Psi)=\rho$. Then the equation*

$$F(z+1)=\Psi(z)F(z)$$

admits a meromorphic solution of order $\rho(F)\le\rho+1$.

In a recent paper [4], Chiang and Feng have improved Theorem D by showing that $\rho(F)\le\rho+1$ can be replaced by $\rho(F)=\rho+1$ (see [4], Corollary 9.3). In fact, they have investigated meromorphic solutions of the linear difference equation

$$\sum_{j=0}^{n} a_j(z)f(z+j)=0, \tag{1.1}$$

where $a_0(z), a_1(z), \cdots, a_n(z)$ are entire functions such that $a_0(z)a_n(z)\not\equiv 0$, and proved the following two results in [4].

Theorem E. *Let $a_0(z), a_1(z), \cdots, a_n(z)$ be polynomials such that there exists an integer l, $0\le l\le n$ such that*

$$\deg(a_l)>\max_{0\le j\le n,\ j\ne l}\{\deg(a_j)\}.$$

If $f(z)$ is a meromorphic solution of (1.1), then $\rho(f)\ge 1$.

Theorem F. *Let $a_0(z), a_1(z), \cdots, a_n(z)$ be entire functions such that there exists an integer l, $0\le l\le n$ such that*

$$\rho(a_l)>\max_{0\le j\le n,\ j\ne l}\{\rho(a_j)\}.$$

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*) LMIB & School of Mathematics and Systems Science, Beihang University, Beijing, 100191, China.

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If $f(z)$ is a meromorphic solution of (1.1), *then* $\rho(f) \geq \rho(a_l) + 1$.

Theorem E can be also found in [5]. Our aim in this paper is to present some generalizations of Theorem E and Theorem F.

2. Main results. In what follows, we will use the notation $\lambda_f := \max\{\lambda(f), \lambda(1/f)\}$, where $\lambda(f)$ and $\lambda(1/f)$ are, respectively, the exponent of convergence of the zeros and poles of $f(z)$. Considering the growth and the value distribution of meromorphic solution $f(z)$, we improve Theorem E by the following result, in which an interesting relationship between $\rho = \rho(f)$ and λ_f is given.

Theorem 2.1. *Let* $a_0(z), a_1(z), \cdots, a_n(z)$, $b(z)$ *be polynomials such that*

$$a_0(z)a_n(z) \not\equiv 0, \ \deg(\sum_{\deg a_j = d} a_j) = d,$$

where $d = \max_{0 \leq j \leq n}\{\deg a_j\}$. *If $f(z)$ is a transcendental meromorphic solution of*

$$\sum_{j=0}^{n} a_j(z)f(z+j) = b(z), \tag{2.1}$$

then $\rho(f) \geq 1$. *Moreover, if $f(z)$ is of finite order, then* $1 \leq \rho(f) \leq 1 + \lambda_f$.

Remark. Obviously, (2.1) might admit some rational solutions. And by the proof of Theorem 2.1, we see that if $f(z)$ has infinitely many poles, then $\lambda(1/f) \geq 1$.

As shown in Ozawa [13], for any given $\rho \in [1, \infty)$, we can choose a periodic entire function $g(z)$ with period 1 such that $\rho(g) = \rho$, in particular, if $\rho \notin \mathbf{N}$, then $\rho(g) = \lambda(g) = \rho$. This enables us to give some examples for Theorem 2.1 to show the sharpness of estimates for the order of growth of solutions. And we also show that $b(z) \not\equiv 0$ and $\lambda(1/f) \geq 1$ may happen. However, we have not found any example such that $1 < \rho(f) = 1 + \lambda_f$.

Examples. (1) $f(z) = e^z + 1$ is of order $\rho(f) = \lambda(f) = 1$ and solves the equation

$$\begin{aligned} &z^2 f(z+2) - (ez^2+1)f(z+1) - ef(z) \\ &\quad = z^2 - (ez^2+1) - e, \end{aligned}$$

while the gamma function $\Gamma(z)$ is of order $\rho(\Gamma) = \lambda(1/\Gamma) = 1$ and solves the equation

$$\Gamma(z+1) - z\Gamma(z) = 0.$$

(2) $f_1(z) = e^z$ and $f_2(z) = g(z)e^z$, where $g(z)$ is a periodic function with period 1 such that $\rho(g) = \lambda(g) = \rho \in (1, 2)$, are solutions of the equation

$$\begin{aligned} &z^2 f(z+2) - (ez^2+1)f(z+1) \\ &\quad - ef(z) = 0, \end{aligned}$$

and $\rho(f_1) = \lambda_{f_1} + 1$ and $1 < \rho = \rho(f_2) < \lambda_{f_2} + 1$.

For the case that some coefficients are transcendental entire functions, as a continuation of Theorem F, we prove the following

Theorem 2.2. *Let* $a_0(z), a_1(z), \cdots, a_n(z)$ *be entire functions, such that*

$$a_0(z)a_n(z) \not\equiv 0, \ \max_{1 \leq j \leq n}\{\rho(a_j)\} = \sigma < 1.$$

Let $f(z)$ be a nontrivial meromorphic solution of

$$\sum_{j=1}^{n} a_j(z)f(z+j) + a_0(z)e^z f(z) = 0. \tag{2.2}$$

Then $\rho(f) \geq 2$. *Moreover, if $f(z)$ is of finite order, then either* $2 \leq \rho(f) \leq 1 + \lambda_f$ *or* $1 + \lambda_f < \rho(f) = 2$.

Next, we give some examples for Theorem 2.2. Unfortunately, we still wonder whether there exit some examples for Theorem 2.2 such that $f(z)$ satisfying $2 < \rho(f) = \lambda_f + 1$.

Examples. (1) $f_1(z) = e^{(z^2-1)/2}$ is of order $\rho(f_1) = 2 > \lambda_{f_1} + 1$ and $f_2(z) = e^{(z^2-1)/2}\sin(2\pi z)$ is of order $\rho(f_2) = 2 = \lambda_{f_2} + 1$ and they solve the equation

$$f(z+1) + e^z f(z) = 0. \tag{2.3}$$

(2) Choose a periodic function $g(z)$ with period 1 such that $\lambda(g) = \rho(g) \in (2, 3)$. Then both $f_1(z) = g(z)e^{(z^2-1)/2}$ and $f_2(z) = e^{(z^2-1)/2}/g(z)$ solve (2.3) such that $2 < \rho(f_1) = \rho(g) < \lambda(g) + 1 = \lambda(f_1) + 1$, and $2 < \rho(f_2) = \rho(g) < \lambda(g) + 1 = \lambda(1/f_2) + 1$.

As an application of Theorem 2.2, we prove the following Theorem 2.3.

Theorem 2.3. *Under the assumption of Theorem* 2.2, *if* $b(z) \not\equiv 0$ *is a meromorphic function, then the equation*

$$\sum_{j=1}^{n} a_j(z)f(z+j) + a_0(z)e^z f(z) = b(z) \tag{2.4}$$

admits at most one meromorphic solution $f(z)$ such that $\rho(f) < 2$.

Remark. In fact, if (2.4) admits two meromorphic solutions $f(z)$, $g(z)$ such that $\max\{\rho(f), \rho(g)\} < 2$, then $h(z) = f(z) - g(z)$ is a meromorphic solutions of (2.2) with order $\rho(h) \leq \max\{\rho(f), \rho(g)\} < 2$. However, by Theorem 2.2 or Theorem F, we have $\rho(h) \geq 2$, a contradiction. Thus we prove Theorem 2.3.

In Theorem 2.3, if $\rho(b) \in [1, 2)$, one can easily give some examples for existence of such meromorphic solution $f(z)$ that $\rho(f) < 2$. However, it seems quite different for the case $\rho(b) < 1$. And we should ask a question: Is it true that all meromorphic solutions of (2.4) are of order ≥ 2 provided that $\rho(b) < 1$?

3. Proofs of results. The Lemma 3.1 below is the Corollary 8.3 in [4].

Lemma 3.1. *Let η_1, η_2 be two arbitrary complex numbers, and let $f(z)$ be a meromorphic function of finite order ρ. Let $\varepsilon > 0$ be given, then there exists a subset $E \subset (0, \infty)$ with finite logarithmic measure such that for all $|z| = r \notin E \cup [0, 1]$, we have*

$$\exp\{-r^{\rho-1+\varepsilon}\} \leq \left|\frac{f(z+\eta_1)}{f(z+\eta_2)}\right| \leq \exp\{r^{\rho-1+\varepsilon}\}.$$

Remark. In Lemma 3.1, if $\rho < 1$, let $\varepsilon = \frac{1-\rho}{2} > 0$, then we can see that

$$\frac{f(z+\eta_1)}{f(z+\eta_2)} \to 1$$

as $|z| = r \notin E \cup [0, 1]$, $r \to \infty$.

The following Lemma is a corollary of the Borel's Theorem on the combination of entire functions (see [17], the corollary of Theorem 1.52).

Lemma 3.2. *If $f_j(z)$ $(j = 1, 2, \ldots, n+1)$ and $g_j(z)$ $(j = 1, 2, \ldots, n)$ $(n \geq 1)$ are entire functions satisfying*

(i) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} = f_{n+1}(z)$;

(ii) *the order of f_j is less than e^{g_k} for $1 \leq j \leq n+1$, $1 \leq k \leq n$; And furthermore, the order of f_j is less than $e^{g_k - g_h}$ for $n \geq 2$ and $1 \leq j \leq n+1$, $1 \leq h < k \leq n$,*

Then $f_j(z) \equiv 0$ $(j = 1, 2, \ldots, n+1)$.

Proof of Theorem 2.1.

Step 1: We prove that $\rho(f) \geq 1$. Otherwise, we have $\rho(f) = \rho < 1$. Then we should first show that $f(z)$ has finitely many poles. Assume that $f(z)$ has infinitely many poles. Since $a_0(z), \cdots, a_n(z)$ are all polynomials, we can choose a point z_0 such that $f(z_0) = \infty$ and for each $t \in \mathbf{N}$, $\Pi_{j=0}^{n} a_j(z_0 + t) \neq 0$. This and (2.1) implies that there is at least one point $j \in \{1, \cdots, n\}$ such that $f(z_0 + j) = \infty$. Denote $l_0 = \max\{j : f(z_0 + j) = \infty, 1 \leq j \leq n\}$ and $z_1 = z_0 + l_0$. Then from

$$\sum_{j=0}^{n} a_j(z_1) f(z_1 + j) = b(z_1),$$

we see that there is at least one point $j \in \{1, \cdots, n\}$ such that $f(z_1 + j) = \infty$. By induction, there is an infinite sequence $\{z_0 + l_t\}_{t=1}^{\infty}$ such that $t \leq l_t \leq nt$, and $f(z_0 + l_t) = \infty$. This yields that

$$N(|z_0| + nt, f) \geq \frac{t}{2} \log t,$$

and thus we get $\lambda(1/f) \geq 1$. This contradicts $\lambda(1/f) \leq \rho(f) < 1$. Therefore, without loss of generality, we can assume that $f(z)$ has no poles in what follows.

By Lemma 3.1 and its remark, for each $j \in \{1, 2, \cdots, n\}$, there exists a set $E \subset (1, \infty)$ of finite logarithmic measure, so that

$$\frac{f(z+j)}{f(z)} \to 1, \tag{3.1}$$

for all z satisfying $|z| = r \notin E$, as $r \to \infty$.

Set $I = \{0, 1, \cdots, n\}, \Lambda = \{j \in I : \deg a_j = d\}$. Fix a point $l \in \Lambda$, and we rewrite (2.1) as follows:

$$\sum_{j \in \Lambda} \frac{a_j(z)}{a_l(z)} \frac{f(z+j)}{f(z)} + \sum_{j \in I \setminus \Lambda} \frac{a_j(z)}{a_l(z)} \frac{f(z+j)}{f(z)} = \frac{b(z)}{a_l(z) f(z)}. \tag{3.2}$$

Now choose an infinite sequence $z_k = r_k e^{i\theta_k}$, $\theta_k \in [0, 2\pi)$, $|z_k| = r_k \notin E$ such that $|f(z_k)| = M(r_k, f)$, $r_k \to \infty$ as $k \to \infty$. Noticing now $f(z)$ is a transcendental entire function, with (3.1) and (3.2), we get

$$\frac{d_\Lambda}{d_l}(1 + o(1)) + o(1) = o(1),$$

where d_l and d_Λ is, respectively, the leading coefficient of $a_l(z)$ and $\sum_{j \in \Lambda} a_j(z)$. That implies $d_\Lambda = 0$, which contradicts our assumption. Thus we have $\rho(f) \geq 1$.

Step 2: We show that $\rho \leq \lambda_f + 1$ if $f(z)$ is of finite order. Otherwise, we have $\lambda_f + 1 < \rho(f) = \rho < \infty$. Assume that $z = 0$ is a zero (or pole) of $f(z)$ of order K. Applying Hadamard factorization Theorem of meromorphic function (see [17], Theorem 2.7), we write $f(z)$ as follows:

$$f(z) = z^K \frac{P_1(z)}{P_2(z)} e^{Q(z)},$$

where $P_1(z), P_2(z)$ are entire functions such that $\rho(P_1) = \lambda(P_1) = \lambda(f)$, $\rho(P_2) = \lambda(P_2) = \lambda(1/f)$, and $Q(z)$ is a polynomial such that $\deg Q(z) = q$. Since $\rho(f) > \lambda_f + 1$, we see that $q = \rho(f) > \lambda_f + 1$.

Denote

$$A_j(z) = \frac{a_j(z)P_1(z+j)\Pi_{k=0}^n P_2(z+k)}{P_2(z+j)},$$
$$A_{n+1}(z) = b(z)\Pi_{k=0}^n P_2(z+k).$$

Then A_j ($j = 0, 1, \ldots, n+1$) are all entire functions such that $\rho(A_j) < q - 1$. We obtain from (2.1) that

$$\sum_{j=0} A_j(z)e^{Q(z+j)} = A_{n+1}(z). \tag{3.3}$$

Notice that $\deg(Q(z+h) - Q(z+k)) = q - 1$. Thus Lemma 3.2 is valid for (3.3) and hence $A_j(z) \equiv 0$ for $j = 0, 1, \ldots, n$. However, $a_0 a_n \not\equiv 0$ yields that $A_0 A_n \not\equiv 0$, a contradiction.

Proof of Theorem 2.2.

We get $\rho(f) = \rho \geq \rho(e^z) + 1 = 2$ by Theorem F immediately. Now if $2 \leq \rho \leq \lambda_f + 1$ or $\rho(f) = 2$, then our second assertion is also true. Otherwise, we have $\max\{2, \ \lambda_f + 1\} < \rho < \infty$. However, in this case, with a similar reasoning as in Step 2 in the proof of Theorem 2.1, we can deduce a similar contradiction. We omit all those details.

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