Proper actions of $SL(2, \mathrm{R})$ on semisimple symmetric spaces

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(Communicated by Masaki KASHIWARA, M.J.A., Feb. 14, 2011)

Abstract: We classify semisimple symmetric spaces G/H for which there exist proper $SL(2,\mathbf{R})$-actions via G. This leads us to the classification of semisimple symmetric spaces that admit surface groups as discontinuous groups.

Key words: Proper action; symmetric space; surface group; nilpotent orbit; weighted Dynkin diagram; Satake diagram.

1. Introduction and statement of main results. We consider one of fundamental problems on locally symmetric spaces as follows:

Problem 1.1 (See [14]). *Fix a simply connected symmetric space M_0 as a model space. What discrete groups can arise as the fundamental groups of a complete affine manifold M which is locally isomorphic to the space M_0?*

By a theorem of É. Cartan, such M is represented as the Clifford–Klein form $\Gamma\backslash G/H$. Here $M_0 = G/H$ is a simply connected symmetric space and $\Gamma \simeq \pi_1(M)$ a discrete subgroup of G acting as a discontinuous group for G/H. Then Problem 1.1 may be reformalized as:

Problem 1.2. *What discrete subgroups of G can act as discontinuous groups for G/H?*

For a compact subgroup H of G, any discrete subgroup Γ in G acts as a discontinuous group for G/H. Thus, our interest in this article is in non-compact H, for which not all discrete subgroups Γ of G act properly discontinuously on G/H. Even for Lorentz symmetric spaces $SO(n+1,1)/SO(n,1)$, Problem 1.2 is non-trivial as was shown by the Calabi–Markus phenomenon [5]. A systematic study of Problem 1.2 for the general non-Riemannian homogeneous space G/H was initiated in the late 1980's by Kobayashi [9–11] followed by [18,21,23,25,29]. See [15,16,24] for the recent development on this topic.

In this paper, we discuss proper actions of $SL(2,\mathbf{R})$ on semisimple symmetric spaces. Among others, we give the classification of semisimple symmetric spaces that admit surface groups as discontinuous groups. (A surface group means the fundamental group of a closed Riemann surface of genus $g \geq 2$.)

The basic setting here is the following

Setting 1.3. *G is a connected linear semisimple Lie group, σ is an involutive automorphism of G, and H is an open subgroup of $G^\sigma := \{g \in G \mid \sigma g = g\}$.*

This setting implies that G/H becomes a symmetric space with respect to the canonical affine connection on G/H. We write $\mathfrak{g}$, $\mathfrak{h}$ for Lie algebras of G, H, respectively. The differential action of σ on $\mathfrak{g}$ will be denoted by the same letter σ, and we set $\mathfrak{q} := \{X \in \mathfrak{g} \mid \sigma X = -X\}$. We denote by $\mathfrak{g}_{\mathbf{C}}$ the complexification of $\mathfrak{g}$, and write the c-dual of $\mathfrak{g}$ for $\mathfrak{g}^c := \mathfrak{h} \oplus \sqrt{-1}\mathfrak{q}$. Both $\mathfrak{g}$ and $\mathfrak{g}^c$ are real forms of $\mathfrak{g}_{\mathbf{C}}$.

Here is our main result:

Theorem 1.4. *In Setting* 1.3*, the following five conditions on a symmetric pair (G,H) are equivalent*:

(i) *There exists a Lie group homomoephism $\Phi : SL(2,\mathbf{R}) \to G$ such that $SL(2,\mathbf{R})$ acts properly on G/H via Φ.*
(ii) *For any $g \geq 2$, G/H admits the surface group of genus g as a discontinuous group.*
(iii) *G/H admits an infinite discontinuous group Γ which is not virtually abelian* (*i.e. Γ has no abelian subgroups of finite index*).
(iv) *G/H admits a discontinuous group which is a free group generated by a unipotent element in G.*
(v) *There exists a nilpotent orbit $\mathcal{O}$ of* $\mathrm{Int}\,\mathfrak{g}_{\mathbf{C}}$ *in $\mathfrak{g}_{\mathbf{C}}$ such that $\mathcal{O}$ meets $\mathfrak{g}$ but does not meet $\mathfrak{g}^c$.*

2000 Mathematics Subject Classification. Primary 22F30; Secondary 22E40, 53C30, 53C35, 57S30.

doi: 10.3792/pjaa.87.35

We will use the following convention: Suppose L is an abstract Lie group. Given a homomorphism $\Phi : L \to G$, we can define the action of L on G/H by $xH \mapsto \Phi(l)xH$ ($l \in L$), which will be referred to simply as an *L-action via G*.

Our theorem generalizes Teduka [28] who studied proper $SL(2, \mathbf{R})$-actions on complex symmetric spaces G/H.

Remark 1.5. For any complex nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}_{\mathbf{C}}$, the intersection $\mathcal{O} \cap \mathfrak{g}$ splits into at most finitely many real nilpotent orbits of G. By the Jacobson–Morozov theorem, there is a one-to-one correspondence between Lie group homomorphisms $\Phi : SL(2, \mathbf{R}) \to G$ up to inner automorphisms of G and real nilpotent orbits $\mathcal{O}_{0,\Phi}$ of G in $\mathfrak{g}$. As a refinement of (i) $\Leftrightarrow$ (v) in Theorem 1.4, we can prove that the complex nilpotent orbit $\mathcal{O}$ containing $\mathcal{O}_{0,\Phi}$ determines whether or not the $SL(2, \mathbf{R})$-action on G/H via Φ is proper.

The key ingredient of Theorem 1.4 is to show the equivalence (i) $\Leftrightarrow$ (v), and we will indicate an idea of the proof in Section 3. The implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are deduced from the lifting theorem of a surface group (see [20]). The equivalence (i) $\Leftrightarrow$ (iv) was previously proved in [12]. We use Benoist's results [3] for the proof of the equivalence (iii) $\Leftrightarrow$ (v).

The full detail will be reported elsewhere.

2. Algorithm and Classification.

Theorem 1.4 may be in a good comparison with the fact below for proper actions by an abelian group $\mathbf{R}$:

Fact 2.1. *In Setting* 1.3*, the following six conditions on a symmetric pair* (G, H) *are equivalent*:

(i) *There exists a Lie group homomorphism* $\rho : \mathbf{R} \to G$ *such that* $\mathbf{R}$ *acts properly on* G/H *via* ρ.
(ii) G/H *admits* $\mathbf{Z}$ *as a discontinuous group.*
(iii) G/H *admits an infinite discontinuous group.*
(iv) G/H *admits a discontinuous group which is a free group generated by a hyperbolic element in* G.
(v) *There exists a hyperbolic orbit* $\mathcal{O}$ *of* $\mathrm{Int}\,\mathfrak{g}_{\mathbf{C}}$ *in* $\mathfrak{g}_{\mathbf{C}}$ *such that* $\mathcal{O}$ *meets* $\mathfrak{g}$ *but does not meet* $\mathfrak{g}^c$.
(vi) $\mathrm{rank}_{\mathbf{R}}\,\mathfrak{g} > \mathrm{rank}_{\mathbf{R}}\,\mathfrak{h}$.

Here an element $X \in \mathfrak{g}$ [*resp.* $x \in G$] *is said to be hyperbolic if* $\mathrm{ad}(X) \in \mathrm{End}(\mathfrak{g})$ [*resp.* $\mathrm{Ad}(x) \in GL(\mathfrak{g})$] *is diagonalizable with only real eigenvalues.*

The equivalence among (i), (ii), (iii), (iv) and (vi) in Fact 2.1 was proved in a more general setting (see [9]). The proof of (i) $\Leftrightarrow$ (v) in Fact 2.1 is similar to the proof of Theorem 1.4. The real rank condition (vi) in Fact 2.1 serves as a criterion for the Calabi–Markus phenomenon ((iii) in Fact 2.1, cf. [5,9]).

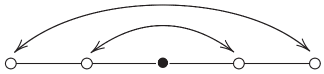
In order to verify the condition (v) in Theorem 1.4 for a symmetric pair $(\mathfrak{g}, \mathfrak{h})$, we give an algorithm which is based on structural results on nilpotent orbits (see Theorem 2.4 below). To be precise, we recall briefly the definition of weighted Dynkin diagrams of $\mathfrak{g}_{\mathbf{C}}$ and the Satake diagrams of $\mathfrak{g}$ and $\mathfrak{g}^c$.

Let $\mathfrak{g}$ be a semisimple Lie algebra with a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Take a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{p}$, and extend it to a Cartan subalgebra $\mathfrak{j}$ in $\mathfrak{g}$. The complexification, denoted by $\mathfrak{j}_{\mathbf{C}}$, of $\mathfrak{j}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbf{C}}$. Let $\Delta(\mathfrak{g}_{\mathbf{C}}, \mathfrak{j}_{\mathbf{C}}) \subset \mathfrak{j}_{\mathbf{C}}^*$ be the root system of $(\mathfrak{g}_{\mathbf{C}}, \mathfrak{j}_{\mathbf{C}})$ and $\Sigma(\mathfrak{g}, \mathfrak{a}) \subset \mathfrak{a}^*$ the restricted root system of $(\mathfrak{g}, \mathfrak{a})$. We fix a positive system $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ of $\Sigma(\mathfrak{g}, \mathfrak{a})$. Then we can and do take a positive system $\Delta^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{j}_{\mathbf{C}})$ of $\Delta(\mathfrak{g}_{\mathbf{C}}, \mathfrak{j}_{\mathbf{C}})$ such that, for any $\alpha \in \Delta^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{j}_{\mathbf{C}})$, the restriction of α to $\mathfrak{a}$ is in $\Sigma^+(\mathfrak{g}, \mathfrak{a}) \cup \{0\}$. We write Π for the fundamental system of $\Delta^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{j}_{\mathbf{C}})$. A weighted Dynkin diagram of $\mathfrak{g}_{\mathbf{C}}$ is defined as a map $\Pi \to \{0, 1, 2\}$. Let Π_0 be the set of all simple roots in Π whose restriction to $\mathfrak{a}$ is zero. The Satake diagram S of $\mathfrak{g}$ consists of the following data: the Dynkin diagram of $\mathfrak{g}_{\mathbf{C}}$ with nodes Π, black nodes Π_0 in S, and arrows joining $\alpha \in \Pi \setminus \Pi_0$ and $\beta \in \Pi \setminus \Pi_0$ in S whose restrictions to $\mathfrak{a}$ are the same.

Combining the Jacobson–Morozov theorem, Kostant [19] with Malcev [22], we have a one-to-one correspondence between the set of nilpotent orbits in $\mathfrak{g}_{\mathbf{C}}$ to a subset of weighted Dynkin diagrams of $\mathfrak{g}_{\mathbf{C}}$.

Definition 2.2. Let D be a weighted Dynkin diagram of $\mathfrak{g}_{\mathbf{C}}$ and S the Satake diagram of $\mathfrak{g}$. We say that D *matches* S if all the weights on black nodes are zero and any pair of nodes joined by an arrow has the same weights.

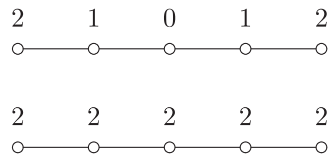
Example 2.3. We consider a semisimple Lie algebra $\mathfrak{g} = \mathfrak{su}(4, 2)$, a real form of $\mathfrak{g}_{\mathbf{C}} = \mathfrak{sl}(6, \mathbf{C})$. The Satake diagram S of $\mathfrak{su}(4, 2)$ is given by


.

Suppose D_1 and D_2 are the following weighted Dynkin diagrams of $\mathfrak{sl}(6, \mathbf{C})$:

$D_1 :=$,

$D_2 :=$.



According to Definition 2.2, D_1 matches S, but D_2 does not match S.

By using results of [6,27], we obtain the following theorem:

Theorem 2.4. *Let $\mathfrak{g}$ be a real form of a complex semisimple Lie algebra $\mathfrak{g}_{\mathbf{C}}$. For a nilpotent orbit $\mathcal{O}$ of* $\operatorname{Int}\mathfrak{g}_{\mathbf{C}}$ *in $\mathfrak{g}_{\mathbf{C}}$, $\mathcal{O}$ meets $\mathfrak{g}$ if and only if the weighted Dynkin diagram corresponding to $\mathcal{O}$ matches the Satake diagram of $\mathfrak{g}$ in the sense of Definition* 2.2.

Nilpotent orbits in a complex semisimple Lie algebra are classified by Dynkin–Kostant [7,19] and Bala–Carter [1,2]. The classification of semisimple symmetric pairs $(\mathfrak{g},\mathfrak{h})$ is known by Berger [4]. Applying the criterion in Theorem 2.4 to the real forms $\mathfrak{g}$ and $\mathfrak{g}^c$ in $\mathfrak{g}_{\mathbf{C}}$, respectively, we obtain the following classification:

Theorem 2.5. *In Setting* 1.3*, suppose G is a simple Lie group. Then G/H admits a proper* $\mathbf{R}$*-action via G (cf. Fact* 2.1*) but does not admit proper $SL(2,\mathbf{R})$-actions via G (cf. Theorem* 1.4*) if and only if $(\mathfrak{g},\mathfrak{h})$ is one of the following*

Table 2.6

$\mathfrak{g}$	$\mathfrak{h}$
$\mathfrak{sl}(2k,\mathbf{R})$	$\mathfrak{sp}(k,\mathbf{R})$
$\mathfrak{sl}(2k,\mathbf{R})$	$\mathfrak{so}(k,k)$
$\mathfrak{su}^*(4m+2)$	$\mathfrak{sp}(m+1,m)$
$\mathfrak{su}^*(4m)$	$\mathfrak{sp}(m,m)$
$\mathfrak{su}^*(2k)$	$\mathfrak{so}^*(2k)$
$\mathfrak{so}(2k+1,2k+1)$	$\mathfrak{so}(i+1,i)\oplus\mathfrak{so}(j,j+1)$ $(i+j=2k)$
$\mathfrak{e}_{6(6)}$	$\mathfrak{f}_{4(4)}$
$\mathfrak{e}_{6(6)}$	$\mathfrak{sp}(4,\mathbf{R})$
$\mathfrak{e}_{6(-26)}$	$\mathfrak{sp}(3,1)$
$\mathfrak{e}_{6(-26)}$	$\mathfrak{f}_{4(-20)}$
$\mathfrak{sl}(n,\mathbf{C})$	$\mathfrak{so}(n,\mathbf{C})$
$\mathfrak{sl}(2k,\mathbf{C})$	$\mathfrak{sp}(k,\mathbf{C})$
$\mathfrak{sl}(2k,\mathbf{C})$	$\mathfrak{su}(k,k)$
$\mathfrak{so}(4m+2,\mathbf{C})$	$\mathfrak{so}(i,\mathbf{C})\oplus\mathfrak{so}(j,\mathbf{C})$ $(i+j=4n+2, i,j$ are odd$)$
$\mathfrak{so}(4m+2,\mathbf{C})$	$\mathfrak{so}(2m+2,2m)$
$\mathfrak{e}_{6,\mathbf{C}}$	$\mathfrak{sp}(4,\mathbf{C})$
$\mathfrak{e}_{6,\mathbf{C}}$	$\mathfrak{f}_{4,\mathbf{C}}$
$\mathfrak{e}_{6,\mathbf{C}}$	$\mathfrak{e}_{6(2)}$

Remark 2.7. The symmetric spaces G/H listed in Table 2.6 do not admit cocompact discontinuous groups (cf. [3,9–11,13,18,21,23,25,29]).

A part of Table 2.6 was previously known in the special case where $\mathfrak{g}$ and $\mathfrak{h}$ are both complex Lie algebras by Teduka [28].

By Theorem 1.4 and Theorem 2.5, we also obtain the classification result below:

Theorem 2.8. *In Setting* 1.3*, suppose G is a simple Lie group. Then, the following three conditions on a symmetric pair (G,H) are equivalent*:

(i) *There exists a Lie group homomorphism $\Phi : SL(2,\mathbf{R})\to G$ such that $SL(2,\mathbf{R})$ acts properly on G/H via Φ.*

(ii) *For any $g\geq 2$, G/H admits the surface group of genus g as a discontinuous group.*

(iii) $\operatorname{rank}_{\mathbf{R}}\mathfrak{g} > \operatorname{rank}_{\mathbf{R}}\mathfrak{h}$ *and $(\mathfrak{g},\mathfrak{h})$ is not in Table* 2.6.

3. Ideas of proof of (i) ⇔ (v) in Theorem 1.4. We will explain the proof of the equivalence (i) ⇔ (v) in Theorem 1.4.

The proof will be devided into two steps (step 1 and step 2 below). In step 1, by using results of Kobayashi [9], we will obtain a purely Lie algebraic condition equivalent to the topological condition (i) in Theorem 1.4 (see Proposition 3.4 (vi)).

In step 2, by using Lemma 3.6 and Lemma 3.7, we will show that the condition (vi) in Proposition 3.4 is equivalent to a new condition on $(\mathfrak{g}_{\mathbf{C}},\mathfrak{g},\mathfrak{g}^c)$ (see Proposition 3.5 (vii)). Finally, by using [27, Proposition 1.11], we will obtain the equivalence between the condition (vii) in Proposition 3.5 and the condition (v) in Theorem 1.4.

Step 1: Kobayashi's criterion. We recall from [9] the criterion for proper actions on a homogeneous space G/H with G and H reductive (Fact 3.2 below).

Let G be a linear reductive Lie group with the Lie algebra $\mathfrak{g}$. We adopt the following definition of reductiveness of subalgebras (or subgroups). Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. We say $\mathfrak{h}$ is a *reductive subalgebra* of $\mathfrak{g}$ if there exists a Cartan involution θ of $\mathfrak{g}$ such that $\mathfrak{h}$ is θ-stable. Then $\mathfrak{h}$ is a reductive Lie algebra with a Cartan involution $\theta|_{\mathfrak{h}}$. We say a closed subgroup H of G is a *reductive subgroup* of G if H has finitely many connected components and its Lie algebra is a reductive subalgebra of $\mathfrak{g}$. Then G/H is said to be a *homogeneous space of reductive type.*

We consider the following

Setting 3.1. *G is a linear reductive Lie group, H and L are reductive subgroups in G.*

We write $\mathfrak{g}$, $\mathfrak{h}$, and $\mathfrak{l}$ for Lie algebras of G, H and L, respectively. Take a Cartan involution θ of $\mathfrak{g}$ which preserves $\mathfrak{h}$. We write the Cartan decompositions $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and $\mathfrak{h} = \mathfrak{k}(\mathfrak{h}) \oplus \mathfrak{p}(\mathfrak{h})$ corresponding to θ and $\theta|_{\mathfrak{h}}$, respectively. We fix a maximal abelian subspace $\mathfrak{a}_{\mathfrak{h}}$ of $\mathfrak{p}(\mathfrak{h})$, and extend it to a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{p}$. We write K for the maximal compact subgroup of G with Lie algebra $\mathfrak{k}$ and denote the Weyl group acting on $\mathfrak{a}$ by $W(\mathfrak{g}, \mathfrak{a}) := N_K(\mathfrak{a})/Z_K(\mathfrak{a})$. Since $\mathfrak{l}$ is a reductive subalgebra of $\mathfrak{g}$, we can take a Cartan involution θ' of $\mathfrak{g}$ preserving $\mathfrak{l}$. We write $\mathfrak{l} = \mathfrak{k}'(\mathfrak{l}) \oplus \mathfrak{p}'(\mathfrak{l})$ for the Cartan decomposition corresponding to $\theta'|_{\mathfrak{l}}$, and fix a maximal abelian subspace $\mathfrak{a}'(\mathfrak{l})$ of $\mathfrak{p}'(\mathfrak{l})$. Then there exists $g \in G$ such that $\mathrm{Ad}(g) \cdot \mathfrak{a}'(\mathfrak{l})$ is contained in $\mathfrak{a}$, and we put $\mathfrak{a}_{\mathfrak{l}} := \mathrm{Ad}(g) \cdot \mathfrak{a}'(\mathfrak{l})$. The subset $W(\mathfrak{g}, \mathfrak{a}) \cdot \mathfrak{a}_{\mathfrak{l}}$ of $\mathfrak{a}$ does not depend on a choice of $g \in G$. Then, the following fact holds:

Fact 3.2 ([9, Theorem 4.1])**.** *In Setting* 3.1*, L acts on G/H properly if and only if*

$$W(\mathfrak{g}, \mathfrak{a}) \cdot \mathfrak{a}_{\mathfrak{l}} \cap \mathfrak{a}_{\mathfrak{h}} = \{0\}.$$

We go back to Setting 1.3. Set an element:

$$X_0 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{sl}(2, \mathbf{R}) \subset \mathfrak{sl}(2, \mathbf{C}).$$

Building on Fact 3.2, we obtain the following lemma:

Lemma 3.3. *In Setting* 1.3*, let $\Phi : SL(2, \mathbf{R}) \to G$ be a Lie group homomorphism. The differential of Φ will be denoted by $\phi : \mathfrak{sl}(2, \mathbf{R}) \to \mathfrak{g}$. Then, $SL(2, \mathbf{R})$ acts properly on G/H via Φ if and only if $\mathrm{Ad}(G) \cdot \phi(X_0)$ does not meet $\mathfrak{h}$.*

Any Lie algebra homomorphism $\phi : \mathfrak{sl}(2, \mathbf{R}) \to \mathfrak{g}$ can be lifted to $\Phi : SL(2, \mathbf{R}) \to G$ because G is linear. Thus, we obtain the proposition below:

Proposition 3.4. *The condition* (i) *in Theorem* 1.4 *is equivalent to the following condition on a symmetric pair $(\mathfrak{g}, \mathfrak{h})$:*

(vi) *There exists a Lie algebra homomorphism $\phi : \mathfrak{sl}(2, \mathbf{R}) \to \mathfrak{g}$ such that $(\mathrm{Int}\, \mathfrak{g}) \cdot \phi(X_0)$ does not meet $\mathfrak{h}$.*

Step 2: Complexification. We recall $\mathfrak{g}^c = \mathfrak{h} \oplus \sqrt{-1}\mathfrak{q}$. Conversely, $\mathfrak{h}$ is recovered from the c-dual $\mathfrak{g}^c$ by $\mathfrak{h} = \mathfrak{g} \cap \mathfrak{g}^c$. The key proposition in Step 2 is the following

Proposition 3.5. *The condition* (vi) *in Proposition* 3.4 *is equivalent to the following condition on $(\mathfrak{g}_{\mathbf{C}}, \mathfrak{g}, \mathfrak{g}^c)$:*

(vii) *There exists a complex Lie algebra homomorphism $\psi : \mathfrak{sl}(2, \mathbf{C}) \to \mathfrak{g}_{\mathbf{C}}$ such that $(\mathrm{Int}\, \mathfrak{g}_{\mathbf{C}}) \cdot \psi(X_0)$ meets $\mathfrak{g}$ but does not meet $\mathfrak{g}^c$.*

To prove Proposition 3.5, we use the following two lemmas:

Lemma 3.6. *For any complex Lie algebra homomorphism $\psi : \mathfrak{sl}(2, \mathbf{C}) \to \mathfrak{g}_{\mathbf{C}}$, if $\psi(X_0) \in \mathfrak{g}$, then there exists a Lie algebra homomorphism $\phi : \mathfrak{sl}(2, \mathbf{R}) \to \mathfrak{g}$ such that $\phi(X_0) = \psi(X_0)$.*

Lemma 3.7. *In Setting* 1.3*, for any hyperbolic element $X \in \mathfrak{g}$, $(\mathrm{Int}\, \mathfrak{g}) \cdot X$ meets $\mathfrak{h}$ in $\mathfrak{g}$ if and only if $(\mathrm{Int}\, \mathfrak{g}_{\mathbf{C}}) \cdot X$ meets $\mathfrak{g}^c$ in $\mathfrak{g}_{\mathbf{C}}$.*

Lemma 3.6 can be proved by using [6,26].

Proof of Proposition 3.5. For (vi) ⇒ (vii), we take $\phi : \mathfrak{sl}(2, \mathbf{R}) \to \mathfrak{g}$ such that $(\mathrm{Int}\, \mathfrak{g}) \cdot \phi(X_0)$ does not meet $\mathfrak{h}$. Then, $(\mathrm{Int}\, \mathfrak{g}_{\mathbf{C}}) \cdot \phi(X_0)$ does not meet $\mathfrak{g}^c$ by Lemma 3.7. Thus, we can take ψ to be the complex linear extension ϕ. For (vii) ⇒ (vi), we can take $\psi : \mathfrak{sl}(2, \mathbf{C}) \to \mathfrak{g}_{\mathbf{C}}$ such that $\psi(X_0) \in \mathfrak{g}$ and $(\mathrm{Int}\, \mathfrak{g}_{\mathbf{C}}) \cdot \psi(X_0)$ does not meet $\mathfrak{g}^c$. Then $\phi : \mathfrak{sl}(2, \mathbf{R}) \to \mathfrak{g}$ in Lemma 3.6 satisfies the condition (vi). □

Therefore, to prove the equivalence (i) ⇔ (v) in Theorem 1.4, it is enough to show that (vii) ⇔ (v). This equivalence can be proved by using [27, Proposition 1.11] for $\mathfrak{g}$ and $\mathfrak{g}^c$ as real forms of $\mathfrak{g}_{\mathbf{C}}$.

Remark 3.8. In Theorem 2.8, we have given a classification of semisimple symmetric spaces G/H on which surface groups $\pi_1(\Sigma_g)$ act as discontinuous groups via G. Our classification factors through a Lie group homomorphism $SL(2, \mathbf{R})$ to G. It is an open question to find all homomorphisms $\pi_1(\Sigma_g)$ to G which induce proper actions on G/H. This question is intimately related to the deformation of discontinuous groups for G/H, see [12,13] for the general definition of the deformation space of discontinuous groups for G/H with H non-compact, and [8,17] for concrete examples for specific (G, H).

Acknowledgement. I would like to give heartful thanks to Prof. Toshiyuki Kobayashi whose suggestions were of inestimable value for this paper.

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