Discrete torsion and its application for a generalized van der Waerden's theorem

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(Communicated by Kenji FUKAYA, M.J.A., Nov. 14, 2011)

Abstract: In this article, we study discrete curvature and torsion for spatial polygonal lines of unit sides. We express geometric conditions on polygons by using inner products of oriented sides. As a application, we prove a generalization of van der Waerden's theorem. The theorem given in this paper and its proof clarify how the conditions on the sides affect the polygon being planar from the discrete torsion point of view.

Key words: Discrete curvature; discrete torsion; polygonal line.

1. Introduction. A celebrated theorem of B. L. van der Waerden [5] states that an equilateral and equiangular pentagon in Euclidean 3-space, $\mathbf{E}^3$, must be planar and hence a standard regular pentagon. On the other hand it is easy to construct an equilateral and equiangular quadrilateral in $\mathbf{E}^3$ which is not planar. In Section 2, after fixing the notation used in this paper, we will discuss a family of polygons with any even number of sides, "crown type polygons", which are equilateral and equiangular but not planar. This raises the question of polygons with an odd number of sides and in Example 2 we describe a non-planar equilateral and equiangular heptagon. In this paper we study the notion of the torsion of a side of an equilateral and equiangular polygon in $\mathbf{E}^3$ with torsion zero corresponding to a side and its two adjacent sides being coplanar. The main result of this paper is the following theorem.

Theorem. *Let n be an integer greater than or equal to 4, and $\{a_1, a_2, \cdots, a_n\}$ the oriented sides of a closed spatial polygon P with n sides. If the inner products $a_i \cdot a_{i+l}$, $1 \leq i \leq n$, are constant for each fixed integer l ranging from 0 to $\min\{[n/2]-1,\ 3\}$, then the polygon P is either planar or of crown type with even sides, where the subscripts of a_i are regarded in terms of modulo n and the symbol [ ] denotes the Gauss bracket.*

This theorem is a direct generalization of the theorem given by van der Waerden in [5]. When $n = 5$, the statement is equivalent to the van der Waerden's theorem, and especially when $n = 7$, we have the fact that *an equilateral and equiangular heptagon in $\mathbf{E}^3$ is planar, if the absolute value of the torsion of each side is the same.*

2. Discrete curvatures and torsions. In this article, the lengths of the sides of an equilateral polygon will always be normalized, to be of unit length. Discrete curvatures and torsions are studied and applied to various fields such as not only discrete geometry in [4], but also chemistry and physics in [1] and [3]. In the following, we redefine discrete curvature and torsion to fix the notation used in this paper.

For the vertices $\{A_1, A_2, \cdots, A_n\}$ of a given polygon P with n-sides of unit length, set $a_i = \overrightarrow{A_iA_{i+1}}$ $(i = 1, 2, \cdots, n)$ where we identify A_{n+1} with A_1. In this article, we also assume that the adjacent two vectors a_i and a_{i+1} are linearly independent for $i = 1, 2, \cdots, n$ where we also identify a_{n+1} with a_1. Describing the definitions briefly, the discrete curvature at vertex A_{i+1} is the angle between the two vectors a_i and a_{i+1}, on the other hand the discrete torsion of side $A_{i+1}A_{i+2}$ is the angle between the two planes $\triangle A_iA_{i+1}A_{i+2}$ and $\triangle A_{i+1}A_{i+2}A_{i+3}$. The precise definitions of these notion are given as follows:

Definition. For a normalized equilateral polygon P with vertices $\{A_1, A_2, \cdots, A_n\}$ $(A_{n+1} = A_1)$, we define an orthonormal frame $\{X_i, Y_i, Z_i\}$ at each vertex A_{i+1}, by

$$X_i = a_i,\ Y_i = \frac{a_{i+1} - (a_{i+1} \cdot a_i)a_i}{|a_{i+1} - (a_{i+1} \cdot a_i)a_i|},\ Z_i = X_i \times Y_i.$$

2000 Mathematics Subject Classification. Primary 51M04; Secondary 51M20, 52C30, 53A04.

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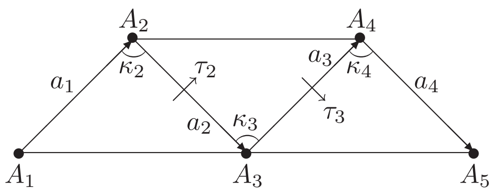
doi: 10.3792/pjaa.87.209

Then, we define a *discrete curvature* κ_{i+1} ($0 < \kappa_{i+1} < \pi$) at each vertex A_{i+1} and a *discrete torsion* τ_{i+1} ($-\pi < \tau_{i+1} \leq \pi$) for each side $A_{i+1}A_{i+2}$ by the following formulas.

$$\kappa_{i+1} = \cos^{-1}(X_i \cdot X_{i+1}),$$

$$\tau_{i+1} = \begin{cases} \cos^{-1}(Z_i \cdot Z_{i+1}) & \text{if } Z_i \cdot a_{i+2} \geq 0, \\ -\cos^{-1}(Z_i \cdot Z_{i+1}) & \text{if } Z_i \cdot a_{i+2} < 0. \end{cases}$$

These are defined regardless of orientation of the curve.



From the Definition above, we can immediately derive the following expression for a normalized equilateral polygon, using inner products.

$$\cos\kappa_{i+1} = X_i \cdot X_{i+1} = a_i \cdot a_{i+1},$$

$$\begin{aligned} \cos\tau_{i+1} &= Z_i \cdot Z_{i+1} \\ &= -\frac{|a_{i+1}|^2 a_i - (a_i \cdot a_{i+1})a_{i+1}}{||a_{i+1}|^2 a_i - (a_i \cdot a_{i+1})a_{i+1}|} \\ &\quad \cdot \frac{|a_{i+1}|^2 a_{i+2} - (a_{i+2} \cdot a_{i+1})a_{i+1}}{||a_{i+1}|^2 a_{i+2} - (a_{i+2} \cdot a_{i+1})a_{i+1}|} \\ &= \frac{(a_i \cdot a_{i+1})(a_{i+1} \cdot a_{i+2}) - a_i \cdot a_{i+2}}{\sqrt{1 - (a_i \cdot a_{i+1})^2}\sqrt{1 - (a_{i+1} \cdot a_{i+2})^2}}. \end{aligned} \tag{1}$$

Remark. The notion of *torsion angle* has been also defined in structural chemistry merely as the angle between two planes ([1], note 5). The discrete torsion with explicit signatures is defined in [3] to study discrete Frenet frames.

By comparison with the curvature and torsion of smooth curves we note the following; see e.g. [2], Sec. 6.1 and Sec. 6.3. Let p be a point on a smooth curve and ω the angle between the tangent at p and the tangent at a nearby point q. Let $|h|$ be the distance from q to p along the curve. Then the curvature at p is the limit of the ratio ω/h as $h \longrightarrow 0$. Similarly let ψ be the oriented angle between the osculating plane at p and the osculating plane at q. Then the torsion of the curve at p is the limit of ψ/h as $h \longrightarrow 0$.

With regard to the constant case for discrete curvatures and torsions, we have the following proposition. This fact will be used in the proof of the Theorem in Section 3.

Proposition. *Any broken curve C of equilateral line segments does not close if the discrete curvatures and torsions are constants excluding* 0 *and* π.

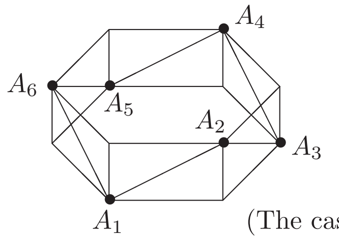
Proof. Let $\{a_1, a_2, \cdots, a_m\}$, $m \geq 3$, denote the oriented sides of the given curve C. Without losing general conditions, we may assume that the length of the sides are equal to one. When we regard the vectors $\{a_1, a_2, \cdots, a_m\}$ as position vectors in $\mathbf{E}^3$, the points $\{a_1, a_2, \cdots, a_m\}$ also define an equilateral curve C' in $\mathbf{E}^3$ whose curvatures are equal to the discrete torsions of C when the sides of C' are normalized. Therefore, all the points $\{a_1, a_2, \cdots, a_m\}$ exist in a small circle on the unit sphere with the origin as its center. Hence, the center of gravity of the points $\{a_1, a_2, \cdots, a_m\}$ satisfies that $\sum_{i=1}^{m} a_i/m \neq 0$, which means that the curve C does not close. □

Definition. A *crown type polygon* is defined to be a spatial equilateral and equiangular polygon whose discrete torsions $\{\tau_i\}$ have the same absolute value and the signs of $\{\sin\tau_i\}$ alternate with respect to the subscript i.

The number of sides of any crown type polygon is even. Moreover, there exists a crown type polygon for any even number of sides greater than or equal to four. In fact, we can realize any crown type $2m$-gon ($m \geq 2$) in a $2m$-gon prism as follows: Let w be a natural number which is less than and relatively prime to $2m$, and r a positive real number less than $1/\sqrt{2 - 2\cos\frac{w\pi}{m}}$. Then, for $i = 1, 2, \cdots, m$, the $2m$ points

$$A_i\left(r\cos\theta_i,\ r\sin\theta_i,\ \frac{(-1)^i}{2}\sqrt{2r^2(\cos\theta_i - 1) + 1}\right),$$

where we have set $\theta_i := iw\pi/m$, are vertices of a normalized crown type $2m$-gon.



(The case of $m = 3$, $w = 1$)

The discrete curvatures and torsions of this polygon satisfy

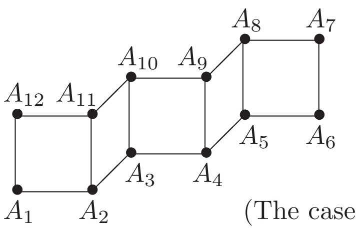
$$\cos \kappa_i = 1 - 2r^2 \sin^2 \frac{w\pi}{m},$$
$$\cos \tau_i = \frac{r^2 \sin^2 \frac{w\pi}{m} - \cos \frac{w\pi}{m}}{1 - r^2 \sin^2 \frac{w\pi}{m}}.$$

Conversely, we have from these formulas that this class of given crown type polygons can be uniquely determined by the discrete curvature and torsion.

Example 1. Any equilateral and equiangular polygon with four sides is either planar or of crown type as stated in the Theorem. On the other hand, there exists a non-crown type spatial equilateral and equiangular polygon for any even number of sides greater than or equal to six: For example,

$$A_i\left(i-1-\left[\frac{i-1}{2}\right], \left[\frac{i-1}{2}\right], 0\right),$$
$$A_{2m+1-i}\left(i-1-\left[\frac{i-1}{2}\right], \left[\frac{i-1}{2}\right], 1\right),$$

$i = 1, 2, \cdots, m$ ($m \geq 3$), is an equilateral and equiangular spatial $2m$-gon of non-crown type, where the symbol [ ] denotes the Gauss bracket.



(The case of $m = 6$)

The discrete torsions of this polygon are written as follows:

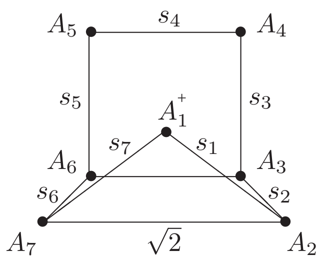
$$m:\ \text{odd} \begin{cases} \tau_1 = -\frac{\pi}{2}, \quad \tau_2 = \cdots = \tau_{m-2} = 0, \\ \tau_{m-1} = \frac{\pi}{2}, \quad \tau_m = 0, \\ \tau_{m+1} = -\frac{\pi}{2}, \quad \tau_{m+2} = \cdots = \tau_{2m-2} = 0, \\ \tau_{2m-1} = \frac{\pi}{2}, \quad \tau_{2m} = 0. \end{cases}$$

$$m:\ \text{even} \begin{cases} \tau_1 = -\frac{\pi}{2}, \quad \tau_2 = \cdots = \tau_{m-2} = 0, \\ \tau_{m-1} = -\frac{\pi}{2}, \quad \tau_m = 0, \\ \tau_{m+1} = \frac{\pi}{2}, \quad \tau_{m+2} = \cdots = \tau_{2m-2} = 0, \\ \tau_{2m-1} = \frac{\pi}{2}, \quad \tau_{2m} = 0. \end{cases}$$

Example 2. Here, we show the necessity of an additional condition to generalize van der Waerden's theorem by constructing an equilateral, equiangular heptagon which is not planar. Begin with three sides, s_3, s_4, s_5, of a square of unit length and consider the plane perpendicular to the parallel sides (s_3, s_5) at the open end of the square. In this plane we have considerable latitude in choosing two unit segments, s_2, s_6, attached to the sides of the open square, s_3, s_5, respectively. Imagine these symmetrically placed and diverging from a direction perpendicular to the plane of the square so that the distance between the endpoints is $\sqrt{2}$. Next form a right angle with two unit segments, s_1, s_7, which will form the remaining sides of the heptagon. We may now attach s_1 to s_2 and s_7 to s_6 at right angles.

Using coordinates, vertices of such a heptagon can be given by the following points:

$$A_1^{\pm}\left(\frac{3+\sqrt{2}}{2\sqrt{1+2\sqrt{2}}}, 0, \frac{\pm\sqrt{7}}{1+2\sqrt{2}}\right),$$
$$A_2\left(\frac{1}{2}\sqrt{1+2\sqrt{2}}, \frac{\sqrt{2}}{2}, 0\right), \quad A_3\left(0, \frac{1}{2}, 0\right),$$
$$A_4\left(0, \frac{1}{2}, 1\right), \quad A_5\left(0, -\frac{1}{2}, 1\right), \quad A_6\left(0, -\frac{1}{2}, 0\right),$$
$$A_7\left(\frac{1}{2}\sqrt{1+2\sqrt{2}}, -\frac{\sqrt{2}}{2}, 0\right).$$



Moreover, maintaining the required property, this heptagon can be prolonged to $(2m+1)$-gon, $m \geq 4$, by replacing side s_4 with sides of unit cubes appropriately. (Compare with the construction in Example 1.)

3. Proof of the Theorem. In this section, we prove the Theorem stated in Introduction. After that, we will give a supplementary explanation of the requisite conditions of the Theorem.

We suppose in this section that the sides have been normalized. From the given condition concerning inner products, we may set $p_l := a_i \cdot a_{i+l}$ for i, l ranging $1 \leq i \leq n$ and $0 \leq l \leq \min\{[n/2] - 1, 3\}$. To prove the Theorem in a simple manner, we state the following lemma.

Lemma. *Let n be an integer between 4 and 7, inclusive, and $\{a_1, a_2, \cdots, a_n\}$ the oriented sides of a closed spatial polygon P with n sides. If the inner products $a_i \cdot a_{i+l}$, $1 \leq i \leq n$, are constant for each fixed integer l, ranging $0 \leq l \leq [n/2]-1$, then the inner products $a_i \cdot a_{i+l}$, $1 \leq i \leq n$, are also constant for each fixed l, ranging $[n/2] \leq l \leq 3$, where the subscripts of a_i are regarded in terms of modulo n.*

Proof. When $n = 4$, the formula $a_i \cdot a_{i+3} = p_1$ is obvious. From

$$(a_i + a_{i+1} + a_{i+2} + a_{i+3}) \cdot a_i = 0$$

we have that $p_0 + p_1 + a_{i+2} \cdot a_i + p_1 = 0$, that is $a_{i+2} \cdot a_i = -p_0 - 2p_1$.

When $n = 5$, the formula

$$a_i \cdot a_{i+2} = a_i \cdot a_{i+3} = -\frac{1}{2}p_0 - p_1$$

is proved in [1], so we omit the proof.

When $n = 6$, from

$$(a_i + a_{i+1} + a_{i+2} + a_{i+3} + a_{i+4} + a_{i+5}) \cdot a_i = 0$$

we have that

$$p_0 + p_1 + p_2 + a_{i+3} \cdot a_i + p_2 + p_1 = 0,$$

that is $a_{i+3} \cdot a_i = -p_0 - 2p_1 - 2p_2$.

When $n = 7$, we can obtain the formula

$$a_i \cdot a_{i+3} = -\frac{1}{2}p_0 - p_1 - p_2 \tag{2}$$

as follows: Since the polygon P is closed, we have

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 = 0.$$

Taking the inner products with the vectors a_i, we have

$$p_0 + 2p_1 + 2p_2 + a_1 \cdot a_4 + a_1 \cdot a_5 = 0, \tag{3}$$
$$p_0 + 2p_1 + 2p_2 + a_2 \cdot a_5 + a_2 \cdot a_6 = 0, \tag{4}$$
$$p_0 + 2p_1 + 2p_2 + a_3 \cdot a_6 + a_3 \cdot a_7 = 0, \tag{5}$$
$$p_0 + 2p_1 + 2p_2 + a_4 \cdot a_7 + a_4 \cdot a_1 = 0, \tag{6}$$
$$p_0 + 2p_1 + 2p_2 + a_5 \cdot a_1 + a_5 \cdot a_2 = 0, \tag{7}$$
$$p_0 + 2p_1 + 2p_2 + a_6 \cdot a_2 + a_6 \cdot a_3 = 0, \tag{8}$$
$$p_0 + 2p_1 + 2p_2 + a_7 \cdot a_3 + a_7 \cdot a_4 = 0. \tag{9}$$

Half of the sum of the above seven equalities, (3) to (9), gives rise to

$$\begin{aligned}\frac{7}{2}p_0 + 7p_1 + 7p_2 + a_1 \cdot a_4 + a_1 \cdot a_5 + a_2 \cdot a_5 \\ + a_2 \cdot a_6 + a_3 \cdot a_6 + a_3 \cdot a_7 + a_4 \cdot a_7 = 0.\end{aligned} \tag{10}$$

The sum of the three equalities (7), (8), and (9) is

$$\begin{aligned}3p_0 + 6p_1 + 6p_2 + a_5 \cdot a_1 + a_5 \cdot a_2 + a_6 \cdot a_2 \\ + a_6 \cdot a_3 + a_7 \cdot a_3 + a_7 \cdot a_4 = 0.\end{aligned} \tag{11}$$

From the difference between the formulas (10) and (11), we obtain that

$$a_1 \cdot a_4 = -\frac{1}{2}p_0 - p_1 - p_2.$$

By substituting this formula in (3), we also have

$$a_1 \cdot a_5 = -\frac{1}{2}p_0 - p_1 - p_2.$$

Likewise we can know the other value of the inner product $a_i \cdot a_{i+3}$ and thus obtain the formula (2). ☐

Proof of the Theorem. From the Lemma above, we may set $p_l := a_i \cdot a_{i+l}$ for i, l ranging $1 \leq i \leq n$ and $0 \leq l \leq 3$. Set $\tilde{a}_i := a_i + a_{i+1}$ for $1 \leq i \leq n$. Since $\sum_{i=1}^n \tilde{a}_i = 0$, the vectors $\{\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n\}$ can be regarded as the oriented sides of a spatial polygon. We denote this polygon by $\tilde{P}$. Then we have

$$\tilde{a}_i \cdot \tilde{a}_i = 2p_0 + 2p_1, \tag{12}$$
$$\tilde{a}_i \cdot \tilde{a}_{i+1} = p_0 + 2p_1 + p_2, \tag{13}$$
$$\tilde{a}_i \cdot \tilde{a}_{i+2} = p_1 + 2p_2 + p_3. \tag{14}$$

Formulas (12), (13), and (14) imply that the discrete curvatures of the polygon $\tilde{P}$ are constant and that the absolute values of the torsions for $\tilde{P}$ are also constant. Let $\tilde{\tau}_i$, $i = 1, 2, \cdots, n$, denote the discrete torsions of $\tilde{P}$. If $\{\sin \tilde{\tau}_i\} = \{0\}$, then $\tilde{P}$ is planar. Otherwise, there must exist consecutive two discrete torsions $\tilde{\tau}_k$ and $\tilde{\tau}_{l+k}$ for which the signs of $\sin \tilde{\tau}_k$ and $\sin \tilde{\tau}_{k+1}$ are the opposite of each other due to the Proposition. In this case, the configuration of the five points

$$\{A_{k-1}, A_k, A_{k+1}, A_{k+2}, A_{k+3}\} \subset \mathbf{E}^3$$

has a symmetry with respect to the 2-plane defined by

$$\left\{X \in \mathbf{E}^3;\ \overrightarrow{A_{k+1}X} \cdot \overrightarrow{A_k A_{k+2}} = 0\right\}.$$

Therefore, the vectors $\{\tilde{a}_{k-1}, \tilde{a}_k, \tilde{a}_{k+1}\}$ are linearly dependent, and hence $\sin \tilde{\tau}_k = 0$. Here we note that, from the formulas (1), (12), (13), and (14), the formula for $\sin^2 \tilde{\tau}_i$ can be written in terms of p_0, p_1, p_2, and p_3, so their values are independent of the subscript i, which implies that $\sin \tilde{\tau}_i$ is identically zero for $i = 1, 2, \cdots, n$, and hence $\tilde{P}$ must be planar. Therefore, the rank of the $3 \times n$ matrix $({}^t\tilde{a}_1, {}^t\tilde{a}_2, \cdots, {}^t\tilde{a}_n)$ is two.

Case 1: The case that n is odd number. Since

$$\begin{cases} a_1 = \quad -\tilde{a}_2 - \tilde{a}_4 - \tilde{a}_6 - \cdots - \tilde{a}_{n-1}, \\ a_2 = \quad -\tilde{a}_3 - \tilde{a}_5 - \tilde{a}_7 - \cdots - \tilde{a}_n, \\ a_3 = -\tilde{a}_1 \quad - \tilde{a}_4 - \tilde{a}_6 - \cdots - \tilde{a}_{n-1}, \\ a_4 = -\tilde{a}_2 \quad - \tilde{a}_5 - \tilde{a}_7 - \cdots - \tilde{a}_n, \\ \quad \vdots \\ a_n = -\tilde{a}_1 - \tilde{a}_3 - \tilde{a}_5 - \cdots - \tilde{a}_{n-2}, \end{cases}$$

the rank of the $3 \times n$ matrix $({}^t a_1, {}^t a_2, \cdots, {}^t a_n)$ is also two. Consequently, we have that the polygon P is planar.

Case 2: The case that n is even number. By the same manner as in the proof of the Proposition in Section 2, we regard the vectors $\{a_1, a_2, \cdots, a_n\}$ as position vectors in $\mathbf{E}^3$. Since the lengths

$$\begin{aligned} |a_{i+1} - a_i| &= \sqrt{2p_0 - 2p_1}, \\ |a_{i+2} - a_i| &= \sqrt{2p_0 - 2p_2}, \\ |a_{i+3} - a_i| &= \sqrt{2p_0 - 2p_3}, \\ |a_{i+2} - a_{i+1}| &= \sqrt{2p_0 - 2p_1}, \\ |a_{i+3} - a_{i+1}| &= \sqrt{2p_0 - 2p_2}, \\ |a_{i+3} - a_{i+2}| &= \sqrt{2p_0 - 2p_1} \end{aligned}$$

are constant, either all the points $\{a_1, a_2, \cdots, a_n\}$ exist in a great circle on the unit sphere with the origin as its center for P to be closed and then P is planar, or we know by the symmetry of the configuration of the four points $\{a_i, a_{i+1}, a_{i+2}, a_{i+3}\}$ on the unit sphere that the vector $a_i + a_{i+3}$ is parallel to the vector $a_{i+1} + a_{i+2}$. In the latter case we have that

$$\begin{aligned} 0 &= \{(a_i + a_{i+3}) \times (a_{i+1} + a_{i+2})\} \cdot a_{i+2} \\ &= (a_i \times a_{i+1}) \cdot a_{i+2} + (a_{i+3} \times a_{i+1}) \cdot a_{i+2} \\ &= (a_i \times a_{i+1}) \cdot a_{i+2} + (a_{i+1} \times a_{i+2}) \cdot a_{i+3}. \end{aligned}$$

Hence, the signs of $\{\sin \tau_{i+1}\}$ alternate with respect to the subscript i, and the polygon P is of crown type. In fact, each of $\{A_1, A_3, \cdots, A_{n-1}\}$ and $\{A_2, A_4, \cdots, A_n\}$ forms equilateral and equiangular planar polygon by the formulas (12), (14). □

The requisite conditions in the Theorem, described by using inner products, are equivalent to the conditions that

when $n = 4, 5$:

$|\overrightarrow{A_iA_{i+1}}| = |\overrightarrow{A_{i+1}A_{i+2}}|$, $|\overrightarrow{A_iA_{i+2}}| = |\overrightarrow{A_{i+1}A_{i+3}}|$;

when $n = 6, 7$:

$|\overrightarrow{A_iA_{i+1}}| = |\overrightarrow{A_{i+1}A_{i+2}}|$, $|\overrightarrow{A_iA_{i+2}}| = |\overrightarrow{A_{i+1}A_{i+3}}|$,
$|\overrightarrow{A_iA_{i+3}}| = |\overrightarrow{A_{i+1}A_{i+4}}|$;

when $n \geq 8$:

$|\overrightarrow{A_iA_{i+1}}| = |\overrightarrow{A_{i+1}A_{i+2}}|$, $|\overrightarrow{A_iA_{i+2}}| = |\overrightarrow{A_{i+1}A_{i+3}}|$,
$|\overrightarrow{A_iA_{i+3}}| = |\overrightarrow{A_{i+1}A_{i+4}}|$, $|\overrightarrow{A_iA_{i+4}}| = |\overrightarrow{A_{i+1}A_{i+5}}|$;

for $i = 1, 2, \cdots, n$, due to the following formulas.

$$\begin{aligned} &|\overrightarrow{A_iA_{i+1}}|^2 - |\overrightarrow{A_{i+1}A_{i+2}}|^2 \\ &\quad = |a_i|^2 - |a_{i+1}|^2 = (a_i \cdot a_i - a_{i+1} \cdot a_{i+1}), \\ &|\overrightarrow{A_iA_{i+2}}|^2 - |\overrightarrow{A_{i+1}A_{i+3}}|^2 \\ &\quad = |a_i + a_{i+1}|^2 - |a_{i+1} + a_{i+2}|^2 \\ &\quad = (a_i \cdot a_i - a_{i+2} \cdot a_{i+2}) + 2(a_i \cdot a_{i+1} - a_{i+1} \cdot a_{i+2}), \\ &|\overrightarrow{A_iA_{i+3}}|^2 - |\overrightarrow{A_{i+1}A_{i+4}}|^2 \\ &\quad = |a_i + a_{i+1} + a_{i+2}|^2 - |a_{i+1} + a_{i+2} + a_{i+3}|^2 \\ &\quad = (a_i \cdot a_i - a_{i+3} \cdot a_{i+3}) + 2(a_i \cdot a_{i+1} - a_{i+2} \cdot a_{i+3}) \\ &\qquad + 2(a_i \cdot a_{i+2} - a_{i+1} \cdot a_{i+3}), \\ &|\overrightarrow{A_iA_{i+4}}|^2 - |\overrightarrow{A_{i+1}A_{i+5}}|^2 \\ &\quad = |a_i + a_{i+1} + a_{i+2} + a_{i+3}|^2 \\ &\qquad - |a_{i+1} + a_{i+2} + a_{i+3} + a_{i+4}|^2 \\ &\quad = (a_i \cdot a_i - a_{i+4} \cdot a_{i+4}) + 2(a_i \cdot a_{i+1} - a_{i+3} \cdot a_{i+4}) \\ &\qquad + 2(a_i \cdot a_{i+2} - a_{i+2} \cdot a_{i+4}) \\ &\qquad + 2(a_i \cdot a_{i+3} - a_{i+1} \cdot a_{i+4}). \end{aligned}$$

4. Conclusion. The theorem given in this paper and its proof clarify how the conditions for sides affect the polygon to be planar from the discrete torsion point of view. The constant value of the inner products p_0, p_1, p_2 determine the absolute value of the torsion, moreover the constant inner product p_3 controls the signature of the torsion. If the torsions have the same signature, plus or minus, the polygonal line does not close by the Proposition in Section 2, otherwise the signatures of the torsions should change alternatively, if not, the torsions are equal to zero, and in these cases, the polygon is either of crown type or planar. However, any polygon with odd sides does not admit the crown type, so the polygon with odd sides satisfying the required conditions must be planar. Especially, when the number of the sides is less than eight, some algebraic relations occur on the relations of inner products $a_i \cdot a_j$, which come from the interference among the sides stated in the proof of the Lemma. Hence, for example, in the van der Waerden case, the conditions of p_2, p_3 are not extrinsically required for the polygon to be planar.

Acknowledgement. The authors would like to express their gratitude to the referee for his valuable comments and suggestions to refine this article, especially for the case of an even number of sides.

References

[1] J. D. Dunitz and J. Waser, The planarity of the equilateral, isogonal pentagon, Elem. Math. **27** (1972), no. 2, 25–32.

[2] A. Goetz, *Introduction to differential geometry*, Addison Wesley Publishing Co., Reading, MA, 1970.

[3] S. Hu, M. Lundgren and A. J. Niemi, Discrete Frenet frame, inflection point solitons, and curve visualization with applications to folded proteins, Phys. Rev. E **83** (2011), 061908.

[4] J. M. Sullivan, Curves of finite total curvature, in *Discrete differential geometry*, Oberwolfach Semin., 38, Birkhäuser, Basel, 2008, pp. 137–161.

[5] B. L. van der Waerden, Ein Satz über räumliche Fünfecke, Elem. Math. **25** (1970), no. 4, 73–78.