On Galois cohomology and weak approximation of connected reductive groups over fields of positive characteristic

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Abstract: We consider some function field analogs of some main cohomological results of Kottwitz theory used in stable trace formula and Colliot-Thélène (and Sansuc) theory and give some applications.

Key words: Galois cohomology; local and global fields; algebraic groups.

Introduction. In [Ko1,Ko2], Kottwitz has given a thorough treatment of the stable trace formula and related endoscopy theory. Among other things, some cohomological treatments given there play an essential role in his theory. Then, in [Bo2], Borovoi has given an abstract approach to some invariant introduced by Kottwitz and developed the theory of abelianized Galois cohomology. Also, in [CT], Colliot-Thélène has extended many important arithmetic invariants available in the case of characteristic 0 (among them some due to Borovoi and Kottwitz) to the arbitrary case. We briefly recall some of them, which are our main concerns in this note.

Let G be a connected (smooth) reductive group defined over a field k with a maximal k-torus T. Let k_s be a separable closure of k, $\Gamma := Gal(k_s/k)$ the absolute Galois group and let M be a (discrete) Γ-module; we denote $\mathrm{H}^i(k, M) := \mathrm{H}^i(\Gamma, M)$ the Galois cohomology group in degree i. Denote by $\hat{G}$ the Langlands dual of G, $\pi_0(.)$ the group of connected components of (.) and by $(.)^D := Hom((.), \mathbf{Q}/\mathbf{Z})$, the Pontryagin dual. We denote $\mathrm{H}^1(k, G) := \mathrm{H}^1(\Gamma, G(k_s))$. Let $\tilde{G}$ be the simply connected covering of the semisimple part $G' = [G, G]$ of G, $p : \tilde{G} \to G' \hookrightarrow G$ the canonical morphism, $\tilde{T} := p^{-1}(T)$, a maximal k-torus of $\tilde{G}$. Let $X^*(.)$ (resp. $X_*(.)$) denote the group of characters (resp. co-characters) of (.). We denote $\pi_1^B(G) := X_*(T)/p_*(X_*(\tilde{T}))$, the algebraic fundamental group of G (after Borovoi [Bo2]), where p_* denotes the induced homomorphism. If X is a sheaf of groups over a scheme S, then considered as a functor, $Z(X)$ (or sometime Z_X for short) denotes the center of X, and $\mathrm{H}^i_*(S, X)$ denotes the cohomology of X, where $*$ stands for "étale", "fppf" or "fpqc".

In [Ko1,Ko2], Kottwitz considers the case when G runs over the category $\mathcal{C}$ of connected reductive groups with *normal homomorphisms*, the functors from $\mathcal{C}$ to *Sets* given by $G \mapsto \mathrm{H}^1(k, G)$, and $G \mapsto \mathcal{A}(G) := \pi_0(Z(\hat{G})^\Gamma)^D$.

0.1. Theorem [Ko1, Section 6, Prop. 6.4] (resp. [Ko2, Section 1, Thm 1.2]). *If k is a local field of characteristic* 0*, then there exists a morphism of functors*

$$\varphi_G : \mathrm{H}^1(k, G) \to \mathcal{A}(G),$$

which is an isomorphism of functors if k is non-archimedean.
(Later on, the assumption with respect to normal homomorphisms has been relaxed by [Mi, Appendix B]. See also [Shi, Lem. 2.3]. Thus 0.1 holds when G runs over the category of all connected reductive groups.)

If k is a global field with the set V of all places, $\bar{k}$ an algebraic closure of k, let $\mathbf{A}$ be the adèle ring of k, $\bar{\mathbf{A}} = \mathbf{A} \otimes_k \bar{k}$. Denote by $Ш^1(k, G) := Ker(\mathrm{H}^1(k, G) \to \prod_{v \in V} \mathrm{H}^1(k_v, G))$, the Shafarevich-Tate "group" of G. Then we have the following

0.2. Theorem. *Let k be a number field.*
1) (see [Ko1, 4.2.2]) *There exists an isomorphism of functors*

$$Ш^1(k, G) \simeq Ш^1(k, Z_{\hat{G}})^D$$

when G runs over $\mathcal{C}$.
2) [Ko2, Theorem 2.2] *We have the following exact sequence of pointed sets, which is functorial when G runs over $\mathcal{C}$*

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$$\mathrm{H}^1(k, G^{ad}) \xrightarrow{\alpha_G} \mathrm{H}^1(\Gamma, G(\bar{\mathbf{A}})/Z_G(\bar{k})) \xrightarrow{\beta_G} \mathcal{A}(G),$$

where $G^{ad} := G/Z_G$ *the adjoint group of* G *and* α *is induced from the following composition map*

$$G^{ad}(\bar{k}) = G(\bar{k})/Z_G(\bar{k}) \hookrightarrow G(\bar{\mathbf{A}})/Z_G(\bar{k}).$$

3) [Ko2, Corol. 2.5, Prop. 2.6.] *The composition map*

$$\delta_G : \mathrm{H}^1(k, G(\bar{\mathbf{A}})) \xrightarrow{\gamma_G} \mathrm{H}^1(k, G(\bar{\mathbf{A}})/Z_G(\bar{k})) \xrightarrow{\beta_G} \mathcal{A}(G)$$

is equal to the composition of the following sequence of maps

$$\mathrm{H}^1(k, G(\bar{\mathbf{A}})) \simeq \oplus_v \mathrm{H}^1(k_v, G) \to$$
$$\to \oplus_v \mathcal{A}_v(G \times_k k_v) \to \mathcal{A}(G).$$

There is an exact sequence of pointed sets

$$1 \to Ш^1(G) \to \mathrm{H}^1(k, G) \to \oplus_v \mathrm{H}^1(k_v, G) \to \mathcal{A}(G)$$

and the local map $\mathrm{H}^1(k_v, G) \to \mathcal{A}(G)$ *is obtained as the composition*

$$\mathrm{H}^1(k_v, G) \xrightarrow{\varphi_{G,v}} \mathcal{A}_v(G \times_k k_v) \xrightarrow{\psi_v} \mathcal{A}(G).$$

In the proof of his many results, Kottwitz makes use of some results due to Sansuc [Sa]. Let G be a linear algebraic group defined over a field k with the set V of all places. For $v \in V$ let k_v be the completion of k at v. We say that G has weak approximation property with respect to a finite subset $S \subset V$ if $G(k)$ is dense in the product $\prod_{v \in S} G(k_v)$ via diagonal embedding, and that G has weak approximation property over k if the above holds for any finite set $S \subset V$. One denotes the obstruction to weak approximation at S (resp. over k) by $\mathrm{A}_k(S, G) := \prod_{v \in S} G(k_v)/Cl_S(G(k))$ (resp. $\mathrm{A}_k(G) := \prod_{v \in V} G(k_v)/Cl(G(k))$, where $Cl_S(G(k))$, $Cl(G(k))$ denote the closure of $G(k)$ in $\prod_{v \in S} G(k_v)$ (resp. $\prod_{v \in V} G(k_v)$). Also, we set

$$Ш^i(G) := \mathrm{Ker}\,(\mathrm{H}^i(k, G) \to \prod_v \mathrm{H}^i(k_v, G)), i \geq 0,$$

whenever it makes sense. Let $\mathcal{B}r(X)$ denote the usual Brauer group of (equivalent classes of) Azumaya algebras over a k-variety X, $Br(X)$ denote the cohomological Brauer group of X. For an extension K/k we denote $X \times_k K$ the base change of X from k to K, and if $K = \bar{k}$, $\bar{X} = X \times_k \bar{k}$ for short. Then we have natural homomorphisms $\mathcal{B}r(k) \to \mathcal{B}r(X) \to \mathcal{B}r(\bar{X})$ and also $Br(k) \to Br(X) \to Br(\bar{X})$, which are semi-exact sequences. We set $\mathcal{B}r_1(X) := Ker(\mathcal{B}r(X) \to \mathcal{B}r(\bar{X}))$, $\mathcal{B}r_0(X) := Im(\mathcal{B}r(k) \to \mathcal{B}r(X))$, $\mathcal{B}r_a(X) := \mathcal{B}r_1(X)/\mathcal{B}r_0(X)$. Let $\mathcal{B}(X)$ (resp. $\mathcal{B}_\omega(X)$) be the set of all elements of $\mathcal{B}r_a(X)$ which is locally trivial for all $v \in V$ (resp. locally trivial for almost all $v \in V$), with respect to natural homomorphisms $\mathcal{B}r_a(X) \to \mathcal{B}r_a(X \times_k k_v)$. We use the same notation for Br.

Quite recently, Colliot-Thélène [CT] has extended many important arithmetic results, among them some due to Sansuc [Sa] cited above for connected linear algebraic groups defined over number fields to fields of characteristic 0, and some of them to arbitrary characteristic, and also gives another interpretations (and proofs) of old results. Among many other things, the following result holds.

0.3. Theorem (see [CT, Sec. 9, Thm 9.4] for 1)–3), [Sa, Corol. 8.14] for 4)).
Let $1 \to S \to H \to G \to 1$ *be a flasque resolution of a connected reductive group, all are defined over a number field* k*. This sequence induces*
1) *an isomorphism of finite abelian groups*

$$\mathrm{A}_k(G) \simeq Coker(\mathrm{H}^1(k, S) \to \oplus_v \mathrm{H}^1(k_v, S));$$

2) *a bijection* $Ш^1(G) \simeq Ш^2(S)$ *between finite sets*;
3) *for any smooth* k*-compactification* X *of* G*, an exact sequence of finite abelian groups*

$$1 \to \mathrm{A}_k(G) \to (Br(X)/Br(k))^D \to Ш^1(G) \to 1;$$

4) *an exact sequence of finite abelian groups*

$$1 \to \mathrm{A}_k(G) \to (\mathcal{B}_\omega(G))^D \to Ш^1(G) \to 1.$$

Our aim in this note is to investigate a possible extension of (0.1)–(0.3) to the case of local and global function fields, since, as one may hope, for example, (0.1)–(0.3) should have some applications to the study, which was initiated by Kottwitz in [Ko1, Ko2], when applied to the function field case. Also, the results mentioned above, which are of interest themselves in their own right, should have counterparts in the case of function fields. The only problem with 0.3, 3), 4) above is that in general, we do not know if for any given linear algebraic group G defined over a field k of characteristic $p > 0$, there exists a *smooth* compactification X over k for G. It would be so, if we assume the resolution of singularities over k (however, see 3.3 below). The details of proofs will appear elsewhere.

1. Preliminaries. We keep above notation and assume k to be an arbitrary field. In general we use the same notation as in [Ko1,Ko2]. In particular, we denote by k_s (resp. $\bar{k}$) a fixed separable (resp. algebraic) closure of k. First we need the following notion of the algebraic fundamental groups $\pi_1(G)$ due to Borovoi [Bo2] (in characteristic 0) and Colliot-Thélène [CT] (in any characteristic) and Kottwitz's group $\mathcal{A}(G)$ and relation between them. Let H be a connected linear algebraic group defined over k, supposed reductive if $char.k > 0$. Then H is called *quasi-trivial* (after Colliot-Thélène [CT]), if $k_s[H]^*/k_s^*$ is a permutation Γ-module and the Picard group $Pic(H_{k_s}) = 0$, where $k_s[H]$ stands for the affine algebra of H, and A^* stands for the group of invertible elements of the ring A. Then if H^{tor} denotes the maximal toric quotient of H, $P := H^{tor}$ is an induced k-torus. Then we have

1.1. Proposition ([CT, Prop. 3.1, 4.2]). *For any connected linear algebraic k-group G (supposed to be reductive if $char.k > 0$)*
1) *There exists a flasque k-torus S, a quasi-trivial connected linear algebraic k-group H (which is also reductive, if G is), with the following exact sequence* $1 \to S \to H \to G \to 1$.
2) *There exists an extension H of G by an induced k-torus Z, such that $[H, H]$ is simply connected and $T = H/[H, H]$ is a coflasque k-torus.*
The embedding $S \to H$ induces a morphism $\alpha : S \to P = H/H^{der}$, thus also a homomorphism $\alpha_* : X_*(S) \to X_*(P)$, and we called $\pi_1^{CT}(G) := Coker(\alpha_*)$ the algebraic fundamental group of G (after Colliot-Thélène). Let $\tilde{T}$ and T stand for the tori that we introduce above for the maximal Levi reductive k-subgroup of G (called *the reductive part* of G in the sequel, which is G itself if $char.k > 0$). Let G be a connected linear algebraic group defined over a field k, which is reductive if $char.k > 0$. Then, it is no harm if we define the Langlands dual $\hat{G}$ of G as that of reductive part of G. Then we have

1.2. Proposition. 1) [CT, Prop. A1, A2] *The complexes $[\tilde{T} \to T]$ and $[S \to P]$ are naturally quasi-isomorphic. Thus the Galois Γ-modules $\pi_1^B(G)$ and $\pi_1^{CT}(G)$ are naturally isomorphic.*
2) [Bo2, Prop. 1.10] *Let G be a connected reductive group defined over a field k. Then the Galois Γ-modules $\pi_1^B(G)$ and $X^*(Z(\hat{G}))$ are canonically isomorphic.*
(In [Bo2, Prop. 1.10], the proof of 2) is given for the case $char.k = 0$, but one can check that it also extends verbatim to the case $char.k > 0$. In the sequel, we use the terminology *the algebraic fundamental group* without referring to Borovoi's or Colliot-Thélène's definition. It is worth also of mentioning the following

1.3. Proposition. a) [Mi, Remark B.5] *We have a duality of abelian groups of finite type*

$$(\pi_1(G)_{\Gamma,tors})^D \simeq \mathcal{A}'(G) := \pi_0(Z(\hat{G})^\Gamma).$$

b) [CT, Prop. 6.3.] *We have the following isomorphism*

$$(\pi_1(G)_{\Gamma,tors})^D \simeq Pic(G).$$

The original proof of a) above also works in the case $char.k > 0$.

1.4. Proposition ([Ko1], Sec. 2.4). *Let k be any field, G a connected reductive k-group. Then there are isomorphisms of functors*

$$\pi_0(Z(\hat{G})^\Gamma) \simeq Pic(G),\ \mathrm{H}^1(k, Z(\hat{G})) \simeq \mathcal{B}r_a(G),$$

which is functorial in G when G runs over $\mathcal{C}$.
Thus, one may also write $Pic(.)^D$ instead of $\mathcal{A}(.)$.

1.5. *z*-extensions. Let G be a connected reductive group defined over a field k. A *z-extension* of G is a connected reductive k-group H such that the semisimple part of H (the derived subgroup of H) is simply connected and H is an extension (in the sense of algebraic groups) of G by means of an induced k-torus Z, i.e., we have an exact sequence of k-groups

$$1 \to Z \to H \to G \to 1.$$

This notion was introduced (and the existence of such extensions for any given G was proved) by Langlands in the case of characteristic 0, but one checks that the same also holds in the case of positive characteristic.

2. Extensions of some of Kottwitz results to function fields. First we proceed to extend the first result (0.1) obtained by Kottwitz [Ko2]. We have the following

2.1. Theorem. *Let k be a local function field, G a connected reductive group defined over k. Then there exists an isomorphism of functors*

$$\varphi_G : \mathrm{H}^1(k, G) \simeq \mathcal{A}(G) := \pi_0(Z(\hat{G})^\Gamma)^D \simeq Pic(G)^D.$$

The proof follows the same scheme as in [Ko1], where we need also the following ingredients.

2.2. Lemma ([Ko1, Lemma 6.1]). *Let G be a connected reductive group defined over a field k. Let $DG := [G, G]$ the semisimple part of G, $\tilde{G}$ the simply connected covering of DG over k, $S := (Z(G))^\circ$ the* *identity component of the center of G, $C := S \cap DG$ (the schematic intersection), C' be the inverse image of C via natural k-morphism $p : \tilde{G} \to DG$, and let $i : C' \to Z$ be a k-embedding into a k-torus Z. Then there exists a central extension G_1 of G by Z such that DG_1 is simply connected.*

2.3. Lemma. 1) [Th1, Lemma 2.1] *Let G be a semisimple group defined over a local function field k. Then there exist maximal k-tori of G which are anisotropic over k and thus have trivial Galois cohomology in degree 2.*
2) *Let k be a global function field, G a semisimple k-group, S any finite set of places of k. Then there exists a maximal k-torus T of G such that T is k_v-anisotropic for all $v \in S$.*
3) *Let k be a global function field, G a semisimple k-group. Then there exists a maximal k-torus T of G such that $\text{Ш}^2(T) = 0$.*

2.4. Lemma. *Let k be a local function field of characteristic p, A a finite k-group scheme of multiplicative type. Then A can be embedded over k into an anisotropic k-torus Z.*
This is the function field analog of [Ko1, Lemma 6.2], which was stated for the case k is a $\mathfrak{p}$-adic field.

2.5. Proof of Theorem 2.1.

2.5.1. First proof. The proof of Theorem 2.1 follows the same line as in [Ko1], where the corresponding lemmas are stated as above. □

2.5.2 Second proof. Basically, the proof given in [CT] of Theorem 9.1, (ii), still holds in our case and we give more details since it was a bit sketchy there. Let $1 \to S \to H \to G$ be a flasque resolution of G, $P := H/[H, H]$ is an induced k-torus (see Section 1). We need the following

2.5.2.1. Lemma. *Let k be a local or global function field. Then above resolution induces a natural bijection $\alpha : \mathrm{H}^1(k, G) \simeq Ker(\mathrm{H}^2(k, S) \to \mathrm{H}^2(k, P))$.*
To proceed with the proof, we recall the following results due to Colliot-Thélène [CT]. Let $1 \to S \to H \to G \to 1$ be a flasque resolution of G, all defined over k, $P = H/H' = H^{tor}$ the torus quotient of H, which is an induced k-torus.

2.5.2.2. Theorem. *We keep the above notation, except that k may be any field. Let X be any k-scheme.*
a) (*See* [CT, Prop. 3.3.]) *There exists an exact sequence of abelian groups*

$$X^*(P)^\Gamma \to X^*(S)^\Gamma \to Pic(G) \to 0.$$

b) (*See* [CT, Prop. 8.2.]) *There exists following commutative diagram*

$$\begin{array}{ccccc} \mathrm{H}^2_{et}(X, S) & \times & X^*(S)^\Gamma & \xrightarrow{p} & Br(X) \\ \downarrow & & \uparrow & & \downarrow = \\ \mathrm{H}^2_{et}(X, P) & \times & X^*(P)^\Gamma & \xrightarrow{q} & Br(X) \end{array}$$

The proof of 2.1 follows by using Lemmas 2.5.2.1, Theorem 2.5.2.2 combined with an argument given in [CT]. □

2.5.3. *Some applications.* As a consequence of the proof we give a new proof of the following important result on the finiteness of Galois cohomology of connected reductive groups over local fields. In the case k is a local field of characteristic 0, this finiteness result, which holds for any linear algebraic k-group, is well-known and is due to Borel-Serre (see [Se, Chap. III]). In the case $char.k > 0$, it is necessary to restrict to the case of connected reductive groups only. The original proof (loc. cit) makes use of Bruhat-Tits theory [BrT].

2.5.3.1. Theorem (Tits, see [Se, Chap. III, p. 148]). *If k is a local function field and G is a connected reductive k-group, then $\mathrm{H}^1(k, G)$ is finite.*

Proof. Since there is a bijection $\mathrm{H}^1(k, G) \simeq Ker(\mathrm{H}^2(k, S) \to \mathrm{H}^2(k, P))$ (Lemma 2.5.2.1), while the latter is isomorphic to $(Coker(X^*(P)^\Gamma \to X^*(S)^\Gamma))^D \simeq Pic(G)^D$, by Theorem 2.5.2.2, which is finite, so $\mathrm{H}^1(k, G)$ is finite. □

The following result was proved by Borovoi in [Bo1, Corollary 6.4] in the case k is a local non-archimedean field of characteristic 0. We apply the Kottwitz's group $\mathcal{A}(G)$ to give a new proof, which also is valid in the function field case, for connected reductive groups G_i.

2.5.3.2. Corollary (see [Bo1, Corollary 6.4] for $char.k = 0$). *Let k be a non-archimedean local field (of any characteristic). Assume that $1 \to G_1 \to G_2 \to G_3 \to 1$ is an exact sequence of (smooth) connected reductive groups all are defined over k, where G_1 is semisimple. Then the natural map of Galois cohomology*

$$\mathcal{A}(G_2) \simeq \mathrm{H}^1(k, G_2) \to \mathrm{H}^1(k, G_3) \simeq \mathcal{A}(G_3)$$

is surjective.
Next we have the following analog of Theorem 0.2.

2.6. Theorem. *Let k be a field, G a connected reductive k-group. For a group scheme X, let denote Z_X the center of X.*

1) *Let k be a global function field. There exists an isomorphism of functors*

$$\text{III}^1(k, G) \simeq \text{III}^1(k, Z(\hat{G}))^D.$$

2) *Let k be a global function field. We have the following exact sequence of pointed sets, which is functorial in G*

$$\mathrm{H}^1_{fppf}(k, G^{ad}) \xrightarrow{\alpha_G} \mathrm{H}^1_{fppf}(k, G(\bar{\mathbf{A}})/Z_G(\bar{k})) \xrightarrow{\beta_G} \mathcal{A}(G),$$

where $G^{ad} := G/Z_G$ is the adjoint group of G and α_G is induced from the following composition map

$$G^{ad}(\bar{k}) = G(\bar{k})/Z_G(\bar{k}) \hookrightarrow G(\bar{\mathbf{A}})/Z_G(\bar{k}).$$

3) *Let k be a global function field. The composition map (denoted by δ_G)*

$$\mathrm{H}^1_{fppf}(k, G(\bar{\mathbf{A}})) \xrightarrow{\gamma_G} \mathrm{H}^1_{fppf}(k, G(\bar{\mathbf{A}})/Z_G(\bar{k})) \xrightarrow{\beta_G} \mathcal{A}(G)$$

is equal to the composition of the following sequence of maps

$$\mathrm{H}^1_{fppf}(k, G(\bar{\mathbf{A}})) \simeq \oplus_v \mathrm{H}^1_{fppf}(k_v, G) \rightarrow \oplus_v \mathcal{A}_v(G \times_k k_v) \rightarrow \mathcal{A}(G).$$

There is an exact sequence of pointed sets

$$1 \rightarrow \text{III}^1(G) \rightarrow \mathrm{H}^1_{fppf}(k, G) \rightarrow \oplus_v \mathrm{H}^1_{fppf}(k_v, G) \rightarrow \mathcal{A}(G)$$

and the local map $\mathrm{H}^1(k_v, G) \rightarrow A(G)$ is obtained as the composition

$$\mathrm{H}^1_{fppf}(k_v, G) \xrightarrow{\sim} \mathcal{A}_v(G \times_k k_v) \rightarrow \mathcal{A}(G).$$

4) *Let k be a global field. For any connected reductive group G over a global field k, we set*

$$C(G) := Coker(\oplus_v \mathcal{A}_v(G \times k_v) \rightarrow \mathcal{A}(G)).$$

The group $C(G)$ can be computed as follows. Let $\tilde{G}$ be the simply connected covering of the semisimple part G' of G, $F := Ker(\tilde{G} \rightarrow G)$, $G^{tor} := G/G'$ the maximal torus quotient of G and let $\text{III}Pic(G) := Ker(Pic(G) \rightarrow \prod_v Pic(G \times_k k_v))$. Then we have

$$C(G) \simeq \text{III}Pic(G) \simeq \text{III}^2(G^{tor}).$$

In particular, we have the following exact sequence

$$1 \rightarrow \text{III}^1(G) \rightarrow \mathrm{H}^1_{fppf}(k, G) \rightarrow \oplus_v \mathrm{H}^1_{fppf}(k_v, G) \rightarrow \rightarrow \mathcal{A}(G) \rightarrow \text{III}^2(G^{tor}) \rightarrow 1.$$

The proof is lengthy and makes use of 2.1, and the method of z-extensions due to Kottwitz.

3. Weak approximation and related invariants over function fields. Our aim in this section is to extend some of results due to Colliot-Thélène and [Sa] mentioned in Introduction to the case of local and global fields of positive characteristic, which were not considered in [CT] and [Sa]. We have

3.1. Theorem (Cf. [CT, Sec. 9], [Sa, Sec. 2,3], [Th4, Th5] and reference therein in the case of number fields)**.** *Let k be a global field.*

1) *For any z-extension $1 \rightarrow Z \rightarrow H \rightarrow G \rightarrow 1$ of a connected reductive k-group G, $T = H^{tor} = H/[H, H]$, and finite set S of valuations, we have canonical isomorphisms of finite abelian groups*

$$\mathrm{A}_k(S, G) \simeq \mathrm{A}_k(S, H) \simeq \mathrm{A}(S, T),$$
$$\mathrm{A}_k(G) \simeq \mathrm{A}_k(H) \simeq \mathrm{A}_k(T).$$

For any flasque resolution $1 \rightarrow F \rightarrow H \rightarrow G \rightarrow 1$ of G, where F is a flasque k-torus, we have

$$\mathrm{A}_k(G) \simeq Coker\,(\mathrm{H}^1(k, F) \xrightarrow{\gamma} \prod_v \mathrm{H}^1(k_v, F)).$$

2) *$\mathrm{A}_k(S, G)$ is a finite abelian group. There exists a well-defined finite set S_0 of valuations of k, such that for any other finite set S of valuations, we have*

$$\mathrm{A}_k(S, G) = \mathrm{A}_k(S \cap S_0, G).$$

In particular, for any $S \supset S_0$, we have

$$\mathrm{A}_k(S, G) = \mathrm{A}_k(S_0, G) = \mathrm{A}_k(G).$$

3) *For any finite extension l/k, let $G_l = G \times_k l$ be the base change of G from k to l, S_l be the extension of S to l. Then we have natural norm homomorphisms of finite abelian groups $N_{S,l/k} : \mathrm{A}_k(S_l, G_l) \rightarrow \mathrm{A}_k(S, G)$, and $N_{l/k} : \mathrm{A}_k(G_l) \rightarrow \mathrm{A}_k(G)$, which are functorial in G, and for a tower of finite separable extensions $E/K/k$, the norm homomorphisms satisfy*

$$N_{S_E, E/k} = N_{S_K, K/k} \circ N_{S_E, E/K}.$$

We have the following relations with Shafarevich-Tate "group" III.

3.2. Theorem (Cf. [CT, Sec. 9, Thm 9.4] for 1)–3), [Sa, Corol. 8.14] for 4) for the case of number fields)**.** *Let $1 \rightarrow S \rightarrow H \rightarrow G \rightarrow 1$ be a flasque resolution of a connected reductive group, all are defined over a global field k. Then this sequence induces*

1) *a bijection $\text{III}^1(G) \simeq \text{III}^2(S)$ between finite sets*;

2) *for any smooth k-compactification $\mathcal{X}$ of G, an exact sequence of finite abelian groups*

$$1 \rightarrow \mathrm{A}_k(G) \rightarrow (Br_1(\mathcal{X})/Br(k))^D \rightarrow \text{III}^1(G) \rightarrow 1;$$

3) *an exact sequence of finite abelian groups*

$$1 \to \mathrm{A}_k(G) \to (\mathcal{B}_\omega(G))^D \to \mathrm{III}^1(G) \to 1.$$

Note that in 3.2, 2), we have to assume the existence of a smooth compactification $\mathcal{X}$ of G over k. Without such assumption, if G is a k-torus, such $\mathcal{X}$ always exist over any field ([CTHS]), and we have the following

3.3. Theorem (Cf. [Th2, Th3] if k is a number field)**.** *Let k be a global function field, G a connected reductive k-group. Take any z-extension $1 \to Z \to H \to G \to 1$ of G and set $T = H/[H, H]$.*
1) *There are bijections of finite sets*

$$\mathrm{III}(G) \simeq \mathrm{III}(H) \simeq \mathrm{III}(T).$$

2) *We may choose the above z-extension to be also a co-flasque resolution of G and choose a smooth k-compactification $\mathcal{X}$ of T. Then we have the following exact sequence*

$$1 \to \mathrm{A}_k(G) \to \mathrm{H}^1(k, Pic(\bar{\mathcal{X}}))^* \to \mathrm{III}(G) \to 0,$$

which is independent of the choice of coflasque z-extension of G.

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References

[Bo1] M. V. Borovoi, Abelianization of the second nonabelian Galois cohomology, Duke Math. J. **72** (1993), no. 1, 217–239.

[Bo2] M. V. Borovoi, Abelian Galois cohomology of reductive groups, Mem. Amer. Math. **132** (1998), no. 626, 1–50.

[BrT] F. Bruhat and J. Tits, Groupes algébriques sur un corps local. Chapitre III. Compléments et applications à la cohomologie galoisienne, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **34** (1987), no. 3, 671–698.

[CT] J.-L. Colliot-Thélène, Résolutions flasques des groupes linéaires connexes, J. Reine Angew. Math. **618** (2008), 77–133.

[CTHS] J.-L. Colliot-Thélène, D. Harari et A. N. Skorobogatov, Compactification equivariante d'un tore (d'apres Brylinski et Künnemann), Expo. Math. **23** (2005) 161–170.

[Ko1] R. E. Kottwitz, Stable trace formula: cuspidal tempered terms, Duke Math. J. **51** (1984), no. 3, 611–650.

[Ko2] R. E. Kottwitz, Stable trace formula: elliptic singular terms, Math. Ann. **275** (1986), no. 3, 365–399.

[Mi] J. S. Milne, The points on a Shimura variety modulo a prime of good reduction, *The zeta functions of Picard modular surfaces*, Univ. Montréal, Montreal, 1992, pp. 151–253.

[Ngô] B. C. Ngô, Le lemme fondamental pour les algèbres de Lie, Publ. Math. Inst. Hautes Études Sci. No. 111 (2010), 1–169.

[Sa] J.-J. Sansuc, Groupe de Brauer et arithmétique des groupes algébriques linéaires sur un corps de nombres, J. Reine Angew. Math. **327** (1981), 12–80.

[Se] J.-P. Serre, *Cohomologie galoisienne*, fifth edition, Lecture Notes in Math., 5, Springer, Berlin, 1994; *Galois cohomology*, translated from the French by Patrick Ion and revised by the author, Springer, Berlin, 1997.

[Shi] S. W. Shin, Counting points on Igusa varieties, Duke Math. J. **146** (2009), no. 3, 509–568.

[Th1] N. Q. Thǎńg, Weak approximation, R-equivalence and Whitehead groups, in *Algebraic K-theory* (*Toronto, ON, 1996*), 345–354, Fields Inst. Commun., 16, Amer. Math. Soc., Providence, RI, 1997.

[Th2] N. Q. Thǎńg, Weak approximation, Brauer and R-equivalence in algebraic groups over arithmetical fields, J. Math. Kyoto Univ. **40** (2000), no. 2, 247–291.

[Th3] N. Q. Thǎńg, Weak approximation, Brauer and R-equivalence in algebraic groups over arithmetical fields. II, J. Math. Kyoto Univ. **42** (2002), no. 2, 305–316.

[Th4] N. Q. Thǎńg, On corestriction principle in non abelian Galois cohomology over local and global fields, J. Math. Kyoto Univ. **42** (2002), no. 2, 287–304.

[Th5] N. Q. Thǎńg, On Galois cohomology of semisimple groups over local and global fields of positive characteristic, Math. Z. **259** (2008), no. 2, 457–470.