

## On certain extremal pencils of curves with respect to the total reducibility order

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**Abstract:** We consider another application of the Ogg-Shafarevich-Grothendieck formula (abbr. OSG) to pencils of plane curves of degree  $d$  with certain extremal properties of reducibility. Besides some new results for extremal pencils we treat also various aspects of the topics, *e.g.* pencils with small number of special fibres, families with  $(2IV_d)$ , etc.

**Key words:** Extremal pencils; total reducibility order; Ogg-Shafarevich-Grothendieck formula.

**Introduction.** The problem of classifying non-trivial families of curves of genus  $g \geq 1$  over  $\mathbf{P}^1$  with small number of singular fibres was motivated by the function field analogue of the second Shafarevich conjecture stating that there are no such smooth families (*cf.* [11–13]). A new interest to reducible fibres, in particular completely reducible fibres, comes from the other branches of mathematics ([7,10,14–18]). Unless otherwise stated the ground field  $k$  is assumed algebraically closed of any characteristic. By “tame condition” on characteristic we mean either  $\text{char}(k) = 0$ , or  $> 2g + 1$ .

The concept of total order of reducibility  $\rho$  was traced back to Poincaré since 1891. Stein’s theorem ([15]) asserts that  $\rho(p) \leq d - 1$  for a non-composite polynomial  $p \in \mathbf{C}[x, y]$  of degree  $d$ . Lorenzini ([10]) extended this result to pencils of curves in  $\mathbf{P}^2(k)$  with bound  $\rho(p/q) \leq d^2 - 1$ . We are mainly interested in the extremal case in both estimates above. For pencils of complex generically smooth plane curves of degree  $d$  Ruppert ([14]) showed that the number of reducible fibres  $s_{rd} \leq 3d - 3$ . We derive Ruppert’s bound as a direct consequence of the OSG formula. Moreover the equality holds only for Ruppert’s pencils.

Examples of polynomials with (maximal)  $s_{rd} = d - 1$  were constructed in [10]. It will be shown in §2 that these are the only polynomials with  $s_{rd}(p) = d - 1$  (up to coordinate change). We prove also a

uniqueness theorem for extremal polynomial mappings on  $\mathbf{A}^2$  with two reducible fibres.

Along the same line of ideas one can give an algebraic approach to the problem of classifying normal forms of polynomials  $\mathbf{A}^2$  with one singular fibre (*cf.* [19,20] and §3).

By analogy with Kodaira-Néron classification of elliptic degenerations let us denote by  $IV_d$  the degenerate configuration consisting of  $d$  concurrent lines. In §4 we deduce some results for families of curves with  $(2IV_d)$  extending previously known results (*cf.* [17]). In the final paragraph some results related to the Hesse cubic pencil in characteristic  $\neq 3$

$$(\mathcal{H}) \quad 3xy - t(x^3 + y^3 + 1) = 0$$

are given. In particular it is established that  $(\mathcal{H})$  is the only extremal pencil, *i.e.* with maximal  $\rho(p/q) = d^2 - 1$  (Theorem 5.2). It is of interest in view of recent results and “4 completely reducible fibres conjecture” *à la* Libgober, Yuzvinsky *et al.* ([9,17,18]).

### 1. Total reducibility order of a pencil.

Let  $p, q \in k[x, y]$ ,  $\max(\deg p, \deg q) = d$ . Standard homogeneous extensions of these polynomials define two curves in the projective plane  $\mathbf{P}^2$  which are denoted by  $C_0$  and  $C_\infty$  respectively. We assume that  $C_0$  and  $C_\infty$  intersect in finitely many points. Consider a pencil of curves defined by  $p(x, y) - tq(x, y) = 0, t \in k$ , or equally speaking the pencil defined by the rational function  $p/q$ . In homogeneous coordinates it is

$$C_t := \{(x : y : z) \in \mathbf{P}^2 : z^d p(x/z, y/z) - tz^d q(x/z, y/z) = 0\}, \quad t \in k.$$

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This work is dedicated to the memory of people who passed away in the March 11th devastating earthquake and tsunami.

**Definition** ([15]). *The total reducibility order of the pencil  $p/q$  is defined to be*

$$\rho(p/q) := \sum_{t \in k} (r_t - 1),$$

where  $r_t$  denotes the number of irreducible components of  $C_t$ .

**Theorem 1.1** ([10]). *We have  $\rho(p/q) \leq d^2 - 1$ .*

We give here a new interpretation of this result in term of the virtual Mordell-Weil rank  $r_{MW}$  introduced in [11]. The rational function  $p/q$  gives rise to a rational map:  $\mathbf{P}^2 \setminus \{C_0 \cap C_\infty\} \rightarrow \mathbf{P}^1$ . By resolving indeterminacy at intersection points of  $C_0$  and  $C_\infty$  via a finite sequence of blowing-ups we get a morphism  $f: X \rightarrow \mathbf{P}^1$ . The exceptional divisors that map onto  $\mathbf{P}^1$  under  $f$  are called horizontal components of  $f$  and we denote their number by  $n_h$ . From the formula relating the Picard number of  $X$  and  $r_{MW}$  ([11,12]) we infer

$$(1.1) \quad \rho(p/q) = n_h - r_\infty - r_{MW}$$

where  $r_\infty$  - the number of irreducible components of  $C_\infty$ . The conclusion of Theorem 1.1 now follows from (1.1) by remarking that  $n_h \leq \#\{\text{blow-ups}\} \leq d^2$ .

**Remark.** (i) For polynomial mappings on  $\mathbf{A}^2$  it would be interesting to interpret  $r_{MW}$  in (1.1) as an affine counterpart for the generalized Jacobians; (ii) In fact Lorenzini gave in [10] a more precise bound by generalizing Lin's remark (cf. [7]): if general curve  $C_\eta$  is of type  $(g, n(\eta))$ , i.e. a smooth curve of genus  $g$  with  $n(\eta)$  'punctures', then  $n_h \leq n(\eta)$ .

**Corollary 1.2** (cf. [7,15]). *For a non-composite polynomial  $p: \mathbf{A}^2 \rightarrow \mathbf{A}^1$  of degree  $d$  we have  $\rho(p) \leq d - 1$ .*

Polynomials  $p$  with  $\rho(p) = d - 1$  are called *extremal*. Since for non-composite polynomial mappings  $n_h \leq n(\eta) \leq d$ , we come to Corollary 1 in [10] and the following corollaries.

**Corollary 1.3.** *Fibres of extremal polynomials are reduced.*

**Remark.** It should be worth emphasizing the following two extremal subcases: (i)  $x^d - y^d = t$  having only one reducible fibre of type  $IV_d$  over  $t = 0$ ; (ii)  $p_{1,d-1}(x, y) := y \prod_{i=1}^{d-1} (x - a_i) + x = t$  with distinct  $a_i$  having  $d - 1$  reducible fibres over  $t = a_i, i = 1, \dots, d - 1$ .

**Corollary 1.4.** *A non-composite polynomial mapping  $p: \mathbf{A}^2 \rightarrow \mathbf{A}^1$  has at most one completely reducible fibre.*

Indeed if  $p(x, y) = \prod_{i=1}^{r_0} l_i(x, y)^{e_i}$  with linear  $l_i(x, y), i = 1, \dots, r$ , so  $\rho(p) = r_0 - 1$ .

**2. Ruppert's pencils and their affine analogues.** Over  $\mathbf{C}$  Ruppert proved that the number of reducible members in a pencil of non-singular plane curves of degree  $d$  is at most  $3d - 3$  ([14], Satz 6). Furthermore he produced pencils with exactly  $3d - 3$  reducible members as follows: Consider the following net of non-singular plane curves of degree  $d \geq 3$ :  $x(y^{d-1} - 1) + \lambda y(x^{d-1} - 1) + \mu(x^{d-1} - y^{d-1})$ . Then a generic line in the plane of parameters  $(\lambda, \mu)$  gives rise to a pencil with  $3d - 3$  reducible curves as its singular members each of them is a union of a line and a smooth curve of degree  $(d - 1)$  intersecting transversally at  $(d - 1)$  points. We call such a pencil Ruppert's pencil. As in §1 we work over any characteristic.

**Theorem 2.1.** *Let  $f: X \rightarrow \mathbf{P}^1$  be the resulting morphism obtained from a pencil of generically non-singular plane curves of degree  $d$  by a finite sequence of blow-ups as in §1. Let  $s_{rd}(f)$  denote the number of reducible fibres of  $f$ . Then  $s_{rd}(f) \leq 3d - 3$ . Moreover the equality implies that  $f$  is obtained from a Ruppert pencil.*

In the situation above the OSG formula for  $f$  ([12]) gives us

$$(2.1) \quad \sum_{t \in S} [g - g(\tilde{X}_t) + n_t - 1] \leq 3(d - 1)^2$$

where  $S$  denotes the set of singular fibres of  $f$ ,  $g =$  genus of a general fibre,  $\tilde{X}_t$  is the normalization of fibre  $X_t$  over  $t$  and  $n_t$  - the number of its irreducible components. By using the genus formula for plane (reducible) curves one has

$$(2.2) \quad g - g(\tilde{X}_t) + n_t - 1 \geq d - 1.$$

The conclusion of Theorem 2.1 now follows from (2.1), (2.2) and equalities in both of them imply that all singular fibres of  $f$  are semi-stable ([12], Lemma 1) with same configurations, as in a Ruppert pencil.

We have the following affine version of Theorem 2.1.

**Theorem 2.2.** *Up to coordinate change non-composite polynomials in two variables with  $(d - 1)$  reducible fibres are  $p_{1,d-1}(x, y)$ .*

The first statement follows from Corollary 1.2. Moreover if  $p$  has  $(d - 1)$  reducible fibres, then all these fibres are reduced (Corollary 1.3) and each of them consists of two irreducible components. Since intersections of components in reducible fibres give rise to a contribution to the total Milnor number and by Bézout’s theorem it follows that each reducible fibre contains a line, say  $l_i(x, y)$  for the fibre over  $a_i$ ,  $i = 1, \dots, d - 1$ . It can be seen that these  $(d - 1)$  lines have the same direction to infinity. Then by choosing an  $l_j$  as new coordinate  $X$ , all  $l_i$  in the new coordinate system have form  $X - b_i$ . Putting  $P(x, y) = p(X, Y)$ , an arithmetics shows that  $P(X, Y) - X = l(X, Y) \prod_{i=1}^{d-1} (X - b_i)$ .

Extremal polynomials with two reducible fibres have many similar properties of generalized Chebyshev polynomials in one variable, e.g. all their critical points are lying in two fibres and the number of components in these fibres equals to  $(d + 1)$ , e.g.  $p_{1,2}(x, y)$  from the series above. In fact one may slightly modify series  $p_{1,d-1}(x, y)$  to get extremal polynomials with two singular fibres. Let  $q_m(x)$  be a polynomial of degree  $m$  having distinct roots and let  $q_m(x) - 1 = \prod_{i=1}^m (x - a_i)$  with distinct  $a_i$ . For  $1 \leq k \leq m - 1$  consider  $p_{1,m,k}(x, y) := y q_m(x) \prod_{i=1}^k (x - a_i) + q_m(x)$ . Then  $p_{1,m,k}(x, y)$  are extremal with two singular fibres (over  $t = 0, 1$ ). Conversely in a similar way as in the proof of Theorem 2.2 we have

**Theorem 2.3.** *Extremal polynomials with two reducible fibres are obtained as above.*

**Remark.** (i) Theorem 2.3 says that extremal polynomials with two reducible fibres are semi-stable (cf. conjecture after Theorem 7 in [12]); (ii) Examples of semi-stable polynomial mappings with three singular fibres can serve polynomials  $T_n(x) - T_m(y)$ , where  $T_n(x)$  is the Chebyshev polynomial of the first kind defined by  $T_n(x) := \cos(n \arccos x)$ .

**3. Some pencils with small  $s$ .** Let  $f : X \rightarrow \mathbf{P}^1$  be a relatively minimal fibration of curves of genus  $g \geq 1$  over an algebraically closed field  $k$ . Let us assume “tame condition” in this paragraph (and additionally  $f$  has no multiple fibres in non-zero characteristic). Let  $s$  denote the number of singular fibres. It is known that  $s \geq 2$  for a non-trivial family  $f$ .

**Conjecture 3.1** ([12]). *Surfaces  $X$  as above with  $s = 2$  are unirational.*

A result towards this conjecture was reported at the Symposium “AGEA II”.

**Theorem 3.2** ([13]). *Conjecture 1.1 is true in the case  $k = \mathbf{C}$ .*

A deep part of the proof of Theorem 3.2 relies on the theory of complex affine surfaces due to Fujita, Ramanujam, Miyanishi, Gurjar, Shatri et al.

A particular case of polynomial mappings on  $\mathbf{A}^2$  with one singular fibre is quite interesting. In fact Zaidenberg and Lin were able to describe all normal forms of such polynomials over  $\mathbf{C}$  ([19,20]). Let us adopt the following notation:  $p(x, y) = \prod_{i=1}^{r_0} p_i(x, y)^{e_i}$ ,  $\Gamma_{0,i} := \{p_i(x, y) = 0\}$ ,  $i = 1, \dots, r_0$ ;  $a, b, e_i, r \in \mathbf{Z}_+$ ,  $l, m, n, l_i \in \mathbf{N}$ ,  $\sigma(x, y) := x^r y + \rho(x)$ ,  $\deg \rho < r$  and  $\rho(0) \neq 0$  for  $r > 0$ .

**Theorem 3.3** (cf. [20]). *Let  $p : \mathbf{A}^2 \rightarrow \mathbf{A}^1$  be a non-composite polynomial mapping with only one singular fibre, say  $\Gamma_0$  over  $t = 0$ . Then all  $\Gamma_{0,i}$ , possibly but one, are simply connected, or simply connected at infinity. More precisely the classification is divided into 3 classes depending on appearance of ‘exceptional’ components.*

(I) *All  $\Gamma_{0,i}$  are simply connected:  $P_1(x, y) = x^a y^b \prod_i (x^m - \alpha_i y^n)^{e_i}$ , or  $P_2(x, y) = q(x) y^l$ .*

(II) *There are components simply connected at infinity (viz.  $\chi(\Gamma_{0,i}) = 0$ ):  $P_3(x, y) = x^a \sigma^b \prod_i (x^m - \alpha_i \sigma^n)^{l_i}$ ,  $r > 0$ ,  $P_4(x, y) = x^a \sigma^b \prod_i (x^m \sigma^n - \alpha_i)^{l_i}$ ,  $(a > 0, (m, n) = 1)$ , or  $P_5(x, y) = q(x) \sigma^l$ ,  $r > 0$ ,  $q(0) = 0$ .*

(III) *There is a  $\Gamma_{0,i}$  with  $\chi(\Gamma_{0,i}) < 0$ :  $P_6(x, y) = q_1(x) (y \prod_{j=1}^m (x - \beta_j)^{l_j} - q_2(x))^l$ ,  $m \geq 2$ ,  $\beta_j$ -pairwise distinct,  $q_1(\beta_j) = 0$ ,  $q_2(\beta_j) \neq 0$ ,  $j = 1, \dots, m$ ,  $\deg q_2 < \sum_{j=1}^m l_j$ .*

Let  $f : X \rightarrow \mathbf{P}^1$  be the associated morphism obtained after blowing-up base points as in §1. By (1.1) we have  $n_h = r_0$ . Next in view of [12], Theorem 6, the equality on the right-hand side of (3) (Lemma 1, loc.cit.) holds for the fibre  $X_0$ . Hence irreducible components of  $X_0$  are rational with at worst singularities of cuspidal type (unibranch) and the dual graph of  $X_0$  is a generalized tree (e.g. type  $IV_d$ ). This gives us a severe restriction on inter-

section relations of components. So there is at least one simply connected component in  $\Gamma_0$ . It is a line, or quasihomogeneous ([1,19]). Note that the proof of Zaidenberg-Lin theorem in [19] is essentially algebraic, except a step using Milnor's theory, which was filled up in [1] (*cf.* also [20]). A detailed consideration leads to the classes (I)–(III) in the theorem.

**Remark.** The arguments above also show that pencils of plane curves having only two singular fibres, one of them is irreducible, are those coming from polynomial ones.

**4. Pencils with  $(2IV_d)$ .** Under the tame condition consider the following pencil

$$(3IV_d) \quad (x^d - 1) - t(y^d - 1) = 0.$$

Over  $\mathbf{C}$  it is known that up to projective isomorphism pencils with  $(3IV_d)$  are unique with equation above ([17], Prop. 3.3). In the theorem below we strengthen this result.

For a field  $k$  and  $a \in k$  the Dickson polynomial of the first kind  $D_n(x, a)$  of degree  $n$  is defined by the following recurrence relation

$$D_0(x, a) = 2, \quad D_1(x, a) = x, \\ D_n(x, a) = xD_{n-1}(x, a) - aD_{n-2}(x, a), \quad n \geq 2.$$

Sometimes for the definition of  $D_n(x, a)$  one takes the relation  $D_n(x + a/x, a) = x^n + (a/x)^n$  coming from a well-known polynomial identity in the theory of symmetric functions:  $x_1^n + x_2^n = D_n(x_1 + x_2, x_1x_2)$ . Dickson polynomial  $D_n(x, a)$ ,  $a \neq 0$  and Chebyshev polynomial  $T_n(x)$  are related by the formula:  $D_n(x, a) = 2a^{n/2}T_n(x/2a^{1/2})$ , hence  $T_n(x) = D_n(2x, 1)$ .

Consider a pencil of Chebyshev-Dickson type of plane curves of degree  $d$  defined by

$$(\mathcal{D}_d) \quad D_d(x, a) - tD_d(y, a) = 0, \quad a \in k^*.$$

The pencil  $(\mathcal{D}_d)$  has two singular fibres of type  $IV_d$  over  $t = 0, \infty$ . Besides it has two other singular fibres at  $t = \pm 1$  which is a union of a line and  $[d/2]$  conics, if  $d$  odd. In the case  $d > 2$  even, the fibre over  $t = -1$  is a union of  $d/2$  conics; the fibre over  $t = 1$  is a union of two lines and  $(d/2 - 1)$  conics. In both cases fibres over  $t = \pm 1$  are semi-stable.

**Theorem 4.1.** *Let  $f : X \rightarrow \mathbf{P}^1$  be a fibration of curves having two singular fibres of type  $IV_d$ ; (i) If  $f$  has one more completely reducible fibre, then it arises from a pencil of type  $(3IV_d)$  above; (ii) If  $f$*

*has a reducible fibre consisting completely of lines and conics, then it arises from pencil  $(3IV_d)$ , or from pencil  $(\mathcal{D}_d)$  described above.*

Clearly  $f$  arises from a pencil of plane curves with  $(2IV_d)$ . By choosing a suitable coordinate system one may assume that two fibres of type  $IV_d$  have concurrent lines at  $(0:1:0)$  and  $(1:0:0)$  respectively. Thus we are led to the pencil  $p(x) - tq(y) = 0$ , where  $p, q$  are polynomials of degree  $d$  with distinct roots. One may also assume that a third singular fibre in the theorem is over  $t = 1$ . Statement (i) now follows by a direct argument:  $p(x) - q(y)$  has a linear factor if and only if  $p(x) = q(ax + b)$ , and since  $p(x) - q(y)$  is completely factorized we come to the pencil  $(3IV_d)$ . For (ii) one can apply results of [3,5,8] to our situation.

**Remark.** It should be noted that results due to Fried, Bilu, Cassou-Noguès, Couveignes *et al.* (using the classification of finite simple groups) give a complete answer concerning factors of  $p(x) - q(y)$  in the indecomposable case (see [3,4,6,8] and references therein). Along the same line of ideas one can have also a finer result for pencils with  $(2IV_d)$  and a reducible fibre consisting completely of low degree curves.

**5. The Hesse cubic pencil.** Pencils with maximal total order of reducibility and  $(\mathcal{H})$  give rise to families  $f : X \rightarrow \mathbf{P}^1$  with  $r_{MW} = 0$ .

**Theorem 5.1.** *Let  $f : X \rightarrow \mathbf{P}^1$  be a fibration of curves of genus  $g \geq 1$ . Let  $s_0$  denote the number of fibres with  $g(\tilde{X}_t) = 0$ , where  $\tilde{X}_t$  denotes the normalization of a fibre  $X_t$ . Assume that the Mordell-Weil rank  $r_{MW} = 0$ , then  $s_0 \leq 4$  and  $s_0 = 4$  is attained only if  $f$  arises from one of six cubic pencils listed in [2].*

Indeed by the OSG formula ([12])  $\sum_{t \in S} [g - g(\tilde{X}_t)] \leq 4g$ . So  $s_0 \leq 4$ . Equality implies that  $f$  has exactly four singular semi-stable fibres. It remains to refer to [2,11].

**Remark.** (i) Pencil  $(\mathcal{D}_d)$  gives rise to a family  $f : X \rightarrow \mathbf{P}^1$  with  $s_0 = 4$ , but with  $r_{MW} = 2g$ . It has 4 “almost” completely reducible fibres (*cf.* conjecture of Libgober, Yuzvinsky *et al.* mentioned in the Introduction) (ii) Pencil  $D_d(x, a) + 2D_d(y, a) - t[D_d(x, a) - 2D_d(y, a)] = 0$ , has  $\rho(p/q) = 3d - 3$  (iii) Note a trivial case of equality in Theorem 1.1, when  $d = 2$ , Ruppert's pencil, pencil  $(\mathcal{D}_d)$  lead to the same pencil  $(3IV_2)$  with maximal  $\rho(p/q) = 3$ .

**Theorem 5.2.** *Extremal pencil of curves of degree  $d > 2$ , i.e. with  $\rho(p/q) = d^2 - 1$ , arises from the pencil  $(\mathcal{H})$ .*

From Theorem 1.1 of §1 it follows that  $C_0$  and  $C_\infty$  intersect transversally in  $d^2$  points which give rise to  $n_h = d^2$  horizontal components. In particular the general fibres of the pencil are smooth. We need the following technical lemma which strengthens (2.2).

**Lemma 5.3.** *In the situation above*

$$(5.1) \quad g - g(\tilde{X}_t) + n_t - 1 \geq (n_t - 1) \left( d - \frac{n_t}{2} \right)$$

Now (5.1) combined with the OSG formula gives  $d \leq 3$ .

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### References

- [ 1 ] S. S. Abhyankar and A. Sathaye, Uniqueness of plane embeddings of special curves, Proc. Amer. Math. Soc. **124** (1996), no. 4, 1061–1069.
- [ 2 ] A. Beauville, Les familles stables de courbes elliptiques sur  $\mathbf{P}^1$  admettant quatre fibres singulières, C. R. Acad. Sci. Paris Sér. I Math. **294** (1982), no. 19, 657–660.
- [ 3 ] Y. F. Bilu, Quadratic factors of  $f(x) - g(y)$ , Acta Arith. **90** (1999), no. 4, 341–355.
- [ 4 ] P. Cassou-Noguès and J.-M. Couveignes, Factorisations explicites de  $g(y) - h(z)$ , Acta Arith. **87** (1999), no. 4, 291–317.
- [ 5 ] M. Fried, On a conjecture of Schur, Michigan Math. J. **17** (1970), 41–55.
- [ 6 ] M. Fried, Variables separated equations: Strikingly different roles for the branch cycle lemma and the finite simple group classification, arXiv:1012.5297.
- [ 7 ] S. Kaliman, Two remarks on polynomials in two variables, Pacific J. Math. **154** (1992), no. 2, 285–295.
- [ 8 ] M. Kulkarni, P. Müller and B. Sury, Quadratic factors of  $f(X) - g(Y)$ , Indag. Math. (N.S.) **18** (2007), no. 2, 233–243.
- [ 9 ] A. Libgober and S. Yuzvinsky, Cohomology of the Orlik-Solomon algebras and local systems, Compositio Math. **121** (2000), no. 3, 337–361.
- [ 10 ] D. Lorenzini, Reducibility of polynomials in two variables, J. Algebra **156** (1993), no. 1, 65–75.
- [ 11 ] V. Kh. Nguyen, On Beauville’s conjecture and related topics, J. Math. Kyoto Univ. **35** (1995), no. 2, 275–298.
- [ 12 ] V. Kh. Nguyen, On families of curves over  $\mathbf{P}^1$  with small number of singular fibres, C. R. Acad. Sci. Paris Sér. I Math. **326** (1998), no. 4, 459–463.
- [ 13 ] V. Kh. Nguyen, On families of curves over  $\mathbf{P}^1$  with two singular fibres, Abstracts of the Symposium, *Algebraic Geometry in East Asia. II, Hanoi, 2005*.
- [ 14 ] W. Ruppert, Reduzibilität ebener Kurven, J. Reine Angew. Math. **369** (1986), 167–191.
- [ 15 ] Y. Stein, The total reducibility order of a polynomial in two variables, Israel J. Math. **68** (1989), no. 1, 109–122.
- [ 16 ] A. Vistoli, The number of reducible hypersurfaces in a pencil, Invent. Math. **112** (1993), no. 2, 247–262.
- [ 17 ] S. Yuzvinsky, Realization of finite abelian groups by nets in  $\mathbf{P}^2$ , Compositio Math. **140** (2004), no. 6, 1614–1624.
- [ 18 ] S. Yuzvinsky, A new bound on the number of special fibers in a pencil of curves, Proc. Amer. Math. Soc. **137** (2009), no. 5, 1641–1648.
- [ 19 ] M. Zaidenberg and V. Lin, An irreducible simply connected curve in  $\mathbf{C}^2$  is equivalent to a quasihomogeneous curve, Soviet Math. Dokl. **28** (1983), 200–204.
- [ 20 ] M. Zaidenberg, Rational actions of the group  $\mathbf{C}^*$  on  $\mathbf{C}^2$ , their quasi-invariants, and algebraic curves in  $\mathbf{C}^2$  with Euler characteristic 1, Soviet Math. Dokl. **31** (1985), 57–60.