On a minimal counterexample to the Alperin-McKay conjecture

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Abstract: We show for a minimal counterexample \((G, B)\) to the Alperin-McKay conjecture, the Fitting subgroup of \(G\) is central and \(G\) has a unique \(G\)-conjugacy class of components.

Key word: Alperin-McKay conjecture.

1. Introduction. Let \(G\) be a finite group and \(p\) a prime. Let \((\mathcal{K}, R, k)\) be a \(p\)-modular system ([NT, p.230]); that is, \(R\) is a complete discrete valuation ring with quotient field \(K\) of characteristic 0 and \(k\) is the residue field of \(R\) of characteristic \(p\). We assume that \(K\) contains a primitive \([G]\)-th root of unity and that \(k\) is algebraically closed. Let \(B\) be a block of \(G\). This means that \(B\) is a block ideal of \(RG\). Let \(D\) be a defect group of \(B\). Let \(\bar{B}\) be the Brauer correspondent of \(B\) with respect to \(D\) in \(N_G(D)\). Let \(k_0(B)\) be the number of irreducible characters of height 0 in \(B\). The Alperin-McKay conjecture [Al] (AM-conjecture, for short) states that \(k_0(B) = k_0(\bar{B})\). In a previous paper [Mu], we have studied reduction of this conjecture. In this note, using a result in [Mu], we give further restrictions on a minimal counterexample \((G, B)\) to the AM-conjecture. Here we say \((G, B)\) is a minimal counterexample to the AM-conjecture, if \(B\) is a counterexample to the AM-conjecture and \(G\) is chosen so that \([G : Z(G)]\) is as small as possible. We prove the following.

Theorem. For a minimal counterexample \((G, B)\) to the AM-conjecture, the following holds.

(i) \(O_p(G)\) and \(O_p^0(G)\) are both central in \(G\). In particular, the Fitting subgroup of \(G\) is a central subgroup of \(G\).

(ii) \(G\) has a unique \(G\)-conjugacy class of components.

It is known that the AM-conjecture is a consequence of Dade’s projective conjecture [Da]. Eaton-Robinson [ER, Theorem 1 and Remarks] and Robinson [Ro, Theorem 1] have obtained restrictions similar to Theorem on a minimal counterexample to Dade’s projective conjecture.

For the McKay conjecture, the prototype of the AM-conjecture, Isaacs, Malle and Navarro [IMN] have obtained a reduction theorem. Using this paper as a starting point, Spåth [Sp] has recently obtained a reduction theorem for the AM-conjecture. Her approach is different from ours.

2. Proof of Theorem. For a block \(B\) of a group \(G\), let \(e_B\) be the block idempotent of \(B\). Let \(F^\sigma(G)\) (resp. \(F(G)\)) be the generalized Fitting (resp. Fitting) subgroup of \(G\). We have proved the following, see [Mu, Proposition 9].

Lemma. Let \((G, B)\) be a minimal counterexample to the AM-conjecture. Let \(D\) be a defect group of \(B\). Then the following holds.

(i) For any non-central normal subgroup \(K\) of \(G\), \(G = N_G(D)K\).

(ii) For any normal subgroup \(K\) of \(G\), \(B\) covers a \(G\)-invariant block of \(K\).

(iii) \(G = N_G(D)F^\sigma(G)\).

For the other notation and terminology, see the books [NT, Th].

Proof of Theorem. (i) Assume \(O_p(G)\) is not central in \(G\). By Lemma (i), \(G = N_G(D)O_p(G) = N_G(D)\), a contradiction. Thus \(O_p(G)\) is central in \(G\).

Assume \(K : = O_p(G)\) is not central in \(G\). By Lemma (i), \(G = N_G(D)K\). Put \(H = C_G(D)DK\). Then \(H \trianglelefteq G\). By Lemma (ii) \(B\) covers a \(G\)-invariant block \(b\) of \(H\). Then \(D\) is a defect group of \(b\). Let \(Q\) be a subgroup of \(D\). Then \(N_H(Q) = C_G(D)N_DK(Q)\) and \(C_H(Q) = C_G(D)C_DK(Q)\). So \(N_H(Q)/C_H(Q) \simeq N_DK(Q)/C_DK(Q)\), which is a \(p\)-group, since \(DK\) is \(p\)-nilpotent. Thus \(b\) is \(p\)-nilpotent. Further, since \(C_G(D) \leq H\) for the defect group \(D\) of \(b\), \(B\) is a unique block of \(G\) covering \(b\). Let \(\bar{B}\) (resp. \(\bar{b}\)) be the Brauer correspondent of \(B\) (resp. \(b\)) with respect to \(D\) in \(N_G(D)\) (resp. \(N_H(D)\)). By the Harris-Knörr theorem [HK, Theorem], \(\bar{B}\) is a unique block of...
\[ N_G(D) \text{ covering } \mathfrak{b}. \] Let \( \beta \) be a block of \( C_H(D) \) covered by \( \mathfrak{b} \). Then \( \mathfrak{B} \) is a unique block of \( N_G(D) \) covering \( \beta \). Let \( B_1 \) be the Fong-Reynolds correspondent of \( \mathfrak{B} \) over \( \beta \in N_G(D)_\beta \), where \( N_G(D)_\beta \) is the inertial group of \( \beta \) in \( N_G(D) \). Then \( B_1 \) is a unique block of \( N_G(D)_\beta \) covering \( \beta \) by the Fong-Reynolds theorem. Let \( D_\beta \) be the defect pointed group of the pointed group \( H_{(\alpha)} \) on \( RH \) associated with the subpair \( (D, \sigma_H) \), where \( \sigma_H \) denotes the natural image of \( e_\beta \) in \( kC_H(D) \). Then \( N_G(D_\beta) = N_G(D)_\beta \), cf. [Th, Proposition 40.13(b)]. Therefore, by [KP, 1.20.3], \( B \) and \( B_1 \) are isomorphic to full matrix algebras over the same \( R \)-algebra. Then \( k_0(B) = k_0(B_1) \) by the Morita equivalence between \( B \) and \( B_1 \). By the Fong-Reynolds theorem, \( k_0(B_1) = k_0(\mathfrak{B}) \). Hence \( k_0(B) = k_0(\mathfrak{B}) \), a contradiction. Thus \( O_p(G) \) is central in \( G \).

(ii) Let \( n \) be the number of \( G \)-conjugacy classes of components of \( G \). If \( n = 0 \), then, by (i) and Lemma (iii), \( G = N_G(D)F^*(G) = N_G(D)F(G) = N_G(D) \), a contradiction. If \( n \geq 2 \), let \( N_1 \) be the product of all components in a \( G \)-conjugacy class and let \( N_2 \) be the product of the remaining components. Since \( N_1 \) and \( N_2 \) are non-central normal subgroups of \( G \), \( DN_1 \) and \( DN_2 \) are normal subgroups of \( G \) by Lemma (i). Clearly \( (DN_1 \cap DN_2)/(N_1 \cap N_2) \) is a \( p \)-group. Since \( N_1 \cap N_2 \leq Z(N_1N_2) \leq F(G) \), \( N_1 \cap N_2 \) is central in \( G \) by (i). Hence \( DN_1 \cap DN_2 \) is a direct product of a \( p \)-group and a \( p' \)-group. Then \( D \leq O_p(DN_1 \cap DN_2) \). Since \( O_p(DN_1 \cap DN_2) \triangleleft G \), we have \( O_p(DN_1 \cap DN_2) \leq D \). So \( D = O_p(DN_1 \cap DN_2) \triangleleft G \), a contradiction. Thus \( n = 1 \), as required.

\[ \square \]

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References


