Gromov hyperbolicity and a variation of the Gordian complex

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Abstract: We introduce new simplicial complexes by using various invariants and local moves for knots, which give generalizations of the Gordian complex defined by Hirasawa and Uchida. In particular, we focus on the simplicial complex defined by using the Alexander-Conway polynomial and the Delta-move, and show that the simplicial complex is Gromov hyperbolic and quasi-isometric to the real line.

Key words: Alexander-Conway polynomial; Delta-move; Gromov hyperbolic space; Gordian complex.

1. Introduction. A *knot* is an ambient isotopy class of a simple closed curve smoothly embedded in the 3–sphere. Let $\mathcal{K}$ be the set of all knots. Let λ be a local move on knots, that is a local operation deforming a knot (see [24, Section 2] for the precise definition of a local move). The λ-*Gordian distance* $d^{\lambda}(K, K')$ between knots K and K' is defined to be the minimal number of the local moves λ needed to deform K into K'. If such a minimum does not exist, then we set $d^{\lambda}(K, K') = \infty$. Let x denote the *crossing change* which is a local move as shown in Figure 1. In the case where $\lambda = \mathrm{x}$, d^{x} is called the *Gordian distance.*

Using the Gordian distance, Hirasawa and Uchida [11] defined the *Gordian complex* denoted by $\mathcal{G}^{\mathrm{x}}$. More generally, the λ-*Gordian complex* $\mathcal{G}^{\lambda}$ introduced by Nakanishi and Ohyama [22, Section 1] is the simplicial complex defined by the following;

- the set of vertices (i.e. 0–simplices) of $\mathcal{G}^{\lambda}$ is $\mathcal{K}$, and
- $n+1$ vertices $K_0, \dots, K_n$ span an n–simplex if and only if $d^{\lambda}(K_i, K_j) = 1$ holds for each $i \neq j \in \{0, \dots, n\}$.

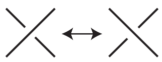

Fig. 1. The crossing change.

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We call the 1–skelton of $\mathcal{G}^{\mathrm{x}}$ (resp. $\mathcal{G}^{\lambda}$) the *Gordian graph* (resp. the λ-*Gordian graph*), and denote it by G^{x} (resp. G^{λ}). Assuming that every edge has length 1, each connected component of G^{λ} is regarded as a metric space which turns to a geodesic space (see Section 2). Then one of the problems we are interested in is to reveal properties of such spaces. In particular, we are interested in global properties, and the Gromov hyperbolicity [7] (for a brief review, see Section 2) is an important one. There are several studies on simplicial complexes arising in geometry and topology. In particular, the curve complex introduced by Harvey [10] is widely studied (see also [9] for a survery). Masur and Minsky proved that curve complexes are Gromov hyperbolic [18]. On the other hand, there is no known fact on the Gromov hyperbolicity of G^{λ} except for the following

Proposition 1.1 [5, Theorem C]**.** *The Gordian graph G^{x} is not Gromov hyperbolic.*

Here we will introduce new simplicial complexes and graphs by using knot invariants and local moves, which give generalizations of the λ-Gordian complex and the λ-Gordian graph. Let ι be a knot invariant, that is, a function on $\mathcal{K}$ such that $\iota(K)$ and $\iota(K')$ coincide if K is equivalent to K'. We write $K \sim_{\iota} K'$ if $\iota(K) = \iota(K')$ holds. Clearly the binary relation $\sim_{\iota}$ provides an equivalence relation on $\mathcal{K}$. Let $[K]_{\iota}$ denote the equivalence class of K, and set $\mathcal{K}_{\iota} = \{[K]_{\iota} \mid K \in \mathcal{K}\}$.

Definition 1.2. Let ι be a knot invariant, and let λ be a local move on knots. The (ι, λ)-*Gordian complex* $\mathcal{G}^{\lambda}_{\iota}$ is defined by the following

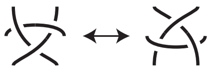

Fig. 2. The Delta-move.

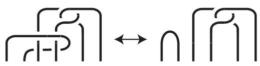

Fig. 3. A C_2-move.

- The set of vertices of $\mathcal{G}_\iota^\lambda$ is $\mathcal{K}_\iota$, and
- $n+1$ vertices $[K_0]_\iota, \dots, [K_n]_\iota$ span an n–simplex if and only if for each $i \neq j \in \{0, \dots, n\}$, there exists a pair of knots $K_{i,j} \in [K_i]_\iota$ and $K_{j,i} \in [K_j]_\iota$ such that $d^\lambda(K_{i,j}, K_{j,i}) = 1$.

We call the 1–skelton of $\mathcal{G}_\iota^\lambda$, denoted by G_ι^λ, the (ι, λ)-*Gordian graph.* Assuming that every edge has length 1, we regard G_ι^λ as a metric space. Then we denote by d_ι^λ the metric on G_ι^λ, and call it the (ι, λ)-*Gordian distance.*

Let ∇_K be the Conway polynomial [3] of a knot K, which is a polynomial in z^2 with integer coefficients. The Conway polynomial is also called the Alexander-Conway polynomial since it is regarded as a normalized Alexander polynomial [1]. We refer the reader to [14] for basic terminologies of knot theory. The *Delta-move*, denoted by the symbol Δ, is a local move on knots as shown in Figure 2, which was introduced by Matveev [19] and Murakami and Nakanishi [20] independently. It is known that the Delta-move is equivalent to a C_2-move (see Figure 3), which is one of C_n-moves introduced by Goussarov [6] and Habiro [8] independently.

Using the Conway polynomial and the Delta move, the (∇, Δ)-Gordian graph G_∇^Δ is defined. In this paper, we show the following

Theorem 1.3. *The* (∇, Δ)-*Gordian graph* G_∇^Δ *is* 2-*hyperbolic. Further it is quasi-isometric to the real line* $\mathbf{R}$.

Remark 1.4. In Section 5 we will see that G_∇^Δ and $\mathcal{G}_\nabla^\Delta$ coincide (see Proposition 1). Thus, $\mathcal{G}_\nabla^\Delta$ is also 2-hyperbolic and quasi-isometric to $\mathbf{R}$.

This paper is constructed as follows: In Section 2, we give a brief review on a Gromov hyperbolic space to study G_∇^Δ. In Section 3, we study the (∇, Δ)-Gordian distance. In Section 4, we prove Theorem 1.3. In Section 5, we give observations on the complexes $\mathcal{G}_\nabla^\mathrm{x}$ and $\mathcal{G}_\nabla^\Delta$. We also give some remarks and questions related to our study.

2. Preliminaries. In this section, we give a brief review on a Gromov hyperbolic space. For details, see [2] or [7]. Let X be a geodesic space, that is, a metric space such that the distance between any two points is equal to the length of a geodesic segment joining them. We denote by $s(x, y)$ a geodesic segment joining two points x and y. A *geodesic triangle* T in X is a triple of points $x, y, z \in X$ together with three geodesic segments $s(x, y)$, $s(y, z)$, and $s(z, x)$ called the *sides* of T. For $\delta \geq 0$, a geodesic triangle is said to be δ-*slim* if each side of a triangle belongs to the δ-neighborhood of the union of the other two sides. We say that X is δ-*hyperbolic* (or *Gromov hyperbolic*) if there exists a constant $\delta \geq 0$ such that any geodesic triangle in X is δ-slim. Clearly if a geodesic space is δ-hyperbolic for a particular δ, then it is also δ'-hyperbolic for all $\delta' \geq \delta$.

Let Γ be a connected graph. We denote by $\overline{vv'}$ an edge connecting vertices v and v'. In general, assuming that each edge has length 1, the graph Γ is regarded as a metric space, and it turns to a geodesic space.

As mentioned in Section 1, we regard G_∇^Δ as a geodesic space with the metric d_∇^Δ induced by the above setting. Note that G_∇^Δ is a connected graph since the Delta-move is an unknotting operation [20]. Recall that the vertex set of G_∇^Δ is $\mathcal{K}_\nabla$. From now, unless otherwise specified, we use the symbol $[K]$ to refer to the vertex $[K]_\nabla$ for brevity. Note that for two vertices $[K]$ and $[K']$ with $d_\nabla^\Delta([K], [K']) = p$, a geodesic segment $s([K], [K'])$ is of the form $\overline{v_0 v_1} \cup \overline{v_1 v_2} \cup \cdots \cup \overline{v_{p-1} v_p}$ for some vertices $v_0, v_1, \dots, v_p \in \mathcal{K}_\nabla$ with $v_0 = [K]$ and $v_p = [K']$.

Let X and X' be metric spaces with metric functions d and d' respectively. A map $f : X \to X'$ is a *quasi-isometry* if there exist constants $A, E \geq 0$, $B, C, D > 0$ such that $Ad(x, y) - B \leq d'(f(x), f(y)) \leq Cd(x, y) + D$ holds for any $x, y \in X$, and for any $x' \in X'$ there exists $x \in X$ such that $d'(x', f(x)) \leq E$. Then we say that X is *quasi-isometric* to X'. It is known that the Gromov hyperbolicity is preserved by quasi-isometries.

3. (∇, Δ)-Gordian distance. In this section, we study the (∇, Δ)-Gordian distance. Let $a_n(K)$ be the n-th coefficient of ∇_K. The following lemma due to Okada shows a fundamental relationship between the Conway polynomials and the Delta-move.

Lemma 3.1 [25]. *For $K, K' \in \mathcal{K}$ with $d^{\Delta}(K, K') = 1$, we have $a_2(K) - a_2(K') = \pm 1$. Furthermore, for any $[K], [K'] \in \mathcal{K}_{\nabla}$, we have*

$$d^{\Delta}_{\nabla}([K], [K']) \geq |a_2(K) - a_2(K')|,$$

and

$$d^{\Delta}_{\nabla}([K], [K']) \equiv |a_2(K) - a_2(K')| \bmod 2.$$

Note that $a_2(K_1) = a_2(K_2)$ holds for any $K_1, K_2 \in [K]$. The following lemma gives the formula to detect the (∇, Δ)-Gordian distance between any pair of vertices.

Lemma 3.2. *For $[K] \neq [K'] \in \mathcal{K}_{\nabla}$, we have the following*

$$d^{\Delta}_{\nabla}([K], [K']) = \begin{cases} |a_2(K) - a_2(K')| & a_2(K) \neq a_2(K'), \\ 2 & a_2(K) = a_2(K'). \end{cases}$$

Proof. It is known that every vertex $[K]$ contains an unknotting number one knot [16,27]. Let $J \in [K]$ be an unknotting number one knot. Then for any sequence of integers $(m_4, \dots, m_{2l})$, there exists a knot J' satisfying $d^{\Delta}(J, J') = 1$, $a_2(J) - a_2(J') = 1$, and $a_{2j}(J) - a_{2j}(J') = m_{2j}$ for $j = 2, \dots, l$ [22, Lemma A].

- If $a_2(K) \neq a_2(K')$, then by the above argument, we have $d^{\Delta}_{\nabla}([K], [K']) \leq |a_2(K) - a_2(K')|$. On the other hand, by Lemma 3.1, we have $d^{\Delta}_{\nabla}([K], [K']) \geq |a_2(K) - a_2(K')|$. Hence we have $d^{\Delta}_{\nabla}([K], [K']) = |a_2(K) - a_2(K')|$.
- If $a_2(K) = a_2(K')$, then by Lemma 3.1 and the assumption $[K] \neq [K']$, $d^{\Delta}_{\nabla}([K], [K'])$ is a non-zero even integer, namely $d^{\Delta}_{\nabla}([K], [K']) \geq 2$. On the other hand, by the above argument, we have $d^{\Delta}_{\nabla}([K], [K']) \leq 2$. Hence we have $d^{\Delta}_{\nabla}([K], [K']) = 2$.

Now we complete the proof of Lemma 3.2. □

4. Proof of Theorem 1.3. For $\varepsilon \geq 0$, let $N(p, \varepsilon)$ be the ε-neighborhood of a point $p \in G^{\Delta}_{\nabla}$, and $N(P, \varepsilon)$ the ε-neighborhood of a subset $P \subset G^{\Delta}_{\nabla}$, that is, $N(p, \varepsilon) = \{q \in G^{\Delta}_{\nabla} \mid d^{\Delta}_{\nabla}(p, q) \leq \varepsilon\}$ and $N(P, \varepsilon) = \bigcup_{p \in P} N(p, \varepsilon)$. Let $V_n = \{[K] \in \mathcal{K}_{\nabla} \mid a_2(K) = n\}$. Then we have the following.

Lemma 4.1. *For any $[K] \in \mathcal{K}_{\nabla}$ with $a_2(K) = n$, we have*

$$N([K], 2) \supset N(V_n, 1).$$

Proof. Recall that a *vertex-induced subgraph* is a subset of the vertices together with all edges whose endpoints are both in this subset. Then we see that $N(V_n, 1)$ is the vertex-induced subgraph which is induced by the subset $V_{n-1} \cup V_n \cup V_{n+1}$ of vertices. By Lemma 3.2, we have $d^{\Delta}_{\nabla}([K], v_n) = 2$ for any $v_n \neq [K] \in V_n$, and $d^{\Delta}_{\nabla}([K], v_{n\pm1}) = 1$ for any $v_{n\pm1} \in V_{n\pm1}$. This completes the proof of Lemma 4.1. □

Now we start the proof of Theorem 1.3.

Proof of Theorem 1.3. Let T be a geodesic triangle in G^{Δ}_{∇} with sides $s(x, y)$, $s(y, z)$, and $s(z, x)$. We only consider the case where x, y, and z are in $\mathcal{K}_{\nabla}$ since the other cases (i.e. the cases where some of x, y, and z are not contained in $\mathcal{K}_{\nabla}$) are proved in a similar way. Let $x = [K]$, $y = [J]$, and $z = [L]$. Without loss of generality, we may assume that $a_2(K) \leq a_2(J) \leq a_2(L)$. Let $k = a_2(J) - a_2(K)$ and $k' = a_2(L) - a_2(J)$. Let

$$\begin{aligned} s(x, y) &= \overline{x_0 x_1} \cup \overline{x_1 x_2} \cup \cdots \cup \overline{x_{p-1} x_p}, \\ s(y, z) &= \overline{y_0 y_1} \cup \overline{y_1 y_2} \cup \cdots \cup \overline{y_{q-1} y_q}, \\ s(z, x) &= \overline{z_0 z_1} \cup \overline{z_1 z_2} \cup \cdots \cup \overline{z_{r-1} z_r}, \end{aligned}$$

where $x_0, \dots, x_p, y_0, \dots, y_q, z_0, \dots, z_r$ are in $\mathcal{K}_{\nabla}$ with $x_0 = x = z_r$, $y_0 = y = x_p$, and $z_0 = z = y_q$. We show that T is 2-slim, that is, G^{Δ}_{∇} is 2-hyperbolic.

Case 1. $k \geq 1$, $k' \geq 1$. By Lemma 3.2, we have $p = k$, $q = k'$, and $r = k + k'$. Figure 4 is an example of a geodesic triangle for $p = q = 4$. (In Figure 4, we plot vertices with respect to the coefficients of the Conway polynomial.) First we show that $N(s(x, y) \cup s(y, z), 2) \supset s(z, x)$. Note that the second coefficients of the polynomials y_j and z_{q-j} coincide. By Lemma 4.1, we have

$$N(y_j, 2) \supset \overline{z_{q-j+1} z_{q-j}},\ \overline{z_{q-j} z_{q-j-1}}$$

for each $j = 1, \cdots, q-1$. Thus, we have

$$\begin{aligned} N(s(y, z), 2) &\supset N(y_1 \cup \cdots \cup y_{q-1}, 2) \\ &\supset \overline{z_q z_{q-1}} \cup \cdots \cup \overline{z_1 z_0}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} N(s(x, y), 2) &\supset N(x_1 \cup \cdots \cup x_{p-1}, 2) \\ &\supset \overline{z_r z_{r-1}} \cup \cdots \cup \overline{z_{q+1} z_q}. \end{aligned}$$

Therefore we have $N(s(x, y) \cup s(y, z), 2) \supset s(z, x)$. Remaining two conditions $N(s(y, z) \cup s(z, x), 2) \supset s(x, y)$ and $N(s(z, x) \cup s(x, y), 2) \supset s(y, z)$ are shown by the similar argument. Therefore the geodesic triangle T is 2-slim.

Case 2. $k = 0$, $k' \geq 1$. By Lemma 3.2, we have $p = 2$, $q = r = k'$. Then by Lemma 4.1, we see that T is 2-slim.

Case 3. $k \geq 1$, $k' = 0$. This case is proved by the same argument applied in Case 2.

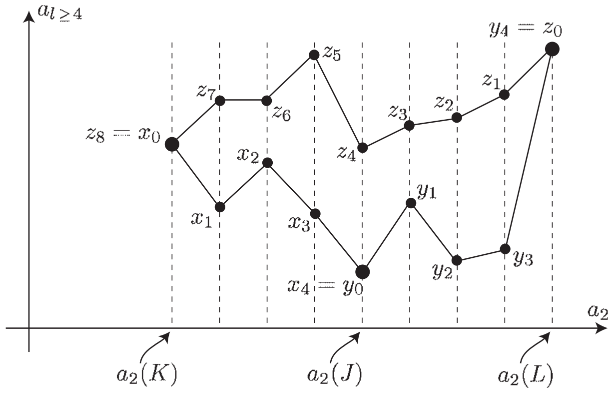

Fig. 4. The case where $p = 4$ and $q = 4$.

Case 4. $\boldsymbol{k = 0,\ k' = 0.}$ By Lemma 3.2, we have $p = q = r = 2$. Then by Lemma 4.1, we see that T is 1-slim.
Therefore G^{Δ}_{∇} is 2-hyperbolic.

Next we show that G^{Δ}_{∇} is quasi-isometric to the real line $\mathbf{R}$. Let $f : G^{\Delta}_{\nabla} \to \mathbf{R}$ be a map defined by the following; $f(\overline{v_{n-1}v_n}) = [n-1, n]$ for $v_{n-1} \in V_{n-1}$ and $v_n \in V_n$. Here $[n-1, n]$ denotes the closed interval bounded by $n-1$ and n. Then, by Lemma 3.2, the map f is a quasi-isometry. $\square$

5. Known facts and questions. In this section, we introduce some facts and remarks related to the diameters and the dimensions of complexes, and propose questions for further studies.

First we focus on the (∇, Δ)-Gordian complex $\mathcal{G}^{\Delta}_{\nabla}$. By Lemma 3.2, there exists no 2–simplex in $\mathcal{G}^{\Delta}_{\nabla}$ (see [23, Proposition 2.3]). This implies the following

Proposition 5.1. *The dimension of* $\mathcal{G}^{\Delta}_{\nabla}$ *is one. Therefore* $\mathcal{G}^{\Delta}_{\nabla}$ *and* G^{Δ}_{∇} *coincide.*

Next we focus on the (∇, x)-Gordian complex $\mathcal{G}^{\mathrm{x}}_{\nabla}$ and the (∇, x)-Gordian graph G^{x}_{∇}. As mentioned in the proof of Lemma 3.2, any Alexander-Conway polynomial of a knot is realized by an unknotting number one knot [16,27]. (There are several studies on the realization problem of the Alexander-Conway polynomial. See [4,12,13,17,21,26,28].) Thus, the diameter of $\mathcal{G}^{\mathrm{x}}_{\nabla}$ and that of G^{x}_{∇} are less than or equal to two. Since a geodesic space with a finite diameter r is r-hyperbolic, and it is quasi-isometric to a point, we have the following

Proposition 5.2. *The* (∇, x)*-Gordian graph* G^{x}_{∇} *is* 2*-hyperbolic, and it is quasi-isometric to a point.*

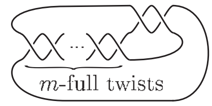

Fig. 5. A twist knot K_m.

Remark 5.3. For any $n \in \mathbf{N}$, there exists vertices $[K]$ and $[K']$ such that $d^{\Delta}_{\nabla}([K], [K']) = n$. Actually, we have $d^{\Delta}_{\nabla}([K_0], [K_n]) = n$, where K_0 and K_n are twist knots depicted in Figure 5. Thus, the diameter of G^{Δ}_{∇} is infinite.

Recently, Kawauchi [15] showed that $d^{\mathrm{x}}_{\nabla}([K_1], [K_{-1}]) = 2$ by using duality theorems on the infinite cyclic covering space of a knot exterior. Here K_1 denotes the trefoil knot and K_{-1} denotes the figure-eight knot as shown in Figure 5. Thus, the diameter of G^{x}_{∇} is just two. Kawauchi also showed that the dimension of the (∇, x)-Gordian complex $\mathcal{G}^{\mathrm{x}}_{\nabla}$ is infinite [15], that is, an n–simplex is contained in $\mathcal{G}^{\mathrm{x}}_{\nabla}$ for any $n \in \mathbf{N}$.

Remark 5.4. The dimension of the Gordian complex $\mathcal{G}^{\mathrm{x}}$ is infinite, which was shown by Hirasawa and Uchida [11]. For a C_n-move with $n \geq 3$, the dimension of the C_n-Gordian complex $\mathcal{G}^{C_n}$ is also infinite, which was shown by Ohyama [23]. Here we note that a C_1-move is equivalent to the crossing change and a C_2-move is equivalent to the Delta-move.

Finally we propose some questions. For several local moves and knot invariants, it is interesting to consider the Gromov hyperbolicity of G^{λ}_{ι}. As mentioned in Remark 5.4, the crossing change is equivalent to a C_1-move and the Delta-move is equivalent to a C_2-move. Then the following question is natural to ask.

Question 5.5. *For* $n \geq 3$*, is each connected component of the* (∇, C_n)*-Gordian graph* $G^{C_n}_{\nabla}$ *Gromov hyperbolic?*

Note that $G^{C_n}_{\nabla}$ consists of infinitely many connected components by results of Goussarov [6] and Habiro [8].

It is also interesting to study the λ-Gordian graph. In particular, G^{x}_{∇} and G^{Δ}_{∇} are Gromov hyperbolic, but G^{x} is not Gromov hyperbolic.

Question 5.6. *Is the* Δ*-Gordian graph* G^{Δ} *Gromov hyperbolic?*

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