# Convergence of a semi-discrete finite difference scheme applied to the abstract Cauchy problem on a scale of Banach spaces

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**Abstract:** We show convergence of a finite difference scheme to solve the abstract Cauchy problem on a scale of Banach spaces which includes that for Kowalevskaya's system. We show convergence of consistent difference schemes even for unstable cases.

**Key words:** Numerical analysis for partial differential equations; finite difference scheme; Cauchy-Kowalevskaya's theorem; ill-posed problems; multiple-precision arithmetic.

**1. Introduction.** We deal with finite difference schemes applied to the Cauchy problems of linear partial differential equations in the normal form, which is called Kowalevskaya's system, and we argue their convergence even without stability. There are two different principles to discuss unique solvability of linear partial differential equations in the normal form; well-posedness and analyticity. In the first category, we know, about difference schemes, a remarkable result of Lax' equivalence theorem [3] that stability and convergence are equivalent to each other for consistent finite difference schemes. The second category recalls to us the Cauchy-Kowalevskaya theorem, and we will show convergence of the consistent scheme in this research within the category. The result contains the case of ill-posed problems in the sense of the first category, and we are able to show convergence of a finite difference scheme to approximate the Cauchy problem of the Cauchy-Riemann equation as an example. In this sense, this work is a generalization of that of K. Hayakawa [2].

We discuss convergence of finite difference schemes in the framework of the Banach scales. L. V. Ovsjannikov [5] and T. Yamanaka [6] succeeded independently in reduction of the Cauchy problem of Kowalevskaya's system to that of an abstract ordinary differential equation on a scale of Banach spaces, and they gave a clear proof of the Cauchy-Kowalevskaya theorem. We follow their discussions and prove our result through discretization of their estimates.

Following Yamanaka [6], we introduce a scale of Banach spaces and the Cauchy problem on it. Let $S$ be a scale of Banach spaces: $S$ is a collection of parametrized Banach spaces $\{X_\rho\}_{0<\rho\le\rho_0}$ satisfying $X_\rho \subset X_{\rho'}$ and $\|x\|_{\rho'} < \|x\|_\rho$ $(x \neq 0, x \in X_\rho)$ for $0 < \rho' < \rho \le \rho_0$. Here $\| \ \|_\rho$ stands for a norm of the Banach space $X_\rho$. Let $A(t)$ be a bounded singular operator with a parameter $t \in [-T_0, T_0]$; $A(t)\colon X_\rho \to X_{\rho'}$ is a bounded linear operator such that

$$\|A(t)u\|_{\rho'} \le \frac{\omega}{\rho-\rho'}\|u\|_\rho \qquad (u \in X_\rho, \quad 0<\rho'<\rho\le\rho_0) \tag{1.1}$$

for some positive number $\omega$, and $\|A(t)u\|_{\rho'}$ is continuous with respect to $t \in [-T_0, T_0]$ for fixed $u \in X_\rho$. We have the following result.

**Theorem** (Ovsjannikov-Yamanaka). *Let $\rho \in (0, \rho_0)$. For given $f \in C^0([-T_0, T_0] : X_{\rho_0})$ and $U \in X_\rho$, there exists, for each $\rho' \in (0, \rho)$, a unique function $u \in C^0([-\delta, \delta] : X_{\rho'})$ satisfying the integral equation*

$$u(t) = U + \int_0^t (A(\tau)u(\tau) + f(\tau))d\tau \quad |t| \le \delta, \tag{1.2}$$

*where* $\delta := \min\Big(T_0, \dfrac{\rho-\rho'}{\omega e}\Big)$.

**Corollary.** *Under the same assumptions above, there exists a unique solution $u \in C^1([-\delta, \delta] : X_{\rho'})$ to the Cauchy problem*

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$$(1.3)\qquad \frac{d}{dt}u(t) = A(t)u(t) + f(t),$$

$$(1.4)\qquad u(0) = U\ (\in X_\rho),$$

*where* $0 < \rho' < \rho \le \rho_0$ *and* $\delta := \min\Big(T_0, \dfrac{\rho - \rho'}{\omega e}\Big)$.

In this paper, we deal with a semi-discrete difference scheme for the equation (1.3), which does not contain finite dimensional approximation of the operator $A(t)$, and we show convergence of the scheme. Since we know the abstract Cauchy problem (1.3)–(1.4) includes that of Kowalevskaya's system

$$(1.5)\qquad \frac{\partial}{\partial t}u_k(t,x) = \sum_{j=1}^{n}\bigg\{a_{kj}(t,x)\frac{\partial}{\partial x_j}u_k(t,x) + b_j(t,x)u_j(t,x)\bigg\} + c_k(t,x) \quad (1 \le k \le n),$$

$$(1.6)\qquad u_k(0,x) = U_k^{(0)}(x) \quad (1 \le k \le n),$$

where $\{a_{kj}(t,x)\}_{1\le k,j\le n}$, $\{b_j(t,x)\}_{1\le j\le n}$ and $\{c_k(t,x)\}_{1\le k\le n}$ are analytic with respect to $x \in \mathbf{R}^n$ and are continuous with respect to $t$, and where $\{U_k^{(0)}(x)\}_{1\le k\le n}$ are analytic, we note that our result guarantees convergence of a semi-discrete finite difference scheme for (1.5). We here remark that the Banach space for (1.5)–(1.6) is that of bounded analytic functions, and an element $u$ of $X_\rho$ is analytic on $D_\rho := \{x \in \mathbf{C}^n \mid |x| < \rho\}$ and $\|u\|_\rho = \sup_{x\in D_\rho}|u(x)|$.

We give our main result and its proof in §2, and we show a numerical example in §3. It is impossible to see convergence of difference schemes without stability through numerical computation using the standard double precision, but it is possible to do it on multiple-precision arithmetic, which can extinguish the rounding errors virtually. In this research, we use the multiple-precision arithmetic *exflib* designed by Hiroshi Fujiwara [1]. The main theorem in §2 was presented orally at the conference "International conference on inverse problems and its applications (17–20 Aug. 2009 at Korea)" without a proof.

**2. Finite difference scheme and its convergence.** Let us consider the Cauchy problem (1.3)–(1.4), and the Ovsjannikov-Yamanaka theorem guarantees its unique solvability in $|t| < \delta$, where $\delta := \min\Big(T_0, \dfrac{\rho-\rho'}{\omega e}\Big)$. We remark that we assume existence of the solution $u \in C^2([-\delta,\delta] : X_{\rho'})$ to the Cauchy problem in our discussion. For a positive number $T$ such that $0 < T < \delta$ and a positive integer $N$, we set $\Delta t := \frac{T}{N}$ and $t_k := k\Delta t$ $(0 \le k \le N)$, and we give a semi-discrete finite difference scheme for (1.3)–(1.4) as follows:

$$\frac{u_{k+1} - u_k}{\Delta t} = A(t_k)u_k + f(t_k) \quad (0 \le k \le N-1),$$
$$u^0 = U.$$

Our difference scheme is

$$(2.1)\qquad u_{k+1} = u_k + \Delta t A(t_k)u_k + \Delta t f(t_k) \quad (0 \le k \le N-1),$$

$$(2.2)\qquad u_0 = U \in X_\rho,$$

and we obtain $\{u_k\}_{k=0}^N \subset X_\rho$. To start analysis for convergence, we should require regularity of the solution of (1.3)–(1.4) in order to apply the Taylor expansion. Suppose $f \in C^1([-T_0,T_0] : X_{\rho_0})$, then $u \in C^2([-\delta,\delta] : X_{\rho'})$ and we have

$$(2.3)\qquad u(t_{k+1}) = u(t_k) + \Delta t A(t_k)u(t_k) + \Delta t f(t_k) + \Delta t^2 v_k,$$

where there exists a positive number $V$ such that $\|v_k\|_{\rho'} \le V$. Let us denote our aimed discretization error $u(t_k) - u^k$ by $e_k$, then we have, from (2.1) and (2.3),

$$(2.4)\qquad e_k = e_0 + \Delta t\sum_{j=0}^{k-1}A(t_j)e_j + \Delta t^2\sum_{j=0}^{k-1}v_j \quad (1 \le k \le N-1),$$

where $e_0 = 0$. We will show the convergence of the scheme through estimation of $\{e_k\}_{k=0}^N$ as $N \to \infty$.

**Main theorem.** *We follow the same hypothesis and notation with the Ovsjannikov-Yamanaka theorem, and we suppose* $f \in C^1([-T_0,T_0] : X_{\rho_0})$. *Let* $T$ *be* $0 < T < \delta$, *and set* $\Delta t = \frac{T}{N}$ *and* $t_k = k\Delta t$ $(0 \le k \le N)$. *Let* $\lambda$ *be a positive number such that* $0 < \lambda < \rho' < \rho \le \rho_0$ *and* $\rho' - \lambda > \rho - \rho'$, *then there exists a positive number* $C$ *such that*

$$\max_{0\le k\le N}\|u(t_k) - u_k\|_\lambda \le C\Delta t.$$

**Lemma 1.** *The radius of convergence of the power series* $\displaystyle\sum_{m=1}^{\infty}\frac{m^m}{m!}z^m$ *is* $\dfrac{1}{e}$.

**Lemma 2.** *For a positive integer* $m$, *we have*

$$\sum_{j=0}^{k-1}j^m \le \frac{1}{m+1}k^{m+1}.$$

The idea of a proof of the main theorem is a discretized analogue of that of Ovsjannikov-Yamanaka theorem, and it is convenient to recall its proof in advance for clear understanding of ours.

*A sketch of a proof of the Ovsjannikov-Yamanaka theorem*: We apply the method of successive approximation to the integral equation (1.2). Define a sequence of functions $\{u^{(p)}\}_{p\geq 0}$ by

$$u^{(p+1)}(t) = U(t) + \int_0^t (A(\tau)u^{(p)}(\tau) + f(\tau))d\tau \qquad (p \geq 0), \tag{2.5}$$

where $u^{(0)} = U$, and we have

$$u^{(p+1)}(t) - u^{(p)}(t) = \int_0^t A(\tau)(u^{(p)}(\tau) - u^{(p-1)}(\tau))d\tau \qquad (p \geq 1). \tag{2.6}$$

In order to estimate $\|u^{(m)} - u^{(m-1)}\|_{\rho'}$, we divide the interval $[\rho', \rho]$ into $m$ equal sub-intervals by $(m-1)$ points

$$\xi_j := \rho - \frac{j}{m}(\rho - \rho') \quad (0 \leq j \leq m). \tag{2.7}$$

Combining (1.1) with (2.6), we have

$$\begin{aligned}\|u^{(1)} - u^{(0)}\|_{\xi_1} &\leq \frac{m\omega}{\rho - \rho'}\|U\|_{\xi_0} t + Ft \\ &= \frac{m\omega}{\rho - \rho'}\|U\|_{\rho} t + Ft,\end{aligned}$$

where $F$ is a positive constant depending on the function $F(t)$, and finally we obtain

$$\begin{aligned}&\|u^{(m)} - u^{(m-1)}\|_{\rho'} \\ &= \|u^{(m)} - u^{(m-1)}\|_{\xi_m} \\ &\leq \|U\|_{\rho}\frac{\omega^m m^m}{(\rho - \rho')^m}\frac{t^m}{m!} + F\frac{\omega^{m-1}m^{m-1}}{(\rho - \rho')^{m-1}}\frac{t^m}{m!}.\end{aligned}$$

By Lemma 1, we can construct a majorant for $\{u^{(m)}\}_{m\geq 0}$ in $|t| < \dfrac{\rho - \rho'}{\omega e}$, and we obtain a solution to the equation (1.2) through completeness of the Banach space $X_{\rho'}$. □

For the sake of a proof of our theorem, we are enough to follow a discretized version of the above proof, but we should note the choice of the scale $\lambda$. In the following proof, we take the scale $\lambda$ such that $\lambda < \rho'$ and $\rho' - \lambda > \rho - \rho'$.

*A proof of the theorem*: Let us regard the formula (2.4) as a kind of discretization of the integral equation (1.2), and we define, following (2.5), a sequence $\{(e_1^{(p)}, \cdots, e_N^{(p)})\}_{p\geq 0}$ by

$$e_k^{(p+1)} = e_0 + \Delta t\sum_{j=0}^{k-1} A(t_j)e_j^{(p)} + \Delta t^2 \sum_{j=0}^{k-1} v_j \qquad (1 \leq k \leq N), \tag{2.8}$$

where $e_k^{(0)} = e_0 \ (= 0)$ for $1 \leq k \leq N$. In order to estimate $\|e_k^{(m)} - e_k^{(m-1)}\|_\lambda$ $(1 \leq k \leq N)$, we divide the interval $[\lambda, \rho']$ into $m$ equal sub-intervals and set

$$\eta_j := \rho' - \frac{j}{m}(\rho' - \lambda) \quad (0 \leq j \leq m).$$

Firstly we have

$$\|e_k^{(1)}\|_{\eta_1} = \Delta t^2 \left\|\sum_{j=0}^{k-1} v_j\right\|_{\eta_1} \leq \Delta t V(k\Delta t). \tag{2.9}$$

The formula (2.8) leads us to

$$e_k^{(p+1)} - e_k^{(p)} = \Delta t\sum_{j=0}^{k-1} A(t_j)(e_j^{(p)} - e_j^{(p-1)}) \qquad (1 \leq k \leq N),$$

and we have, from (2.9),

$$\begin{aligned}\|e_k^{(2)} - e_k^{(1)}\|_{\eta_2} &= \Delta t\left\|\sum_{j=0}^{k-1} A(t_j)e_j^{(0)}\right\|_{\eta_2} \\ &\leq \Delta t\sum_{j=0}^{k-1}\left(\frac{m\omega}{\rho' - \lambda}\Delta t^2 jV\right) \\ &\leq \Delta t\frac{m\omega}{\rho' - \lambda} V\frac{1}{2}(k\Delta t)^2 \quad (1 \leq k \leq N).\end{aligned}$$

Through the inductive estimation, we obtain, from Lemma 2,

$$\|e_k^{(p)} - e_k^{(p-1)}\|_{\eta_p} \leq \Delta t\left(\frac{m\omega}{\rho' - \lambda}\right)^{p-1} V\frac{(k\Delta t)^p}{p!} \qquad (1 \leq k \leq N),$$

since

$$\begin{aligned}\|e_k^{(p)} - e_k^{(p-1)}\|_{\eta_p} &= \Delta t\left\|\sum_{j=0}^{k-1} A(t_j)(e_j^{(p-1)} - e_j^{(p-2)})\right\|_{\eta_p} \\ &\leq \Delta t\frac{m\omega}{\rho' - \lambda}\sum_{j=0}^{k-1}\|e_j^{(p-1)} - e_j^{(p-2)}\|_{\eta_{p-1}}.\end{aligned}$$

Hence we have

$$\begin{aligned}\|e_k^{(m)} - e_k^{(m-1)}\|_\lambda &= \|e_k^{(m)} - e_k^{(m-1)}\|_{\eta_m} \\ &\leq \Delta t\left(\frac{m\omega}{\rho' - \lambda}\right)^{m-1} V\frac{(k\Delta t)^m}{m!} \qquad (1 \leq k \leq N).\end{aligned} \tag{2.10}$$

When $m > l$, we can estimate $\|e_k^{(m)} - e_k^{(l)}\|_\lambda$ by (2.10):

$$\|e_k^{(m)} - e_k^{(l)}\|_\lambda \le \sum_{p=l+1}^{m} \|e_k^{(p)} - e_k^{(p-1)}\|_\lambda$$
$$\le \Delta t V \sum_{p=l+1}^{m} \left(\frac{p\omega}{\rho' - \lambda}\right)^{p-1} \frac{(k\Delta t)^p}{p!}.$$

The radius of convergence of the power series $\sum_{p=1}^{\infty}\left(\frac{p\omega}{\rho' - \lambda}\right)^{p-1}\frac{t^p}{p!}$ is $\frac{\rho' - \lambda}{\omega e}$, and we have assumed that $0 \le k\Delta t \le T < \delta < \frac{\rho - \rho'}{\omega e}$ and $\rho' - \lambda > \rho - \rho'$.

Then we conclude that there exists $e_k \in X_\lambda$ such that $\|e_k - e_k^{(m)}\|_\lambda \to 0$ as $m \to +\infty$ and that

$$\max_{0\le k\le N} \|e_k\| \le C\Delta t.$$

This estimate is equivalent to

$$\max_{0\le k\le N} \|u^k - u(t_k)\| \le C\Delta t,$$

and we complete the proof. $\square$

**Remark.** The result can be generalized for non-linear cases, following the result of T. Nishida [4].

**3. Numerical example.** We show a numerical example to illustrate the main theorem and we apply our theory to the Cauchy problem

$$\frac{\partial}{\partial t}u(t,x) = (5x-3)\frac{\partial}{\partial x}u(t,x) \quad t > 0,\ x \in \mathbf{R}, \tag{3.1}$$
$$u(0,x) = 5x - 3. \tag{3.2}$$

This example is a simple case of Kowalevskaya's system (1.5)–(1.6), and it has a real characteristic line $t + \frac{1}{5}\log|5x-3| = \text{const.}$ and the exact solution $u(t,x) = (5x-3)e^{5t}$. In the following computation, we discretize also the $x$-direction by the forward difference, and our finite difference scheme is

$$\frac{u^{k+1}(x) - u^k(x)}{\Delta t} = (5x-3)\frac{u^k(x+\Delta x) - u^k(x)}{\Delta x}. \tag{3.3}$$

We remark that the scheme (3.3) does not take either the characteristic line or speed of propagation into account and that it is generally an unstable scheme in the usual sense [3]. Our result guarantees convergence of the *exact* numerical solution, which is the exact solution to the difference scheme, for an analytic initial data. Here our numerical results for the case $\Delta x = \Delta t = 0.01$ and $0.1 \le t \le 0.25$.

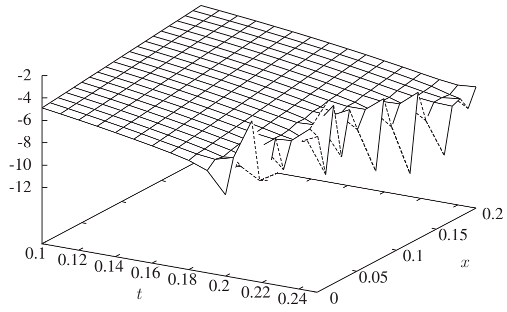


Fig. 1. Numerical results by the double precision.

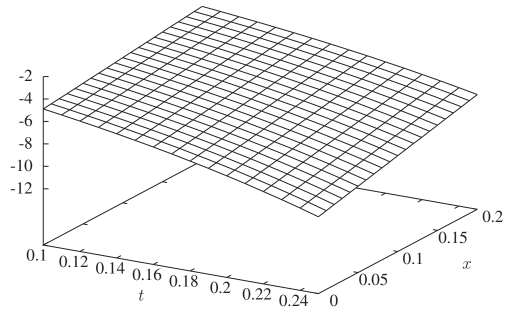


Fig. 2. Numerical results by 50 decimal digits precision.

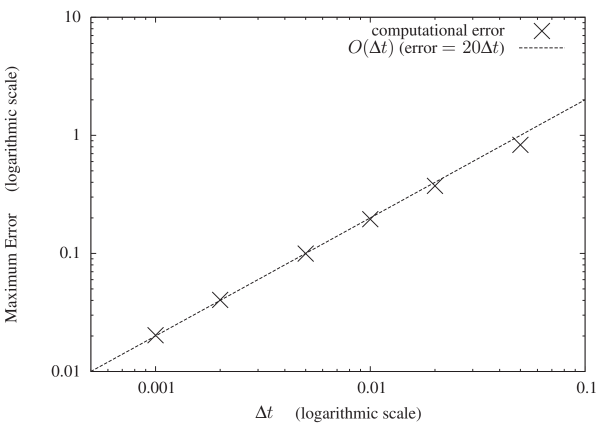


Fig. 3. Rate of convergence by 200 decimal digits precision.

Figure 1 shows the result by the standard double precision, and Fig. 2 does by 50 (decimal) digits precision on *exflib* [1] which enables us to extinguish influence of the rounding errors virtually. The numerical solution in Fig. 1 has oscillation because of instability of the difference scheme (3.3), but the oscillation is due to influence of the rounding errors. We are, as is shown in Fig. 2, able to extinguish influence of the rounding error virtually on *exflib*. Figure 3 shows rate of convergence as $\Delta t \to 0$. We here compute numerical solutions by 200 (decimal) digits precision on *exflib* and show an estimate

$$\max_{\substack{0<t_k<0.2\\0\leq x\leq 1}} |u(t_k,x)-u^k(x)|$$

for the case $\Delta x = \Delta t$. Dotted line in Fig. 3 indicates $O(\Delta t)$, and we notice that it is consistent to the main result. There numerical computations are carried out by Prof. Hiroshi Fujiwara on his system *exflib* [1].

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## References

[ 1 ] H. Fujiwara, `http://www-an.acs.i.kyoto-u.ac.jp/~fujiwara/exflib`
[ 2 ] K. Hayakawa, Convergence of finite difference scheme and analytic data, Publ. Res. Inst. Math. Sci. **24** (1988), no. 5, 759–764.
[ 3 ] P. D. Lax and R. D. Richtmyer, Survey of the stability of linear finite difference equations, Comm. Pure Appl. Math. **9** (1956), 267–293.
[ 4 ] T. Nishida, A note on a theorem of Nirenberg, J. Differential Geom. **12** (1977), 629–633.
[ 5 ] L. V. Ovsjannikov, Singular operator in the scale of Banach spaces, Dokl. Akad. Nauk SSSR **163-4** (1965), 819–822. (Eng. Trans. Soviet Math. Dokl. **6** (1965), 1025–1028).
[ 6 ] T. Yamanaka, Note on Kowalevskaja's system of partial differential equations, Comment. Math. Univ. St. Paul **9** (1961), 7–10.