The number of orientable small covers over cubes

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Abstract: We count orientable small covers over cubes. We also get estimates for O_n/R_n, where O_n is the number of orientable small covers and R_n is the number of all small covers over an n-cube up to the Davis-Januszkiewicz equivalence.

Key words: Orientable small cover; acyclic digraph; real Bott manifold; Toric Topology.

1. Introduction. A *small cover*, defined by Davis and Januszkiewicz [2], is an n-dimensional closed smooth manifold M with an effective real torus $(S^0)^n (=: T^n)$-action such that the action is locally isomorphic to a standard T^n-action on $\mathbf{R}^n$ and the orbit space M/T^n can be identified with a simple combinatorial polytope. For instance, $\mathbf{R}P^n$ with a natural T^n-action is a small cover over an n-simplex. In general, a real toric manifold, the set of real points of a toric manifold, provides an example of small covers. Hence, small covers can be seen as a topological generalization of real toric manifolds in algebraic geometry.

A small cover over a cube is known as a *real Bott manifold* which is obtained as iterated $\mathbf{R}P^1$ bundles starting with a point, where each fibration is the projectivization of a Whitney sum of two real line bundles. These manifolds are well-studied in numerous papers such as [3] and [4]. The author also found a strong relation between small covers and acyclic digraphs, and he calculated the number of them up to several senses in [1].

In the present paper, we restrict our attention to the case of orientable small covers over a cube. Thankfully, Nakayama and Nishimura [5] found a simple criterion for a small cover to be orientable. Using this criterion, we establish the formula of the number of orientable small covers over a cube and show that the ratio O_n/R_n is approximately $\frac{1.262}{2^n}$, where O_n is the number of orientable small covers and R_n is the number of small covers over an n-cube up to the Davis-Januszkiewicz equivalence.

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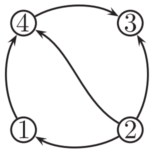
2. Orientable small covers over cubes. Let P be an n-dimensional simple polytope with m facets. Two small covers M_1 and M_2 over P are *Davis-Januszkiewicz equivalent* (or simply, *D-J equivalent*) if there is a weak T^n-equivariant homeomorphism $f\colon M_1 \to M_2$ which makes the diagram commute:

$$\begin{tikzcd} M_1 \arrow[rr, "f"] \arrow[dr] & & M_2 . \arrow[dl] \\ & P & \end{tikzcd}$$

It is well-known by [2] that all small covers over P can be distinguished by the map λ from the set of facets of P to $\mathbf{Z}_2^n = \{0,1\}^n$, called the *characteristic function*, which satisfies the *non-singularity condition*; $\{\lambda(F_{i_1}), \dots, \lambda(F_{i_n})\}$ is a basis of $\mathbf{Z}_2^n$ whenever the intersection $F_{i_1} \cap \cdots \cap F_{i_n}$ is non-empty, where $\{F_1, \dots, F_m\}$ is the set of facets of P. Let M_1, M_2 be two small covers over P corresponding to characteristic functions λ_1, λ_2, respectively. By [2], M_1 is D-J equivalent to M_2 if and only if there is an automorphism $\sigma \in \mathrm{Aut}(\mathbf{Z}_2^n)$ such that $\lambda_1 = \sigma \circ \lambda_2$. Hence, the D-J equivalence classes are independent of the choice of basis for $\mathbf{Z}_2^n$. One may assign an $n \times m$ matrix Λ to λ by ordering the facets and choosing a basis for $\mathbf{Z}_2^n$ as the follow:

$$\Lambda = (\lambda(F_1) \cdots \lambda(F_m)).$$

If we additionally assume that the first n facets meet at a vertex, by the non-singularity condition, we can choose an appropriate basis of $\mathbf{Z}_2^n$ such that $\Lambda = (E_n | \Lambda_*)$, where E_n is the identity matrix of size n and Λ_* is an $n \times (m-n)$ matrix. Hence, the D-J equivalence classes of small covers over P are classified by Λ_*.



$$\xmapsto{\phi} \begin{pmatrix} 1&1&0&0\\ 0&1&0&0\\ 0&1&1&1\\ 1&1&0&1 \end{pmatrix}$$

Fig. 1. A bijection ϕ.

Now, we consider the case where P is an n-cube. Note that P has $2n$ facets. We order the facets of P satisfying $F_j \cap F_{n+j} = \emptyset$ for $1 \leq j \leq n$. Then the first n facets meet at a vertex. Hence, for each λ, the corresponding matrix Λ can be expressed as $\Lambda = (E_n|\Lambda_*)$, where Λ_* is an $n \times n$ matrix. One can check that the non-singularity condition holds if and only if all of principal minors of Λ_* are 1. Therefore, there is a bijection between small covers over cubes up to the D-J equivalence and square $\mathbf{Z}_2$-matrices all of whose principal minors are 1.

Let $M(n)$ be the set of square $\mathbf{Z}_2$-matrices of size n all of whose principal minors are 1 and let $\mathcal{G}_n$ be the set of acyclic digraphs with labelled n vertices. By [1], we have a bijection $\phi : \mathcal{G}_n \to M(n)$ by

$$\phi : G \mapsto A(G)^t + E_n,$$

where $A(G)^t$ is the transpose matrix of the vertex adjacency matrix of G (see Fig. 1).

Remark 2.1. In the classical theory of real Bott manifolds, the representative matrix of real Bott manifold is the transpose matrix of its characteristic function matrix Λ_*. This is a reason why we use $A(G)^t$ instead of $A(G)$ in the definition of ϕ.

On the other hand, we have a nice orientability condition for small covers due to Nakayama and Nishimura in [5].

Theorem 2.2 (Nakayama and Nishimura [5])**.** *Let P be an n-dimensional simple polytope with m facets and let M be a small cover over P with Λ. Then M is orientable if and only if the sum of entries of the i-th column of Λ is odd for all $i = 1, \ldots, m$.*

Corollary 2.3. *The number of orientable small covers over an n-cube up to D-J equivalence is equal to the number of acyclic digraphs with labelled n vertices all of whose vertices have even out-degrees.*

Proof. Let G be a digraph and $A(G)$ its vertex adjacency matrix. Then the sum of entries of the i-th row of $A(G)$ means the out-degree of the i-th vertex of G (see Appendix). Let M be an orientable small cover over an n-cube corresponding to Λ_*. Since $\Lambda_* \in M(n)$, the transpose Λ_*^t of Λ_* is also in $M(n)$. Note that the sum of entries of each row of $\Lambda_*^t - E_n$ is even by Theorem 2.2, and hence, every vertex of $\phi^{-1}(\Lambda_*)$ has an even out-degree. Since the D-J equivalence classes are classified by Λ_* and ϕ is a bijection, we prove the corollary. $\square$

3. The number of orientable small covers. Let R_n be the number of acyclic digraphs with labelled n vertices. The following is the recursive formula for R_n due to R. W. Robinson in [6].

$$R_n = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} 2^{k(n-k)} R_{n-k}.$$

Let $\mathcal{O}_n \subset \mathcal{G}_n$ be the set of acyclic digraphs all of whose vertices have even out-degrees and let O_n be the cardinality of $\mathcal{O}_n$ (we use the alphabet 'O' instead of 'E' although they have only 'even' out-degree vertices, because the 'O' is the abbreviation of the word '*Orientable*').

Theorem 3.1. *Let R_k be the number of acyclic digraphs with labelled k vertices. Then,*

$$O_n = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} 2^{(k-1)(n-k)} R_{n-k}.$$

Proof. We count matrices in $M(n)$ all of whose the sum of entries of each column are odd. Let us denote the sum of entries of the i-th column of an $n \times n$ matrix A by $c_i(A)$. Since an acyclic digraph always has a vertex of out-degree 0, there is at least one i such that $c_i(A) = 1$ for each $A \in M(n)$. Assume $c_{i_1}(A) = \cdots = c_{i_k}(A) = 1$, where $k \geq 1$. Since all principal minors of A are 1, the diagonal entries of A are all 1. Thus, by a replacement of labels, we may assume that A is of the following form:

$$\begin{pmatrix} E_k & S \\ 0 & T \end{pmatrix}, \tag{1}$$

where E_k is the identity matrix of size k, T is an $(n-k) \times (n-k)$-matrix and S is a $k \times (n-k)$-

Table I

n	1	2	3	4	5	6	7
R_n	1	3	25	543	29,281	3,781,503	1,138,779,265
O_n	1	1	4	43	1,156	74,581	11,226,874

matrix. Note that $A \in M(n)$ if and only if $T \in M(n-k)$. Thus we may control only one row of S for making all $c_i(A)$'s are odd. This implies the number of A's of the form (1) whose $c_i(A)$'s are odd for all i is $2^{(k-1)(n-k)}R_{n-k}$. To avoid counting repeatedly, we apply the Principle of Inclusion-Exclusion and we get the formula for O_n. $\square$

We list a few values of R_n and O_n in Table I.

Let us consider the *chromatic generating functions* of R_n and O_n, namely, we set

$$R(x) = \sum_{n=0}^{\infty} R_n \frac{x^n}{n!2^{\binom{n}{2}}} \text{ and } O(x) = \sum_{n=0}^{\infty} O_n \frac{x^n}{n!2^{\binom{n}{2}}}.$$

Corollary 3.2. *Let $F(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!2^{\binom{n}{2}}}$. Then*

$$O(x) = \frac{1-F(-x)}{F\left(-\dfrac{x}{2}\right)}.$$

Proof. Let us consider chromatic generating functions $A(x)$, $B(x)$ and $C(x)$ with respect to the sequences A_n, B_n and C_n, respectively. Note that if $C(x) = A(x)B(x)$, then $C_n = \sum_{k=0}^{n} A_k B_{n-k}\binom{n}{k}2^{k(n-k)}$. Thus, we have $F(-x)R(x) = 1$ (see [7]) and

$$R\left(\frac{x}{2}\right)F(-x) + O(x) = \sum_{n=0}^{\infty} \frac{R_n}{2^n}\frac{x^n}{n!2^{\binom{n}{2}}} = R\left(\frac{x}{2}\right).$$

Hence we have $O(x) = F\left(-\frac{x}{2}\right)^{-1}(1-F(-x))$. $\square$

Let $G(x) = \frac{F(\frac{x}{2})}{1-F(x)}$. We obtain estimates for O_n by analyzing the behavior of the function $G(x)$. Since $F(x)$ has an isolated zero $\alpha \approx -1.488$ (see [7, Section 2]), $G(x)$ has an isolated zero 2α. Hence, standard techniques provide the asymptotic formula

$$G(x) \sim G'(2\alpha)(x-2\alpha).$$

Hence we have

$$O(x) = \frac{1}{G(-x)} \sim \frac{1}{G'(2\alpha)(-x-2\alpha)}.$$

Note that $F'(x) = F(\frac{x}{2})$. Therefore, the following asymptotic formula

$$O(x) \sim -\frac{1-F(2\alpha)}{\alpha F\left(\dfrac{\alpha}{2}\right)} \sum_{n=0}^{\infty}\left(-\frac{x}{2\alpha}\right)^n$$

immediately follows two facts $\frac{1}{G'(2\alpha)} = \frac{2(1-F(2\alpha))}{F'(\alpha)}$ and $\frac{1}{-x-2\alpha} = \frac{1}{-2\alpha}\sum_{n=0}^{\infty}(-\frac{x}{2\alpha})^n$.

Therefore $O_n \sim K2^{\binom{n}{2}}n!(-\frac{1}{2\alpha})^n$, where $K = -\frac{1-F(2\alpha)}{\alpha F(\frac{\alpha}{2})} \approx 2.197$.

Corollary 3.3. *We have estimates for the orientable small covers ratio as*

$$\frac{O_n}{R_n} \sim \frac{K}{C2^n},$$

where $\frac{K}{C} \approx 1.262$.

Proof. Since $R(x)F(-x) = 1$ and $F(x)$ has an isolated zero α, we have $R(x) = \frac{1}{F(-x)} \sim \frac{1}{F'(\alpha)(-x-\alpha)} = \frac{1}{-\alpha F(\frac{\alpha}{2})}\sum_{n=0}^{\infty}(-\frac{x}{\alpha})^n$. Hence, we have $R_n \sim C2^{\binom{n}{2}}n!(-\frac{1}{\alpha})^n$, where $C = -\frac{1}{\alpha F(\frac{\alpha}{2})} \approx 1.739$. Therefore $\frac{O_n}{R_n} \sim \frac{K}{C2^n} \approx \frac{1.262}{2^n}$. $\square$

Appendix. Graph theory terminology. We review the terminology in graph theory, following [8]. A *directed graph* or *digraph* G is a triple (V, E, φ), where $V = \{v_1, \ldots, v_n\}$ is a set of vertices, E is a set of directed edges, and φ is a map from E to $V \times V$. If $\varphi(e) = (u, v)$, then e is called an *edge from u to v* with the initial vertex u and the final vertex v. If $u = v$ then e is called a *loop*. If φ is injective and has no loops, then G is said to be *simple*. In this case, we denote e by (u, v) for simplicity and represent G by (V, E). *Throughout this paper, every graph is simple.* A *walk* of length k from vertex u to v is a sequence $v_0, v_1, \ldots, v_k$ such that $v_0 = u$ and $v_k = v$, where $(v_i, v_{i+1}) \in E$ for all $i = 0, \ldots, k-1$. If all v_i's are distinct except for $v_o = v_k$, then the walk is called a *cycle*. G is *acyclic* if there is no cycle of any length in G. The *out-degree* of a vertex v is the number of edges of G with the initial vertex v. Similarly the in-degree of v is the number of edges of G with the final vertex v.

All digraphs can be represented by matrices. Define an $n \times n$ matrix $A(G) = (A_{ij})$ by

$$A_{ij} = \begin{cases} 1, & \text{if } (v_i, v_j) \in E; \\ 0, & \text{otherwise.} \end{cases}$$

The matrix $A(G)$ is called the *vertex adjacency matrix* of G. We remark that the sum of entries of the i-th column of A is equal to the in-degree of v_i and the sum of entries of the j-th row of A is equal to the out-degree of v_j.

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