Modularity gap for Eisenstein series

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Abstract: We present a formula describing modularity gap for Eisenstein series, which is written in terms of a certain double series. Limit values of the gap at nonzero rational points are expressible by Hurwitz zeta values. Our gap estimates near the origin are applied to examining the asymptotic behaviour of Ramanujan q-series and q-zeta values near the natural boundary $|q| = 1$.

Key words: Eisenstein series; modularity; zeta functions; Ramanujan q-series.

1. Introduction. For $s \in \mathbf{C}$, consider the Ramanujan q-series

$$\Phi_{s-1}(q) := \sum_{n=1}^{\infty} \frac{n^{s-1} q^n}{1 - q^n} = \sum_{n=1}^{\infty} \sigma_{s-1}(n) q^n$$

with $\sigma_{s-1}(n) = \sum_{d|n} d^{s-1}$, which is holomorphic for $|q| < 1$ (cf. [1]). If s is an even positive integer satisfying $s \geq 4$, the Eisenstein series

$$E_s(\tau) = \frac{\zeta(1-s)}{2} + \Phi_{s-1}(e^{2\pi i \tau})$$

for $\tau \in \mathbf{C}$ with $\mathrm{Im}(\tau) > 0$ admits the modularity $E_s(-1/\tau) = \tau^s E_s(\tau)$. If $s = 2$, then $E_2(-1/\tau) = \tau^2 E_2(\tau) - \tau/(4\pi i)$. For each positive integer s, Kurokawa [5] expressed the gap $E_s(-1/\tau) - \tau^s E_s(\tau)$ in terms of a multiple cotangent function, and computed its limit values as $\tau \to 1, 2, 1/2$ ($\mathrm{Im}(\tau) > 0$).

In this paper we make a more direct approach toward this problem, and present a formula describing modularity gap, which is continuous in $s \in \mathbf{C}$ with $\mathrm{Re}(s) > 2$. Our gap formula is written in terms of a certain double series. Using this, we show that the limit values of the gap as $\tau \to \mu \in \mathbf{Q} \setminus \{0\}$ may be expressed by Hurwitz zeta values, in particular, by $\zeta(s-1)$ and $L(s, \chi)$ if μ (or $1/\mu$) $\in \mathbf{N}$. Furthermore, for the gap near $\tau = 0$, order estimates are given.

Kaneko *et al.* [3] introduced a q-analogue of the Riemann zeta function defined by

$$\zeta_q(s) := (1-q)^s \sum_{n=1}^{\infty} \frac{q^{n(s-1)}}{(1-q^n)^s}$$

for $\mathrm{Re}(s) > 1$, and proved that, for each q satisfying $0 < q < 1$, the function $\zeta_q(s)$ is continued meromorphically to the whole complex plane, and that $\lim_{q \to 1-0} \zeta_q(s) = \zeta(s)$ for every $s \in \mathbf{C} \setminus \{1\}$. For every integer $s \geq 2$, the Ramanujan q-series $\Phi_{s-1}(q)$ is related to $\zeta_q(s)$ through the equality

$$(1.1) \qquad \zeta_q(s) = (1-q)^s \sum_{n=1}^{\infty} \binom{n}{s-1} \frac{q^n}{1-q^n} = \frac{(1-q)^s}{(s-1)!} \sum_{r=1}^{s-1} \kappa_r^s \Phi_r(q)$$

with κ_r^s such that

$$n(n-1) \cdots (n-(s-2)) = \sum_{r=1}^{s-1} \kappa_r^s n^r,$$

in particular $\kappa_{s-1}^s = 1$, $\kappa_{s-2}^s = -(s-1)(s-2)/2$ (cf. [3, p. 185]). For q-zeta values $\zeta_q(s_1), \ldots, \zeta_q(s_l)$ (or $\Phi_{s_1-1}(q), \ldots, \Phi_{s_l-1}(q)$) with $s_1, \ldots, s_l \in \mathbf{N}$, Pupyrev [8] discussed linear and algebraic independence over $\mathbf{C}(q)$ as q-series, by using order estimates for them as q tends to a root of unity (see also [9]).

As applications of our gap estimates near $\tau = 0$, we examine the asymptotic behaviour of $\Phi_{s-1}(q)$ and $\zeta_q(s)$ near the natural boundary $|q| = 1$.

2. Results. For $s \in \mathbf{C}$ and for $\tau \in \mathbf{C}$ with $\mathrm{Im}(\tau) > 0$, consider the Eisenstein series

$$E_s(\tau) = \frac{\zeta(1-s)}{2} + \Phi_{s-1}(e^{2\pi i \tau}) = \frac{\Gamma(s)\zeta(s)}{(2\pi)^s} \cos \frac{\pi s}{2} + \Phi_{s-1}(e^{2\pi i \tau}),$$

and let us set

$$\Delta_s(\tau) := \tau^{-s} E_s(-1/\tau) - E_s(\tau).$$

The modularity gap is described as follows:

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Theorem 2.1. *For* $\mathrm{Re}(s) > 2$ *and for* $\mathrm{Im}(\tau) > 0$,

$$\Delta_s(\tau) = -\frac{2i\Gamma(s)}{(2\pi)^s}\sin\frac{\pi s}{2} \times \Bigl(\sum_{(j,k)\in\mathbf{N}^2}\frac{1}{(j\tau+k)^s} + \frac{\tau^{-s}+1}{2}\zeta(s)\Bigr),$$

where the branches of τ *and* $j\tau + k$ *are taken so that* $\arg(\tau)$, $\arg(j\tau + k) \in (0, \pi)$.

Remark 2.1. Substitution $\tau \mapsto e^{\pi i}/\tau$ yields another relation of the form

$$\Delta_s^*(\tau) := (e^{\pi i}\tau^{-1})^s E_s(-1/\tau) - E_s(\tau) = \frac{2i\Gamma(s)}{(2\pi)^s}\sin\frac{\pi s}{2} \times \Bigl(\sum_{(j,k)\in\mathbf{N}^2}\frac{e^{\pi i s}}{(j\tau-k)^s} + \frac{(e^{\pi i}\tau^{-1})^s+1}{2}\zeta(s)\Bigr)$$

with $\arg(e^{\pi i}\tau^{-1})$, $\arg(j\tau - k) \in (0, \pi)$.

For each $\mu > 0$, the sum $\sum_{(j,k)\in\mathbf{N}^2}(j\tau + k)^{-s}$ (respectively, $\sum_{(j,k)\in\mathbf{N}^2}(j\tau - k)^{-s}$) with $\mathrm{Re}(s) > 2$ is holomorphic around $\tau = \mu$ (respectively, $\tau = -\mu$). As an immediate consequence of Theorem 2.1 and Remark 2.1 we obtain the following

Theorem 2.2. *Suppose that* $\mathrm{Re}(s) > 2$. *Then, for each* $\mu > 0$,

$$\Delta_s(\tau) = -\frac{2i\Gamma(s)}{(2\pi)^s}\sin\frac{\pi s}{2} \times \Bigl(\Lambda_\mu(s) + \frac{\mu^{-s}+1}{2}\zeta(s) + O(\tau-\mu)\Bigr)$$

and

$$\Delta_s^*(\tau) = \frac{2i\Gamma(s)}{(2\pi)^s}\sin\frac{\pi s}{2} \times \Bigl(\Lambda_\mu(s) + \frac{\mu^{-s}+1}{2}\zeta(s) + O(\tau+\mu)\Bigr)$$

as $\tau \to \mu$ $(\mathrm{Im}(\tau) > 0)$ *and as* $\tau \to -\mu$ $(\mathrm{Im}(\tau) > 0)$, *respectively, where* $\Lambda_\mu(s) := \sum_{(j,k)\in\mathbf{N}^2}(j\mu + k)^{-s}$.

For each positive rational number μ, the series $\Lambda_\mu(s)$ may be expressed as a finite sum of Hurwitz zeta functions.

Theorem 2.3. *Let* g *and* h *be given relatively prime positive integers, and set* $\lambda_{h,g}(a, b) := a/h + b/g$ *for positive integers* a, b. *Then, for each* $\mu = g/h$, *we have*

$$\begin{aligned}\Lambda_{g/h}(s) = {} & \frac{1}{g^s}\sum_{\lambda_{h,g}(a,b)\le 1}^{(h,g)}\Bigl(\zeta(s-1,\lambda_{h,g}(a,b)) + \bigl(1-\lambda_{h,g}(a,b)\bigr)\zeta(s,\lambda_{h,g}(a,b))\Bigr)\\ & + \frac{1}{g^s}\sum_{\lambda_{h,g}(a,b)>1}^{(h,g)}\Bigl(\zeta(s-1,\lambda_{h,g}(a,b)-1) + \bigl(1-\lambda_{h,g}(a,b)\bigr)\zeta(s,\lambda_{h,g}(a,b)-1)\Bigr),\end{aligned}$$

where $\sum_{(*)}^{(h,g)}$ *is the summation over the pairs* (a, b) *satisfying* $1 \le a \le h$, $1 \le b \le g$ *and the inequality* $(*)$. *Moreover, if* $\mu = g$ *is a positive integer,*

$$\Lambda_g(s) = \frac{1}{g}\zeta(s-1) - \frac{1}{2}(1 + g^{-s})\zeta(s) + H_g(s),$$

$$H_g(s) := \frac{1}{g}\sum_{b=1}^{[g/2]}\Bigl(\frac{g}{2} - b\Bigr) \times \sum_{\nu=0}^{\infty}\Bigl(\frac{1}{(g\nu+b)^s} - \frac{1}{(g\nu+g-b)^s}\Bigr).$$

The quantity $H_g(s)$ *may be represented as a linear combination of Dirichlet L-functions* $L(s,\chi)$ *with coefficients in* $\mathbf{Q}(e^{2\pi i/\varphi(g)}, g_1^{-s}, \dots, g_d^{-s})$, *where* $g_j \,|\, g$ $(1 \le j \le d)$.

Example 2.1. Since $H_1(s) = H_2(s) = 0$, the limit values of $\Delta_s(\tau)$ as $\tau \to 1, 2$ agree with the result of [5]. Moreover,

$$H_3(s) = \frac{1}{6}L(s,\varepsilon_3), \quad H_4(s) = \frac{1}{4}L(s,\varepsilon_4),$$

$$H_5(s) = \frac{1}{20}\Bigl((3-i)L(s,\chi_*) + (3+i)L(s,\overline{\chi}_*)\Bigr),$$

$$H_6(s) = \frac{1}{3}L(s,\varepsilon_6) + \frac{2^{-s}}{6}L(s,\varepsilon_3).$$

Here $\varepsilon_g = \varepsilon_g(n)$ $(g = 3, 4, 6)$ (respectively, $\chi_* = \chi_*(n)$) is the character such that $\varepsilon_g(n) = \pm 1$ if $n \equiv \pm 1 \pmod g$ (respectively, $\chi_*(n) = \pm 1$ if $n \equiv \pm 1 \pmod 5$, $\chi_*(n) = \pm i$ if $n \equiv \pm 2 \pmod 5$) and that $\varepsilon_g(n) = 0$ (respectively, $\chi_*(n) = 0$) otherwise.

Remark 2.2. Let $p_n(x)$ $(n = 1, 2, 3, \dots)$ be polynomials defined by

$$p_1(x) = x, \quad p_{n+1}(x) = (1 + x^2)p_n'(x)/n.$$

If s is an odd integer, it is known [6, §5.3] that

$$\sum_{\nu=0}^{\infty}\Bigl(\frac{1}{(g\nu+b)^s} - \frac{1}{(g\nu+g-b)^s}\Bigr) = \frac{\pi^s}{g^s}p_s\Bigl(\cot\frac{\pi b}{g}\Bigr).$$

Remark 2.3. Since $\Lambda_{1/g}(s) = g^s\Lambda_g(s)$, the value $\Lambda_{1/g}(s)$ is also expressible by $\zeta(s)$, $\zeta(s-1)$ and $L(s,\chi)$. By Theorem 2.2 the limit value of $\Delta_s^*(\tau)$ as $\tau \to -g/h$ is written in terms of Hurwitz zeta values.

In what follows θ_0 denotes a given positive number such that $\theta_0 < \pi/2$. The following gives gap estimates near $\tau = 0$.

Theorem 2.4. *For* $\mathrm{Re}(s) > 2$,

$$\Delta_s(\tau) = -\frac{2i\Gamma(s)}{(2\pi)^s}\sin\frac{\pi s}{2}\Big(\frac{\tau^{-s}+1}{2}\zeta(s) + O(\tau^{-1})\Big)$$

as $\tau \to 0$ *through the sector* $\pi/2 - \theta_0 < \arg(\tau) \le \pi/2$, *and*

$$\Delta_s^*(\tau) = \frac{2i\Gamma(s)}{(2\pi)^s}\sin\frac{\pi s}{2} \times \Big(\frac{(e^{\pi i}\tau^{-1})^s+1}{2}\zeta(s) + O(\tau^{-1})\Big)$$

as $\tau \to 0$ *through the sector* $\pi/2 \le \arg(\tau) < \pi/2 + \theta_0$.

Using these estimates, we obtain asymptotic expressions for $\Phi_{s-1}(q)$ and $\zeta_q(s)$ near the natural boundary $|q| = 1$.

Theorem 2.5. *We have*

$$\begin{aligned}\Phi_1(q) &= \frac{\zeta(2)}{\log^2(1/q)} - \frac{1/2}{\log(1/q)} + \frac{1}{24} + O((1-q)^{-2}e^{-c_0/|1-q|})\\ &= \frac{\zeta(2)}{(1-q)^2} - \frac{\zeta(2)+1/2}{1-q} + \frac{2\zeta(2)+7}{24} + O(1-q)\end{aligned}$$

and, for each $s \in \mathbf{C}$ *such that* $\mathrm{Re}(s) > 2$,

$$\begin{aligned}\Phi_{s-1}(q) &= \Gamma(s)\zeta(s)\Big(\frac{1+O(e^{-c_0/|1-q|})}{\log^s(1/q)} - \frac{e^{-\pi i s/2}}{(2\pi)^s}\Big) + O(\sin(\pi s/2)(1-q)^{-1})\\ &= \frac{\Gamma(s)\zeta(s)}{(1-q)^s}\Big(1 - \frac{s}{2}(1-q) + O((1-q)^2)\Big) + O((1-q)^{-1})\end{aligned}$$

as $q \to 1$ *through the sector* $|\arg(1-q)| < \theta_0$, *where* c_0 *is some positive number depending on* θ_0.

Theorem 2.6. *We have*

$$\begin{aligned}\zeta_q(2) &= (1-q)^2\Big(\frac{\zeta(2)}{\log^2(1/q)} - \frac{1/2}{\log(1/q)} + \frac{1}{24}\Big) + O(e^{-c_0/|1-q|})\\ &= \zeta(2) - \Big(\zeta(2) + \frac{1}{2}\Big)(1-q) + \Big(\frac{\zeta(2)}{12} + \frac{7}{24}\Big)(1-q)^2 + O((1-q)^3)\end{aligned}$$

and, for each integer $s \ge 3$,

$$\begin{aligned}\zeta_q(s) &= \frac{(1-q)^s}{(s-1)!}\sum_{r=1}^{s-1}\frac{\kappa_r^s r!\zeta(r+1)}{\log^{r+1}(1/q)} + O((1-q)^{s-1})\\ &= \zeta(s) - \frac{1}{2}\Big(s\zeta(s) + (s-2)\zeta(s-1)\Big)(1-q) + O((1-q)^2)\end{aligned}$$

as $q \to 1$ *through the sector* $|\arg(1-q)| < \theta_0$.

Let g and h be given relatively prime integers satisfying $|g| \ge 1$ and $h \ge 2$. Set $q = e^{2\pi i g/h}t$ with $|t| < 1$. Then we have the following results, where a constant related to the symbol O may be taken independently of h and g.

Theorem 2.7. *For each integer* $s \ge 2$,

$$\Phi_{s-1}(q) = \Phi_{s-1}(t^h) + O(h^{s+1}(1-t^h)^{-1})$$

as $t \to 1$ *through the sector* $|\arg(1-t)| < \theta_0$.

Theorem 2.8. *For each integer* $s \ge 2$,

$$\zeta_q(s) = \Big(\frac{1-q}{1-t^h}\Big)^s\zeta_{t^h}(s) + O(h^{s+1}(1-t^h)^{-1})$$

as $t \to 1$ *through the sector* $|\arg(1-t)| < \theta_0$.

It is known that the Chazy equation

$$y''' = 2yy'' - 3(y')^2$$

($' = d/dz$) admits a family of solutions

$$y_C(z) := \pi i(1 - 24\Phi_1(e^{2\pi i(z-C)})) \quad (C \in \mathbf{C})$$

(cf. [1, 2]). Note that $y_C(z)$ possesses the natural boundary $\mathrm{Im}\,(z-C) = 0$ corresponding to $|q| = 1$ of $\Phi_1(q)$. As an immediate corollary to Theorem 2.7, we have

Corollary 2.9. *The solution* $y_C(z)$ *is holomorphic in the half-plane* $\mathrm{Im}(z-C) > 0$, *and*

$$y_C(C + g/h + \eta i) = -\pi i h^{-2}\eta^{-2}(1 + O(h^4\eta))$$

as $\eta \to 0$ *through the sector* $|\arg(\eta)| < \theta_0$.

3. Proof of Theorem 2.1. For $\mathrm{Re}\,(s) > 2$ and for $\mathrm{Im}(\tau) > 0$, the function

$$f(\tau) = \sum_{k\in\mathbf{Z}}(\tau+k)^{-s}$$

admits the Fourier expansion $\sum_{m=1}^{\infty} a_m e^{2\pi i m\tau}$, whose coefficients are given by

$$\begin{aligned}a_m &= \int_i^{1+i} e^{-2\pi i m\tau}f(\tau)d\tau = \sum_{k\in\mathbf{Z}}\int_i^{1+i}\frac{e^{-2\pi i m\tau}}{(\tau+k)^s}d\tau\\ &= \int_{-\infty+i}^{\infty+i} u^{-s}e^{-2\pi i m u}du,\end{aligned}$$

where $\arg(u)$ varies from π to 0. Observe that the path of integration may be changed into the contour $C(0)$ that starts from $u = e^{3\pi i/2}\infty$, encircles $u = 0$ in the negative sense, and ends at $u = e^{-\pi i/2}\infty$. By putting $2\pi m u = e^{-\pi i/2} v$, this integral is written in the form

$$\begin{aligned}\int_{C(0)} & u^{-s} e^{-2\pi i m u} du \\ &= -i e^{\pi i s/2} (2\pi m)^{s-1} \int_{C'(0)} v^{-s} e^{-v} dv \\ &= -i e^{\pi i s/2} (1 - e^{-2\pi i s}) (2\pi m)^{s-1} \Gamma(1-s) \\ &= e^{-\pi i s/2} (2\pi)^s m^{s-1} / \Gamma(s),\end{aligned}$$

where $C'(0) := \{v = 2\pi m e^{\pi i/2} u \,|\, u \in C(0)\}$. Thus we have

$$f(\tau) = \frac{e^{-\pi i s/2}(2\pi)^s}{\Gamma(s)} \sum_{m=1}^{\infty} m^{s-1} e^{2\pi i m \tau},$$

and hence

$$\begin{aligned}\sum_{j \in \mathbf{N}, k \in \mathbf{Z}} \frac{1}{(j\tau + k)^s} &= \sum_{j \in \mathbf{N}} f(j\tau) \\ &= \frac{e^{-\pi i s/2}(2\pi)^s}{\Gamma(s)} \sum_{m=1}^{\infty} \sum_{j=1}^{\infty} m^{s-1} e^{2\pi i j m \tau} \\ &= \frac{e^{-\pi i s/2}(2\pi)^s}{\Gamma(s)} \sum_{n=1}^{\infty} \sigma_{s-1}(n) e^{2\pi i n \tau} \\ &= \frac{e^{-\pi i s/2}(2\pi)^s}{\Gamma(s)} \Phi_{s-1}(e^{2\pi i \tau})\end{aligned}$$

(for a positive integer s, the same expression of multiple Eisenstein series is given by [4, Theorem 3]). Then,

$$\begin{aligned}\tau^{-s} & \frac{e^{-\pi i s/2}(2\pi)^s}{\Gamma(s)} \Big(E_s(-1/\tau) - \frac{\Gamma(s)\zeta(s)}{(2\pi)^s} \cos\frac{\pi s}{2} \Big) \\ &= \sum_{j \in \mathbf{N}, k \in \mathbf{Z}} \frac{\tau^{-s}}{(-j/\tau + k)^s} = \sum_{j \in \mathbf{N}, k \in \mathbf{Z}} \frac{1}{(k\tau - j)^s} \\ &= \sum_{(j,k) \in \mathbf{N}^2} \Big(\frac{1}{(k\tau - j)^s} + \frac{1}{(-k\tau - j)^s} \Big) + \sum_{j \in \mathbf{N}} \frac{1}{(-j)^s} \\ &= \sum_{(j,k) \in \mathbf{N}^2} \Big(\frac{1}{(k\tau - j)^s} + \frac{e^{-\pi i s}}{(k\tau + j)^s} \Big) + \sum_{j \in \mathbf{N}} \frac{e^{-\pi i s}}{j^s} \\ &= \sum_{j \in \mathbf{Z}, k \in \mathbf{N}} \frac{1}{(k\tau + j)^s} + \sum_{(j,k) \in \mathbf{N}^2} \frac{e^{-\pi i s} - 1}{(k\tau + j)^s} \\ &\quad + (e^{-\pi i s} - \tau^{-s}) \zeta(s) \\ &= \frac{e^{-\pi i s/2}(2\pi)^s}{\Gamma(s)} \Big(E_s(\tau) - \frac{\Gamma(s)\zeta(s)}{(2\pi)^s} \cos\frac{\pi s}{2} \Big) \\ &\quad + \sum_{(j,k) \in \mathbf{N}^2} \frac{e^{-\pi i s} - 1}{(k\tau + j)^s} + (e^{-\pi i s} - \tau^{-s}) \zeta(s),\end{aligned}$$

from which the desired formula follows. □

4. Proof of Theorem 2.3. For each g/h, we have (cf. [7, p. 290])

$$\begin{aligned}\Lambda_{g/h}(s) &= h^s \sum_{(j,k) \in \mathbf{N}^2} \frac{1}{(gj + hk)^s} \\ &= h^s \sum_{a=1}^{h} \sum_{b=1}^{g} \sum_{j'=0}^{\infty} \sum_{k'=0}^{\infty} \frac{1}{(g(hj' + a) + h(gk' + b))^s} \\ &= h^s \sum_{a=1}^{h} \sum_{b=1}^{g} \sum_{\nu=0}^{\infty} \frac{\nu + 1}{(gh\nu + ag + bh)^s} \\ &= \frac{1}{g^s} \sum_{a=1}^{h} \sum_{b=1}^{g} \sum_{\nu=0}^{\infty} \frac{\nu + \lambda_{h,g}(a,b) + (1 - \lambda_{h,g}(a,b))}{(\nu + \lambda_{h,g}(a,b))^s}.\end{aligned}$$

Dividing the double summation $\sum_{a=1}^{h} \sum_{b=1}^{g}$ into two parts for $\lambda_{h,g}(a,b) \le 1$ and for $\lambda_{h,g}(a,b) > 1$, we obtain the desired equality. If $\mu = g$, then

$$\begin{aligned}\Lambda_g(s) &= \frac{1}{g^s} \sum_{b=1}^{g} \Big(\zeta(s-1, b/g) - \frac{b}{g} \zeta(s, b/g) \Big) \\ &= \frac{1}{g} \sum_{b=1}^{g} \sum_{\nu=0}^{\infty} \Big(\frac{1}{(g\nu + b)^{s-1}} - \frac{b}{(g\nu + b)^s} \Big) \\ &= \frac{1}{g} \sum_{l=1}^{\infty} \frac{1}{l^{s-1}} - \frac{1}{g} \sum_{b=1}^{g} \sum_{\nu=0}^{\infty} \frac{b}{(g\nu + b)^s} \\ &= \frac{1}{g} \zeta(s-1) - \frac{1}{2}(1 + g^{-s}) \zeta(s) - \frac{1}{g} \sum_{b=1}^{g-1} \sum_{\nu=0}^{\infty} \frac{b - g/2}{(g\nu + b)^s},\end{aligned}$$

which implies the expression as in the theorem. To verify the final assertion of this theorem, it is sufficient to consider $\sum_{\nu=0}^{\infty} ((g\nu + b)^{-s} - (g\nu + g - b)^{-s})$ such that g and b are relatively prime. Set

$$(\mathbf{Z}/g\mathbf{Z})^{\times} = \{\pm r_l \pmod g \,|\, 1 \le l \le \varphi(g)/2\},$$

where $r_1 = 1 < r_2 < \cdots < r_{\varphi(g)/2}$. For each l ($1 \le l \le \varphi(g)/2$), let ρ_l be the mapping $(\mathbf{Z}/g\mathbf{Z})^{\times} \to \{0, \pm 1\}$

such that $\rho_l(\pm r_l) = \pm 1$ and that $\rho_l(n) = 0$ otherwise. Then the summation mentioned above is written as $\sum_{n=1}^{\infty} \rho_b(n) n^{-s}$. Let $\chi_1, \ldots, \chi_{\varphi(g)/2}$ be the characters such that $\chi_l(\pm 1) = \pm 1$ $(1 \le l \le \varphi(g)/2)$. Then

$$(\chi_1, \ldots, \chi_{\varphi(g)/2}) = (\rho_1, \ldots, \rho_{\varphi(g)/2})U$$

with the square matrix U whose (α, β)-entry is $\chi_\beta(r_\alpha)$. Since $(1/\sqrt{\varphi(g)/2})U$ is unitary, ρ_b may be written as a linear combination of χ_l $(1 \le l \le \varphi(g)/2)$ with coefficients in $\mathbf{Q}(e^{2\pi i/\varphi(g)})$. This completes the proof. □

5. Proof of Theorem 2.4. To derive the estimate for $\Delta_s(\tau)$ it is sufficient to show

$$\sum_{(j,k) \in \mathbf{N}^2} (j\tau + k)^{-s} = O(\tau^{-1}) \tag{5.1}$$

as $\tau \to 0$ through the sector $\pi/2 - \theta_0 < \arg(\tau) \le \pi/2$. In this sector, $(j\tau + k)^{-s} = O((j\eta + k)^{-\mathrm{Re}(s)})$ uniformly for $(j, k) \in \mathbf{N}^2$, where $\eta = \mathrm{Im}(\tau)$. Since

$$\begin{aligned} &\sum_{j \in \mathbf{N}} (j\eta + k)^{-\mathrm{Re}(s)} \\ &\quad \le (\eta + k)^{-\mathrm{Re}(s)} + \int_1^\infty (\eta x + k)^{-\mathrm{Re}(s)} dx \\ &\quad = O(\eta^{-1}(\eta + k)^{-\mathrm{Re}(s)+1}), \end{aligned}$$

we have

$$\begin{aligned} \eta \sum_{(j,k) \in \mathbf{N}^2} (j\eta + k)^{-\mathrm{Re}(s)} &= O\Big((\eta + 1)^{-\mathrm{Re}(s)+1} \\ &\quad + \int_1^\infty (\eta + x)^{-\mathrm{Re}(s)+1} dx\Big) = O(1), \end{aligned}$$

which implies (5.1). The estimate for $\Delta_s^*(\tau)$ is verified by the same argument. □

6. Proofs of Theorems 2.5 and 2.6. Put $\tau = e^{\pi i/2}(2\pi)^{-1}\log(1/q)$. If $e^{-\pi i/2}\tau \to +0$, then $q \to 1 - 0$. Suppose that $\mathrm{Re}(s) > 2$. By Theorem 2.4,

$$\begin{aligned} \Phi_{s-1}(q) &= E_s(\tau) - \frac{\Gamma(s)\zeta(s)}{(2\pi)^s} \cos\frac{\pi s}{2} \\ &= \tau^{-s} E_s(-1/\tau) - \Delta_s(\tau) - \frac{\Gamma(s)\zeta(s)}{(2\pi)^s}\cos\frac{\pi s}{2} \\ &= \frac{\Gamma(s)\zeta(s)}{(2\pi)^s}\Big(\tau^{-s} e^{\pi i s/2} - e^{-\pi i s/2} \\ &\quad + O(\tau^{-s}\Phi_{s-1}(e^{-2\pi i/\tau}))\Big) + O(\tau^{-1}\sin(\pi s/2)) \end{aligned}$$

as $\tau \to 0$ through the sector $\pi/2 - \theta_0 < \arg(\tau) \le \pi/2$. Here $\Phi_{s-1}(e^{-2\pi i/\tau}) = O(e^{-c_0/|1-q|})$ for some $c_0 > 0$ in this sector. The estimate for $\Delta_s^*(\tau)$ yields the same expression in the sector $\pi/2 \le \arg(\tau) < \pi/2 + \theta_0$. Thus we obtain the asymptotic formula of Theorem 2.5 as $q \to 1$ through $|\arg(1-q)| < \theta_0$. The case $s = 2$ may be treated in a similar way by use of the relation $E_2(-1/\tau) = \tau^2 E_2(\tau) - \tau/(4\pi i)$.

Using Theorem 2.5 and (1.1), we derive the asymptotic formula for $\zeta_q(s)$ as in Theorem 2.6. □

7. Proof of Theorem 2.8. Let g and h be mutually prime integers satisfying $|g| \ge 1$ and $h \ge 2$. Write $\zeta_q(s)$ in the form

$$\zeta_q(s) = (1-q)^s \sum_{l=0}^{h-1} Z_l(q), \tag{7.1}$$

$$Z_l(q) := \sum_{n \equiv l \ (\mathrm{mod}\ h)} \frac{q^{n(s-1)}}{(1-q^n)^s} = \sum_{m=1}^{\infty} \frac{q^{(mh+l)(s-1)}}{(1-q^{mh+l})^s}.$$

Set $q = e^{2\pi i g/h} t$, $|t| < 1$. It is easy to see that

$$\begin{aligned} (1-q)^s Z_0(q) &= (1-q)^s \sum_{m=1}^{\infty} \frac{t^{mh(s-1)}}{(1-t^{mh})^s} \\ &= (1-q)^s (1-t^h)^{-s} \zeta_{t^h}(s). \end{aligned} \tag{7.2}$$

For each l satisfying $1 \le l \le h-1$, observing that

$$\begin{aligned} q^{mh+l} &= (e^{2\pi i g/h} t)^{mh+l} \\ &= e^{2\pi i g l/h} t^{mh+l} = e^{2\pi i g_0(l)/h_0(l)} t^{mh+l}, \end{aligned}$$

we have $|1 - q^{mh+l}| \ge \sin(\pi/h_0(l)) \ge \sin(\pi/h)$ near $t = 1$, where $h_0(l)$ and $g_0(l)$ are relatively prime integers such that $2 \le h_0(l) \le h$. Hence, for $1 \le l \le h-1$, we have

$$\begin{aligned} Z_l(q) &\ll \sin^{-s}(\pi/h) \sum_{m=1}^{\infty} t^{(s-1)(mh+l)} \\ &\ll h^s (1 - t^{(s-1)h})^{-1} \ll h^s (1-t^h)^{-1} \end{aligned} \tag{7.3}$$

as $t \to 1$ through the sector $|\arg(1-t)| < \theta_0$. Here $\phi(t) \ll \psi(t)$ means $\phi(t) = O(\psi(t))$, whose related constant is independent of h and l. Substituting (7.3) and (7.2) into (7.1), we obtain Theorem 2.8. □

8. Proof of Theorem 2.7. For each integer $s \ge 2$, relation (1.1) implies

$$D_s(q)\mathbf{z}_s(q) = K_s \mathbf{x}_s(q).$$

Here $\mathbf{z}_s(q)$ and $\mathbf{x}_s(q)$ are column vectors of the form

$$\mathbf{z}_s(q) := {}^T(\zeta_q(2), \ldots, \zeta_q(s)),$$
$$\mathbf{x}_s(q) := {}^T(\Phi_1(q), \ldots, \Phi_{s-1}(q)),$$

$D_s(q)$ is a diagonal matrix of the form

$$D_s(q) := \mathrm{diag}[1!(1-q)^{-2}, \ldots, (s-1)!(1-q)^{-s}],$$

and K_s is a lower triangular matrix whose (α, β)-entry $(1 \le \beta \le \alpha \le s-1)$ is $\kappa_\beta^{\alpha+1}$. Suppose that $qe^{-2\pi i g/h} = t \to 1$ through the sector $|\arg(1-t)| < \theta_0$. By Theorem 2.8, for $2 \le \nu \le s$, we have $\zeta_q(\nu) = (1-q)^\nu (1-t^h)^{-\nu} \zeta_{t^h}(\nu) + O(h^{\nu+1}(1-t^h)^{-1})$. Hence

$$\begin{aligned}
\mathbf{x}_s(q) &= K_s^{-1} D_s(q) \mathbf{z}_s(q) \\
&= K_s^{-1} D_s(q) \Big(D_s(q)^{-1} D_s(t^h) \mathbf{z}_s(t^h) \\
&\quad + O(h^{s+1}(1-t^h)^{-1}) \Big) \\
&= K_s^{-1} D_s(t^h) \mathbf{z}_s(t^h) + O(h^{s+1}(1-t^h)^{-1}) \\
&= \mathbf{x}_s(t^h) + O(h^{s+1}(1-t^h)^{-1}),
\end{aligned}$$

which implies Theorem 2.7. □

Appendix. Remark on $\mathbf{\Phi_0(q)}$. Let $q = e^{2\pi i g/h} t$ be as in Theorem 2.7. Then

$$\Phi_0(q) = -\frac{\log(1-q) + O(1)}{1-q}$$

as $q \to 1$ through the sector $|\arg(1-q)| < \theta_0$, and

$$\Phi_0(q) = \Phi_0(t^h) + O(h^2(1-t^h)^{-1})$$

as $t \to 1$ through the sector $|\arg(1-t)| < \theta_0$.

To derive these, we note that

$$\begin{aligned}
(1-q)\sum_{n=1}^{\infty} (\log n) q^n &= \sum_{n=1}^{\infty} (\log(n+1) - \log n) q^{n+1} \\
&= q \sum_{n=1}^{\infty} (n^{-1} + O(n^{-2})) q^n = -\log(1-q) + O(1)
\end{aligned}$$

as $q \to 1$, $|q| < 1$, which implies

$$\sum_{n=1}^{\infty} (\log n) q^n = -\frac{\log(1-q)}{1-q} + \rho(q)$$

with $\rho(q) = O((1-q)^{-1})$. Observing that

$$\rho'(q) = \frac{1}{2\pi i} \int_{\gamma_q} \frac{\rho(z)}{(z-q)^2} dz = O((1-q)^{-2})$$

with $\gamma_q :\ |z-q| = (1/2)|1-q|\cos\theta_0$ in the sector $|\arg(1-q)| < \theta_0$, we have

$$\sum_{n=1}^{\infty} (n \log n) q^n = -\frac{\log(1-q) + O(1)}{(1-q)^2}$$

as $q \to 1$ through this sector. Hence

$$\begin{aligned}
\frac{\Phi_0(q)}{1-q} &= \sum_{n=1}^{\infty} \frac{d(n) q^n}{1-q} = \sum_{n=1}^{\infty} \sum_{\nu=1}^{n} d(\nu) q^n \\
&= \sum_{n=1}^{\infty} (n \log n + O(n)) q^n = -\frac{\log(1-q) + O(1)}{(1-q)^2}
\end{aligned}$$

as $q \to 1$ through $|\arg(1-q)| < \theta_0$. The second expression is obtained by the same argument as in the proof of Theorem 2.8.

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