On meromorphic functions sharing five one-point or two-point sets IM

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Abstract: We show that if two meromorphic functions sharing five one-point or two-point sets two points IM, then one of them is a Möbius transformation of the other.

Key words: Uniqueness theorem; sharing sets; Nevanlinna theory.

1. Introduction. For nonconstant meromorphic functions f and g on $\boldsymbol{C}$ and a finite set S in $\overline{\boldsymbol{C}} = \boldsymbol{C} \cup \{\infty\}$, we say that f and g share S CM (counting multiplicities) if $f^{-1}(S) = g^{-1}(S)$ and if for each $z_0 \in f^{-1}(S)$ two functions $f - f(z_0)$ and $g - g(z_0)$ have the same multiplicity of zero at z_0, where the notations $f - \infty$ and $g - \infty$ mean $1/f$ and $1/g$, respectively. Also, if $f^{-1}(S) = g^{-1}(S)$, then we say that f and g share S IM (ignoring multiplicities). In particular if S is a one-point set $\{a\}$, then we say also that f and g share a CM or IM.

In [N1] and [N2], R. Nevanlinna showed the following two theorems:

Theorem A1. *Let f and g be two distinct nonconstant meromorphic functions on $\boldsymbol{C}$ and $a_1, \cdots, a_4$ four distinct points in $\overline{\boldsymbol{C}}$. If f and g share $a_1, \cdots, a_4$ CM, then f is a Möbius transformation of g, i.e. $f = (ag + b)/(cg + d)$ for some complex numbers a, b, c, d with $ad - bc \neq 0$, and there exists a permutation σ of $\{1, 2, 3, 4\}$ such that $a_{\sigma(3)}, a_{\sigma(4)}$ are Picard exceptional values of f and g and the cross ratio $(a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}, a_{\sigma(4)}) = -1$.*

Theorem A2. *Let f and g be two nonconstant meromorphic functions on $\boldsymbol{C}$ sharing distinct five points in $\overline{\boldsymbol{C}}$ IM, then $f = g$.*

In [T] Tohge considered two meromorphic functions sharing $1, -1, \infty$ and a two-point set containing none of them and Theorem 4 in [T] induces the following

Theorem B. *Let S_1, S_2, S_3 be one-point sets in $\overline{\boldsymbol{C}}$ and let S_4 be a two-point set in $\overline{\boldsymbol{C}}$. Assume that S_1, S_2, S_3, S_4 are pairwise disjoint. If two nonconstant meromorphic functions f and g on $\boldsymbol{C}$ share S_1, S_2, S_3, S_4 CM, then f is a Möbius transformation of g.*

Also, Theorem 1.2 in [ST] and its proof induce

Theorem C. *Let S_1, S_2 be one-point sets in $\overline{\boldsymbol{C}}$ and let S_3, S_4 be a two-point set in $\overline{\boldsymbol{C}}$. Assume that S_1, S_2, S_3, S_4 are pairwise disjoint. If two nonconstant meromorphic functions f and g on $\boldsymbol{C}$ share S_1, S_2, S_3, S_4 CM, then f is a Möbius transformation of g.*

Moreover, in [S] the author considered meromorphic functions sharing two-point sets CM and Theorem 1.1 in [S] and its proof induce

Theorem D. *Let $S_1, \cdots, S_6$ be pairwise disjoint two-point sets in $\overline{\boldsymbol{C}}$. If two nonconstant meromorphic functions f and g on $\boldsymbol{C}$ share $S_1, \cdots, S_6$ CM, then f is a Möbius transformation of g.*

In this paper we consider two meromorphic functions on $\boldsymbol{C}$ sharing five one-point or two-point sets in $\overline{\boldsymbol{C}}$ IM.

Theorem 1. *Let $S_1, \cdots, S_5$ be pairwise disjoint one-point or two-point sets in $\overline{\boldsymbol{C}}$. If two nonconstant meromorphic functions f and g on $\boldsymbol{C}$ share $S_1, \cdots, S_5$ IM, then f is a Möbius transformation of g and hence f and g share each S_j CM.*

This result is much better than that of Theorem D, whereas the proof of the formar is much easier than that of the latter.

The following corollary is induced from the result of Theorem 1 and the little Picard Theorem.

Corollary 2. *Let $S_1, \cdots, S_5$ be pairwise disjoint one-point or two-point sets in $\overline{\boldsymbol{C}}$. Assume that there is no Möbius transformation T except the indentity with at most two points z in $\boldsymbol{C}$ satisifying one of the following conditions: (i) $z \in S_j$ and $T(z) \notin S_j$ for some $j = 1, \cdots, 5$; (ii) $z \notin \cup_{j=1}^5 S_j$ and $T(z) \in \cup_{j=1}^5 S_j$. Then two nonconstant meromorphic functions on $\boldsymbol{C}$ sharing $S_1, \cdots, S_5$ IM are identical.*

How about the case of sharing CM? We give a conjecure.

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Conjecture. Let $S_1, \cdots, S_4$ be pairwise disjoint one-point or two-point sets in $\overline{\boldsymbol{C}}$. If two nonconstant meromorphic functions f and g share $S_1, \cdots, S_4$ CM, then there exists a Möbius transformation T such that $f = T \circ g$.

This conjecture is true for the cases that the number of one-point sets is four, three or two, and so the remaining problem is the case that the number of one-point sets is one or zero.

We assume that the reader is familiar with the standard notations and results of the value distribution theory (see, for example, [H]). In particular, we express by $S(r, f)$ quantities such that $\lim_{r\to\infty, r\notin E} S(r, f)/T(r, f) = 0$, where E is a subset of $(0, \infty)$ with finite linear measure and it is variable in each cases.

2. Proof of Theorem 1. Before beginning the proof of Theorem 1, we show the following

Lemma 3. *For any distinct four point* $\xi_1, \eta_1, \xi_2, \eta_2$ *in* $\overline{\boldsymbol{C}}$*, there exists a Möbius transformation* T *such that* $T(\eta_j) = -T(\xi_j)$ *for* $j = 1, 2$.

Proof. We may assume that the four points are $\xi_1 = 0, \xi_2 = 1, \eta_1 = \infty$ and $\eta_2 = a$, where $a \neq 0$, 1, ∞. One of the Möbius transformation desired is given by

$$T(z) = \frac{z - \sqrt{a}}{z + \sqrt{a}}.$$

Indeed, $T(\infty) = 1 = -T(0)$, $T(a) = \dfrac{a - \sqrt{a}}{a + \sqrt{a}} = \dfrac{\sqrt{a} - 1}{\sqrt{a} + 1} = -T(1)$. □

Now we start the proof of Theorem 1. We assume, more generally, that f and g share one-point sets $S_1, \cdots, S_p$ and two-point sets $S_{p+1}, \cdots, S_{p+q}$ IM, where these sets are pairwise disjoint and p and q are non-negative integers with $p + q \geq 5$. However, we may assume that $p \leq 4$ by Theorem A2. Also, if $f = g$, then there is nothing to prove. Therefore we assume that $f \not\equiv g$.

Let T be a Möbius transfomation. Then $T \circ f$ and $T \circ g$ share $T(S_j)$ IM, and if $T \circ f$ is a Möbius transformation of $T \circ g$, then f is a Möbius transformation of g. Therefore we may assume that any set S_j does not contain ∞.

By the second main theorem and the first main theorem we have

$$(1) \quad \begin{aligned}(p + 2q - 2)T(r, f) &\leq \sum_{j=1}^{p+q} \sum_{\xi\in S_j} \overline{N}\left(r, \frac{1}{f - \xi}\right) + S(r, f) \\ &= \sum_{j=1}^{p+q} \sum_{\xi\in S_j} \overline{N}\left(r, \frac{1}{g - \xi}\right) + S(r, f) \\ &\leq (p + 2q)T(r, g) + S(r, f)\end{aligned}$$

and, by the same way,

$$(2) \quad (p + 2q - 2)T(r, g) \leq (p + 2q)T(r, f) + S(r, g).$$

Hence there is no need to distinguish $S(r, f)$ and $S(r, g)$, and so we denote them by $S(r)$.

By $\overline{N}_E\left(r, \dfrac{1}{f - \xi}\right)$ and $\overline{N}_N\left(r, \dfrac{1}{f - \xi}\right)$ we denote the counting functions which count the point z such that $f(z) = \xi = g(z)$ and $f(z) = \xi \neq g(z)$ counted once, respectively, and we define $\overline{N}_E\left(r, \dfrac{1}{g - \xi}\right)$ and $\overline{N}_N\left(r, \dfrac{1}{g - \xi}\right)$ by the same way. It is easy to see that $\overline{N}_N\left(r, \dfrac{1}{f - \xi}\right) = \overline{N}_N\left(r, \dfrac{1}{g - \xi}\right) = 0$ for $\xi \in S_1 \cup \cdots \cup S_p$ and that

$$(3) \quad \begin{aligned}&\sum_{\xi\in S_j} \overline{N}_E\left(r, \frac{1}{f - \xi}\right) = \sum_{\xi\in S_j} \overline{N}_E\left(r, \frac{1}{g - \xi}\right), \\ &\sum_{\xi\in S_j} \overline{N}_N\left(r, \frac{1}{f - \xi}\right) = \sum_{\xi\in S_j} \overline{N}_N\left(r, \frac{1}{g - \xi}\right)\end{aligned}$$

for $j = p + 1, \cdots, q$. Since $f - g \not\equiv 0$, we have

$$\begin{aligned}\sum_{j=1}^{p+q} \sum_{\xi\in S_j} \overline{N}_E\left(r, \frac{1}{f - \xi}\right) &\leq \overline{N}\left(r, \frac{1}{f - g}\right) \\ &\leq T(r, f) + T(r, g) + O(1)\end{aligned}$$

and

$$\begin{aligned}&\sum_{j=p+1}^{p+q} \sum_{\xi\in S_j} \overline{N}_N\left(r, \frac{1}{f - \xi}\right) \\ &\quad = \sum_{j=1}^{p+q} \sum_{\xi\in S_j} \overline{N}\left(r, \frac{1}{f - \xi}\right) - \sum_{j=1}^{p+q} \sum_{\xi\in S_j} \overline{N}_E\left(r, \frac{1}{f - \xi}\right) \\ &\quad \geq (p + 2q - 3)T(r, f) - T(r, g) + S(r).\end{aligned}$$

By the same way and (3) we have

$$\sum_{j=p+1}^{q}\sum_{\xi\in S_j}\overline{N}_N\left(r,\frac{1}{f-\xi}\right)$$
$$\geq (p+2q-3)T(r,g)-T(r,f)+S(r).$$

Adding these two inequalities we obtain

$$\sum_{j=p+1}^{p+q}\sum_{\xi\in S_j}\overline{N}_N\left(r,\frac{1}{f-\xi}\right) \geq \frac{1}{2}(p+2q-4)(T(r,f)+T(r,g))+S(r). \tag{4}$$

(i) The case $q\geq 2$.

From (4) we see that there exist distinct j_1 and j_2 in $\{p+1,\cdots,q\}$ and a subset I of $(0,+\infty)$ of infinite linear measure such that

$$\frac{1}{q}(p+2q-4)(T(r,f)+T(r,g))+S(r) \leq \sum_{\xi\in S_{j_1}\cup S_{j_2}}\overline{N}_N\left(r,\frac{1}{f-\xi}\right) \tag{5}$$

holds for $r\in I$. Put $S_{j_1}=\{\xi_1,\eta_1\}$, $S_{j_2}=\{\xi_2,\eta_2\}$. Then by Lemma 3 there exists a Möbius transformation T such that $T(\eta_j)=-T(\xi_j)$ for $j=1,2$, and we put $F=T\circ f$, $G=T\circ g$. Of couse $F\neq G$ by assumption, and assume $F\neq -G$. Then since the points counted in $\overline{N}_N\left(\dfrac{1}{f-\xi}\right)$ for some $\xi\in S_{j_1}\cup S_{j_2}$ are zeros of $F+G$,

$$\sum_{\xi\in S_j\cup S_{j_2}}\overline{N}_N\left(r,\frac{1}{f-\xi}\right)\leq \overline{N}\left(r,\frac{1}{F+G}\right)$$
$$\leq T(r,f)+T(r,g)+O(1)$$

holds for $r\in I$. By connecting this lefthand side with the righthand side of (5), we get $p+q\leq 4$, which contradicts the hypothesis. Therefore we conclude that $F=-G$, which induces that f is a Möbius transformation of g.

(ii) The case $q=1$.

In this case we have $p=4$. Put $S_j=\{a_j\}$ for $j=1,\cdots,4$, then by the second main theorem and the first main theorem we get

$$2T(r,f)\leq\sum_{j=1}^{4}\overline{N}\left(r,\frac{1}{f-a_j}\right)+S(r)$$
$$\leq \overline{N}\left(r,\frac{1}{f-g}\right)+S(r)$$
$$\leq T(r,f)+T(r,g)+S(r)$$

and

$$2T(r,g)\leq\sum_{j=1}^{4}\overline{N}\left(r,\frac{1}{g-a_j}\right)+S(r)$$
$$\leq \overline{N}\left(r,\frac{1}{f-g}\right)+S(r)$$
$$\leq T(r,f)+T(r,g)+S(r).$$

Hence we obtain

$$T(r,f)=T(r,g)+S(r) \tag{6}$$

and

$$\sum_{j=1}^{4}\overline{N}\left(r,\frac{1}{f-a_j}\right)=2T(r,f)+S(r) = \overline{N}\left(r,\frac{1}{f-g}\right)+S(r). \tag{7}$$

Now put $S_5=\{a_5,b_5\}$, then by the second main theorem and (7) we have

$$4T(r,f)\leq\sum_{j=1}^{5}\overline{N}\left(r,\frac{1}{f-a_j}\right)+\overline{N}\left(r,\frac{1}{f-b_5}\right)+S(r)$$
$$=2T(r,f)+\overline{N}\left(r,\frac{1}{f-a_5}\right)+\overline{N}\left(r,\frac{1}{f-b_5}\right)$$
$$+S(r),$$

and hence

$$2T(r,f)\leq\overline{N}\left(r,\frac{1}{f-a_5}\right)+\overline{N}\left(r,\frac{1}{f-b_5}\right)+S(r). \tag{8}$$

Since

$$\overline{N}_E\left(r,\frac{1}{f-a_5}\right)+\overline{N}_E\left(r,\frac{1}{f-b_5}\right)$$
$$\leq\overline{N}\left(r,\frac{1}{f-g}\right)-\sum_{j=1}^{4}\overline{N}\left(r,\frac{1}{f-a_j}\right)$$
$$\leq T(r,f)+T(r,g)-2T(r,g)+S(r)=S(r)$$

holds by using (6) and (7), we have

$$\overline{N}_E\left(r,\frac{1}{f-a_5}\right)+\overline{N}_E\left(r,\frac{1}{f-b_5}\right)\leq S(r).$$

This and (8) yield

$$\overline{N}_N\left(r,\frac{1}{f-a_5}\right)+\overline{N}_N\left(r,\frac{1}{f-b_5}\right)\geq 2T(r,f)+S(r). \tag{9}$$

On the other hand it follows from (7) that there exists some j_0 in $\{1,\cdots,4\}$ such that

$$(10)\qquad \frac{1}{2}T(r,f) \le \overline{N}\left(r, \frac{1}{f-a_{j_0}}\right) + S(r)$$

holds for $r \in I$, where I is a subset of $(0,+\infty)$ of infinite linear measure. We take a Möbius transformation T such that $T(b_5) = -T(a_5)$ and $T(a_{j_0}) = 0$ and put $F = T \circ f$ and $G = T \circ g$. Assume that $F \neq -G$. Then since the points z such that $f(z) = a_{j_0}$ or that $f(z)$ and $g(z)$ are distinct points in S_5 are zeros of $F+G$, by (9), (10) and (6)

$$\begin{aligned}
&\frac{1}{2}T(r,f) + 2T(r,f) \\
&\quad \le \overline{N}\left(r, \frac{1}{f-a_{j_0}}\right) + \overline{N}_N\left(r, \frac{1}{f-a_5}\right) \\
&\qquad + \overline{N}_N\left(r, \frac{1}{f-b_5}\right) + S(r) \\
&\quad \le \overline{N}\left(r, \frac{1}{F+G}\right) + S(r) \le T(r,f) + T(r,g) + S(r) \\
&\quad = 2T(f,r) + S(r)
\end{aligned}$$

holds for $r \in I$, which is a contradiction. Hence we conclude that $F = -G$, which induces that f is a Möbius transformation of g.

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