Branching laws for square integrable representations

By Michel DUFLO*) and Jorge Antonio VARGAS**)

(Communicated by Masaki KASHIWARA, M.J.A., Feb. 12, 2010)

Abstract: In this note, we study square integrable representations of a real reductive Lie group with admissible restriction to some reductive subgroup. We give a simple condition which insures admissibility of the restriction, and which allows to compute the branching numbers in a simple explicit manner by means of partition functions, generalizing the multiplicity formulas due to Kostant-Heckman and Hecht-Schmid. We consider also the semi-classical analogue of these results for coadjoint orbits.

Key words: Discrete series; square integrable representations; branching laws; multiplicity formulas.

1. Introduction. Let G be a connected reductive Lie group, and H a closed connected reductive subgroup of G. Recall that a square integrable representation (also called a discrete series) of G is an irreducible unitary representation of G which can be realized as a subspace of the left regular representation in $L^2(G)$. We consider a square integrable representation π of G which has an admissible restriction to H. This means that we may express the restriction $\pi_{|_H}$ as a Hilbert sum

$$\pi_{|_H} = \sum_{i \in \mathbf{N}} n_i \sigma_i,$$

(here, the σ_i are classes of irreducible unitary representations of H) with $0 \leq n_i < \infty$. The σ_i for which $n_i > 0$ are known to be square integrable [11], 8.7. In this note we present some results on the multiplicity (also called branching number)

$$m(\pi, \sigma_i) := n_i$$

of a square integrable representation σ_i of H in π. Proofs will appear elsewhere.

A theorem of Harish-Chandra insures that a connected reductive Lie group admits square integrable representations if and only if it has a compact Cartan subgroup. Henceforth, we assume that G and H have a compact Cartan subgroup. We fix a compact Cartan subgroup T of G, and a maximal compact subgroup K of G containing T so that:

$L := H \cap K$ is a maximal compact subgroup of H, $U := H \cap T$ is a compact Cartan subgroup of H. Let W_K denote the Weyl group of K. We use a similar notation for other compact connected groups. We denote by V^* the dual of a vector space V. We denote by $\mathfrak{g}_{\mathbf{R}}$ the Lie algebra of G and $\mathfrak{g}$ the complexified Lie algebra. We use the same system of notations for any Lie group. We denote by $\Phi(\mathfrak{g}, \mathfrak{t}) \subset i\mathfrak{t}^*_{\mathbf{R}} \subset \mathfrak{t}^*$ the set of roots. If $\Psi \subset \Phi(\mathfrak{g}, \mathfrak{t})$ is a positive system of roots, we consider the corresponding sets of positive compact roots $\Psi_c := \Psi \cap \Phi(\mathfrak{k}, \mathfrak{t})$, and of noncompact roots $\Psi_n := \Psi \backslash \Psi_c$. We denote as usual by ρ, ρ_c and ρ_n the corresponding half-sums.

Harish-Chandra parameterized the set of equivalence classes of square integrable representations by means of the set of $\mathfrak{g}$-regular $\lambda \in i\mathfrak{t}^*_{\mathbf{R}}$ such that $e^{\lambda+\rho}$ is a character of T. We call such a λ a G-Harish-Chandra parameter, and denote by π_λ the corresponding equivalent class of square integrable representation. Moreover, $\pi_\lambda = \pi_{\lambda'}$ if and only if $W_K\lambda' = W_K\lambda$.

In the special case of K, we denote by τ_ξ the irreducible representation of K associated with the K-Harish-Chandra parameter $\xi \in i\mathfrak{t}^*_{\mathbf{R}}$. It is the irreducible representation of K with infinitesimal character defined by ξ.

Similarly, we denote by σ_μ the square integrable representation of H associated with a H-Harish-Chandra parameter $\mu \in i\mathfrak{u}^*_{\mathbf{R}}$, and by κ_ν the irreducible representation of L associated with the L-Harish-Chandra parameter $\nu \in i\mathfrak{u}^*_{\mathbf{R}}$.

Let λ be a G-Harish-Chandra parameter. This defines a positive system of roots

$$\Psi^\lambda = \{\alpha \in \Phi(\mathfrak{g}, \mathfrak{t}) : \lambda(h_\alpha) > 0\},$$

2000 Mathematics Subject Classification. Primary 22E46; Secondary 17B10, 05E15.

*) Université Paris Diderot-Paris 7, UFR de Mathématiques, Case 7012, 75205 PARIS CEDEX 13, France.

**) FAMAF-CIEM, Universidad Nacional de Córdoba, 5000 Córdoba, Argentina.

doi: 10.3792/pjaa.86.49

where $h_\alpha \in \mathfrak{t}$ is the coroot. We recall the formula of Blattner (proven by Hecht-Schmid [4]) for the multiplicity $m(\pi_\lambda, \tau_\xi)$ of τ_ξ in π_λ. We denote by $p_{\Psi_n^\lambda}$ the partition function associated with Ψ_n^λ. Assuming that ξ is in the positive Weyl Chamber for Ψ_c^λ, we have:

$$m(\pi_\lambda, \tau_\xi) = \sum_{w \in W_K} \epsilon(w)\, p_{\Psi_n^\lambda}(w\xi - \lambda - \rho_n).$$

There are similar formulas in terms of partition functions, due to Heckman and Kostant [5], for the restriction of an irreducible representation of K to L.

We prove that for a square integrable representation π_λ, its restriction $\pi_{\lambda_{|_H}}$ to H is admissible if and only if its restriction $\pi_{\lambda_{|_L}}$ to L is admissible. This allows a strategy to compute multiplicities in $\pi_{\lambda_{|_H}}$ in case of admissibility. First, we compute multiplicities in $\pi_{\lambda_{|_L}}$, combining the formulas of Blattner and Heckman-Kostant, and deduce from it the multiplicities in $\pi_{\lambda_{|_H}}$. We explain the latter step in theorem 5. However, we obtain nice formulas for the multiplicities in $\pi_{\lambda_{|_L}}$ only with a supplementary assumption which we call *condition* (C).

In section 2 we introduce this *condition* (C) for Ψ and L. Under this condition, we prove that $\pi_{\lambda_{|_L}}$ is admissible. In sections 3 and 4 we give formulas for the multiplicities in $\pi_{\lambda_{|_L}}$ and $\pi_{\lambda_{|_H}}$, in term of a linear combination of partition functions. *Condition* (C) is only a sufficient condition of admissibility, but it allows a tremendous simplification in the computation of multiplicities. However, for (G, H) a symmetric pair we obtain the equivalence: $\pi_{\lambda_{|_H}}$ is admissible if and only if condition (C) holds for Ψ and L. An interesting fact is the role of a particular invariant connected subgroup $Z_1K_1 \subset K$ associated to Ψ. We compare our results with previous works, notably by Gross-Wallach [3] on small representations, and Kobayashi [9] on symmetric spaces.

In section 5, we give semi-classical analogues of these results. We consider the coadjoint orbit $\Omega := G\lambda \subset i\mathfrak{g}_{\mathbf{R}}^*$. The admissibility of $\pi_{\lambda_{|_H}}$ is equivalent to the fact that the restriction $q_{\mathfrak{h}}$ from Ω to $i\mathfrak{h}_{\mathbf{R}}^*$ is proper, so we may consider the measure on $i\mathfrak{h}_{\mathbf{R}}^*$ which is the push-forward of the Liouville measure β_Ω. Under *condition* (C), we give a formula for this measure in terms of convolution product of Heaviside functions, generalizing formulas given in [5] for the pair (K, L) and in [2] for the pair (G, K).

We say that two square integrable representations π_λ and $\pi_{\lambda'}$ of G are in the same family if Ψ^λ and $\Psi^{\lambda'}$ are W_K-conjugate. There is a similar definition for the square integrable representations of H. We conjecture that, for a square integrable representation π of G such that $\pi_{|_H}$ is admissible, all the representations σ_i which occur in $\pi_{|_H}$ belong to a unique family of square integrable representations of H. We prove that this is true under *condition* (C): this follows from our formulas for multiplicities and for Liouville measures, a convexity result of Weinstein [17], and deep results on partition functions, in particular those of Szenes and Vergne [15].

The problem of finding branching numbers for discrete series, or more generally for $A_{\mathfrak{q}}(\lambda)$ modules, has received much attention. Let us mention Schmid [14], Kobayashi [6, 8], Gross-Wallach [3]. For an update on the subject we refer to Kobayashi [12, 13] and references therein. There exists different types of formulas for branching numbers. In this note we consider only formulas in terms of partition functions. Moreover, we do not expect that our results extend to more singular $A_{\mathfrak{q}}(\lambda)$ modules. For instance, in this larger context, the conjecture on the unique family stated in the previous paragraph is not always true: see examples in [6].

2. Condition (C). We shall use multisets S of vectors of $i\mathfrak{u}_{\mathbf{R}}^*$, that is non ordered lists of vectors, allowing repetitions. We say that a multiset S is strict if it is contained in an open half-subspace of $i\mathfrak{u}_{\mathbf{R}}^*$; in particular all elements of S are non zero. A submultiset $S' \subset S$ is called a positive system for S if S' is the intersection of S with an open half-subspace.

Let $q_{\mathfrak{u}} : \mathfrak{t}^* \to \mathfrak{u}^*$ be the restriction map. Let Z be the connected component of the centralizer of $\mathfrak{u}$ in K. We define $\Phi_z := \Phi(\mathfrak{z}, \mathfrak{t})$. It is the set of $\alpha \in \Phi(\mathfrak{k}, \mathfrak{t})$ such that $q_{\mathfrak{u}}(\alpha) = 0$.

For any space V in which $\mathfrak{u}$ acts, we denote by $\Phi(V, \mathfrak{u}) \subset \mathfrak{u}^*$ the multiset of non zero roots. For instance, we have $\Phi(\mathfrak{k}/\mathfrak{l}, \mathfrak{u}) = q_{\mathfrak{u}}(\Phi(\mathfrak{k}, \mathfrak{t}) \backslash \Phi_{\mathfrak{z}}) \backslash \Phi(\mathfrak{l}, \mathfrak{u})$.

Definition 1. *Consider a positive system* $\Psi \subset \Phi(\mathfrak{g}, \mathfrak{t})$. *Condition* (C) *holds for* Ψ *and* L *if there exists a positive system* $\Delta(\mathfrak{k}/\mathfrak{l}, \mathfrak{u}) \subset \Phi(\mathfrak{k}/\mathfrak{l}, \mathfrak{u})$ *such that, for each* $w \in W_K$, *the multiset*

$$q_{\mathfrak{u}}(w\Psi_n) \cup \Delta(\mathfrak{k}/\mathfrak{l}, \mathfrak{u})$$

is strict.

Theorem 1. *Let* λ *be a* G*-Harish-Chandra parameter. If condition* (C) *holds for* Ψ^λ *and* L, *then* π_λ *restricted to* H, *as well to* L, *is admissible.*

Remark 1. *Condition* (C) *implies in par-*

ticular that $q_{\mathfrak{u}}(\alpha) \neq 0$ for all non compact roots of $\Phi(\mathfrak{g}, \mathfrak{t})$. This is in fact always satisfied when π_λ restricted to L is admissible. The rest of condition (C) is only a sufficient condition of admissibility—see the examples given below after the definition of $K_2(\Psi)$. However it is a natural one to consider for the problem of expressing branching numbers in term of partition functions.

Hereinafter, for simplicity, we assume that G is a simple group. We consider some examples. We denote by Z_K the connected component of the center of K, and $\mathfrak{z}_K$ its Lie algebra. Recall that $\dim \mathfrak{z}_K \leq 1$, and that if $\dim \mathfrak{z}_K = 1$—this is the Hermitian symmetric case—, there are (up to W_K-conjugacy) exactly two positive systems $\Psi \subset \Phi(\mathfrak{g}, \mathfrak{t})$ such that Ψ_n is W_K invariant. They are called holomorphic.

- We suppose that $\dim \mathfrak{z}_K = 1$.

Let U be a subtorus of T and Ψ^λ a holomorphic system. Then, condition (C) holds for Ψ^λ and U if and only if π_λ restricted to U is admissible. Moreover, condition (C) holds if U is sufficiently close to Z_K (see [16]).

For an arbitrary system Ψ condition (C) holds for Ψ and Z_K if and only if Ψ is a holomorphic system.

- Let K_s denote the semisimple factor of K. Then, condition (C) holds for Ψ and K_s if and only if $\mathbf{R}^+\Psi_n \cap i\mathfrak{z}_K^* = \{0\}$.
- The fact that condition (C) holds for Ψ and L implies that condition (C) holds for Ψ and any subgroup between L and K.
- Let G be a noncompact real form for G_2, and $\Psi_c = \{\alpha_1, \alpha_2\}$, with α_1 short.

Let $L = SU_2(\alpha_2)$, $\Psi \supset \Psi_c$ be the positive system of roots for which α_1 is simple. Then condition (C) holds and Ψ and L. This is the quaternionic case, and in particular Ψ is small in the sense of [3].

Let $L = SU_2(\alpha_1)$ and $\Psi \supset \Psi_c$ be the positive system of roots for which α_2 is simple. Then condition (C) holds for Ψ and L. In this case Ψ is not small in the sense of [3].

In the next paragraph we analyze the constraints on K when we assume that condition (C) holds for certain subgroup. To Ψ we associate two ideals of $\mathfrak{k}$. We define

$$\mathfrak{k}_1(\Psi)_{\mathbf{C}} := \text{ideal spanned by } \sum\nolimits_{\alpha,\beta\in\Psi_n} \mathbf{C}X_{\alpha+\beta}.$$

Still assuming for simplicity that $\mathfrak{g}$ is simple, we define $\mathfrak{z}_1(\Psi) = \mathfrak{z}_K$ if $\mathbf{R}^+\Psi_n \cap i\mathfrak{z}_K^* \neq \{0\}$, and $\mathfrak{z}_1 = \{0\}$ otherwise. We denote by $Z_1(\Psi) \subset Z_K$ and $K_1(\Psi) \subset K_s$ the corresponding invariant connected subgroups of K.

Proposition 1. (i) *If condition (C) holds for Ψ and L, then we have $K_1(\Psi) \subseteq L$. If we assume moreover that $\mathfrak{l} = \mathfrak{l} \cap \mathfrak{z} \oplus \mathfrak{l} \cap \mathfrak{k}_s$, then we have $Z_1(\Psi)K_1(\Psi) \subseteq L$.*

(ii) *When $K_1(\Psi)\,Z_1(\Psi)$ is included in L, then condition (C) holds for Ψ and L.*

When $Z_1(\Psi) \neq 1$, condition (C) alone does not imply that L contains $Z_1(\Psi)$. This can been seen on the holomorphic example with $L = U$ considered above.

When Ψ is small in the sense of [3], Gross and Wallach define a specific invariant proper subgroup of K, to which the corresponding discrete series of G restrict in an admissible manner. In this case, we check that the group $K_1(\Psi)\,Z_1(\Psi)$ coincides with the group of Gross and Wallach. However, we computed all the positive systems Ψ for which the group $K_1(\Psi)$ is a proper subgroup of $Z_K K_s$, and (for classical $\mathfrak{g}$) those for which $Z_1(\Psi)$ is a proper subgroup of Z_K. Many of them are not small. We gave above an example for $G = G_2$.

Let $K_2(\Psi) \subset K_s$ denote the invariant connected subgroup complementary to $K_1(\Psi)$. We show that $\mathfrak{k}_2(\Psi)$ is the largest semisimple ideal of $\mathfrak{k}_s$ contained in the subalgebra generated by the root vectors corresponding to the simple compact roots of Ψ, and their opposite.

The above results imply that π_λ has an admissible restriction to the group $K_1(\Psi^\lambda)\,Z_1(\Psi^\lambda)$. It is an interesting fact that it may have also an admissible restriction to $K_2(\Psi^\lambda)$. This provides examples where there is admissibility of restriction, but where condition (C) is not satisfied.

Suppose first that $K_2(\Psi^\lambda) = K_s$. Then Ψ^λ is holomorphic and one can show that π_λ restricted to K_s is admissible if and only if G/K is not a tube domain.

Suppose now that $K_2(\Psi^\lambda)$ is a proper subgroup of K_s. We show that π_λ restricted to $K_2(\Psi^\lambda)$ (and also to $Z_K K_2(\Psi^\lambda)$) is admissible if and only if $\mathfrak{g} = \mathfrak{sp}(p, 1)$ and Ψ is quaternionic (that is, $\mathfrak{k}_1(\Psi^\lambda) = \mathfrak{sp}(1)$ and $\mathfrak{k}_2(\Psi^\lambda) = \mathfrak{sp}(q)$). Note that in the particular case of $\mathfrak{sp}(1, 1) \simeq \mathfrak{so}(4, 1)$, all Ψ are quaternionic.

Next we relate condition (C) to some of the theorems of Kobayashi on admissible restrictions. *For the rest of this section we assume (G, H) is a symmetric* *pair*. Let σ denote the involution determined by the symmetric pair (G, H). We may and do assume that σ stabilizes K and T. Thus (K, L) is again a symmetric pair. We choose a σ-stable compact Cartan subgroup $B \subset K$ such that the space $\mathfrak{b}_- := \{X \in \mathfrak{b} : \sigma X = -X\}$ is of maximal dimension: it is a Cartan subspace for the symmetric pair (K, L). We fix a system of positive roots $\Delta \subset \Phi(\mathfrak{k}, \mathfrak{b})$ such that $\{\alpha_{|_{\mathfrak{b}_-}} \neq 0 : \alpha \in \Delta\}$ is a system of positive roots for $\Phi(\mathfrak{k}, \mathfrak{b}_-)$.

Let $\tilde{\Psi} \subset \Phi(\mathfrak{g}, \mathfrak{b})$ be a positive system. We say that $\tilde{\Psi}$ satisfies Kobayashi's condition if we have:

$$\tilde{\Psi} \supset \Delta \quad \text{and} \quad \mathbf{R}^+ \tilde{\Psi}_n \cap \mathfrak{b}_-^* = \{0\}.$$

Consider a G-Harish-Chandra parameter $\lambda \in \mathfrak{b}^*$ such that the positive system of positive roots $\tilde{\Psi}^\lambda \subset \Phi(\mathfrak{g}, \mathfrak{b})$ contains Δ. The importance of Kobayashi condition comes from the following criterium of admissibility:

Theorem 2. (*Kobayashi* [7, 9, 10]) *The restriction of π_λ to H is admissible if and only if $\tilde{\Psi}^\lambda$ satisfies Kobayashi's condition.*

We prove:

Proposition 2. *Assume $\mathbf{R}^+ \tilde{\Psi}_n \cap \mathfrak{b}_-^* = \{0\}$. Then $K_1(\tilde{\Psi})$ is a subgroup of L.*

Condition (C) is stated by means of the σ-stable compact Cartan subgroup T, for which the dimension of $\mathfrak{t}_-$ is minimal. Kobayashi's condition is expressed through the σ-stable compact Cartan subgroup B, for which the dimension of $\mathfrak{b}_-$ is maximal. We relate $\mathfrak{b}$ and $\mathfrak{t}$ via a Cayley transform. Consider a positive system $\tilde{\Psi} \subset \Phi(\mathfrak{g}, \mathfrak{b})$ which contains contains Δ, and let $\Psi \subset \Phi(\mathfrak{g}, \mathfrak{t})$ be the positive system obtained from $\tilde{\Psi}$ by Cayley transform. Using proposition 2, we prove:

Proposition 3. *Assume (G, H) is a symmetric pair. Then condition (C) for Ψ and L holds if and only if Kobayashi's condition holds for $\tilde{\Psi}$.*

Corollary 1. *Let (G, H) be a symmetric pair.*

(i) *The restriction of π_λ to L and H is admissible if and only if condition (C) holds for Ψ^λ and L.*

(ii) *The restriction of π_λ to L and H is admissible if and only $Z_1(\Psi^\lambda) K_1(\Psi^\lambda) \subset L$.*

3. Multiplicity formulas. To express multiplicities, it is convenient to use centered partition functions, that is partitions functions appropriately shifted. Let $S = \{\gamma_1, \cdots, \gamma_q\}$ be a strict multiset of elements of $i\mathfrak{u}_{\mathbf{R}}^*$. We denote by p_S the corresponding partition function, by ρ_S the half-sum of elements of S, and by q_S the partition function shifted by ρ_S. Thus, for $\nu \in i\mathfrak{u}_{\mathbf{R}}^*$, we have $q_S(\nu) = p_S(\nu - \rho_S)$.

We choose a positive system $\Psi_{\mathfrak{z}} \subset \Phi_{\mathfrak{z}}$, and denote by $\varpi_{\mathfrak{z}}$ the Weyl dimension polynomial for Z: for $\gamma \in \mathfrak{t}^*$

$$\varpi_{\mathfrak{z}}(\gamma) = \frac{\prod_{\alpha \in \Psi_{\mathfrak{z}}} \gamma(h_\alpha)}{\prod_{\alpha \in \Psi_{\mathfrak{z}}} \rho_{\Psi_{\mathfrak{z}}}(h_\alpha)}.$$

Let λ be a G-Harish-Chandra parameter. Let $m(\pi_\lambda, \sigma_\mu)$ denote the multiplicity of σ_μ in $\pi_{\lambda_{|_H}}$, for all H-Harish-Chandra parameters μ, and $m(\pi_\lambda, \kappa_\nu)$ the multiplicity of κ_ν in $\pi_{\lambda_{|_L}}$, for all L-Harish-Chandra parameters ν.

Assume that Ψ^λ satisfies condition (C) for L, and choose $\Delta(\mathfrak{k}/\mathfrak{l}, \mathfrak{u})$ like in definition 1. For each $w \in W_K$, we define $S_w^L := q_{\mathfrak{u}}(w\Psi_n) \cup \Delta(\mathfrak{k}/\mathfrak{l}, \mathfrak{u})$ and $S_w^H := S_w^L \backslash \Phi(\mathfrak{h}/\mathfrak{l}, \mathfrak{u})$. Remark that that for all $w \in W_K$, S_w^L is a positive system for $\Phi(\mathfrak{g}/\mathfrak{l}, \mathfrak{u})$, and S_w^H a positive system for $\Phi(\mathfrak{g}/\mathfrak{h}, \mathfrak{u})$.

Theorem 3. *Assume that condition (C) holds for Ψ^λ and L. Then we have*:

$$m(\pi_\lambda, \sigma_\mu) = \pm \sum_{w \in W_Z \backslash W_K} \epsilon(w) \varpi_{\mathfrak{z}}(w\lambda) q_{S_w^H}(\mu - q_{\mathfrak{u}}(w\lambda)).$$

Note that when $H = L$ and $\mu = \nu$, the theorem gives the multiplicities $m(\pi_\lambda, \kappa_\nu)$.

We may rewrite theorem 3 using the decomposition $W_K = W_1 \times W_2$, where $W_i = W_{K_i(\Psi^\lambda)}$. We do it in the simplest case, when H and G have the same rank. Assume that $T = U$. We may choose $\Delta(\mathfrak{k}/\mathfrak{l}, \mathfrak{t}) \subset \Psi^\lambda$. Define $S_\lambda^H := \Psi^\lambda \backslash \Phi(\mathfrak{h}, \mathfrak{t})$. We obtain

$$m(\pi_\lambda, \sigma_\mu) = \pm \sum_{s \in W_1,\, t \in W_2} \epsilon(st)\ q_{S_\lambda^H}(s\mu - t\lambda).$$

4. Partition functions via discrete Heaviside measures. To make the comparison with the results of the next paragraph easier, we restate theorem 3 in terms of measures on $i\mathfrak{u}_{\mathbf{R}}^*$. For $\gamma \in i\mathfrak{u}_{\mathbf{R}}^*$ let δ_γ denote the Dirac delta function attached to γ. If $\gamma \neq 0$, we consider the measure

$$y_\gamma = \sum_{n \geq 0} \delta_{\frac{\gamma}{2} + n\gamma} = \delta_{\frac{\gamma}{2}} + \delta_{\frac{\gamma}{2} + \gamma} + \delta_{\frac{\gamma}{2} + 2\gamma} + \cdots$$

Let $S = \{\gamma_1, \cdots, \gamma_q\}$ be a strict multiset of elements of $i\mathfrak{u}_{\mathbf{R}}^*$. We write $*$ for the convolution product of measures on $i\mathfrak{u}_{\mathbf{R}}^*$. We consider the measure

$$y_S := y_{\gamma_1} * \cdots * y_{\gamma_q} = \sum_{\mu \in i\mathfrak{u}_{\mathbf{R}}^*} q_S(\mu)\ \delta_\mu.$$

Let λ be a G-Harish-Chandra parameter. Assume that Ψ^λ satisfy condition (C) for L, and choose $\Delta(\mathfrak{k}/\mathfrak{l},\mathfrak{u})$ like in definition 1. We use the notations of section 3. For $\nu \in i\mathfrak{u}^*_{\mathbf{R}}$, we define $m^L(\lambda,\nu)$ by the formula

$$\sum_{\nu\in i\mathfrak{u}^*_{\mathbf{R}}} m^L(\lambda,\nu)\delta_\nu = \sum_{w\in W_Z\backslash W_K} \epsilon(w)\varpi_{\mathfrak{z}}(w\lambda)\delta_{q_{\mathfrak{u}}(w\lambda)} * y_{S^L_w}.$$

For $\mu \in i\mathfrak{u}^*_{\mathbf{R}}$, we define $m^H(\lambda,\mu)$ similarly.

The function $m^H(\lambda,.)$ and $m^L(\lambda,.)$ are skew invariant under the action of W_L. The following is essentially a reformulation of theorem 3.

Theorem 4. *Assume that condition* (C) *holds for* Ψ^λ *and* L.

(i) $m^L(\lambda,\nu)=0$ *if* ν *is not a L-Harish-Chandra parameter. If* ν *is a L-Harish-Chandra parameter, we have* $m(\pi_\lambda,\kappa_\nu)=|m^L(\lambda,\nu)|$.

(ii) $m^H(\lambda,\mu)=0$ *if* μ *is not a H-Harish-Chandra parameter. If* μ *is a H-Harish-Chandra parameter, we have* $m(\pi_\lambda,\sigma_\mu)=|m^H(\lambda,\mu)|$.

Let us give an outline of the proof of theorem 4. We consider first some special connected subgroups $L'\subset K$ such that condition (C) is satisfied for Ψ^λ and L', and prove theorem 4 by applying it to the pairs (G,K) and (K,L') (where it is known, by [4] and [5]). Then, as in [5], we use a trick which allows to deduce the multiplicities for the pair (G,H) from the multiplicities for the pair (G,L'), whenever H contains L' and L' contains U, by using product of difference operators associated to the roots in some positive system $\Psi(\mathfrak{h}/\mathfrak{l}',\mathfrak{u})\subset\Phi(\mathfrak{h}/\mathfrak{l}',\mathfrak{u})$. In the particular case $L=L'$, it follows from theorem 5 below. To state this theorem, we introduce some notations.

For $\gamma\in i\mathfrak{u}^*_{\mathbf{R}}$ and S as above, let $d_\gamma := \delta_{-\frac{\gamma}{2}} - \delta_{\frac{\gamma}{2}}$, and

$$d_S := d_{\gamma_1} * \cdots * d_{\gamma_q}.$$

The operation of convolution by d_γ is a difference operator, and we have $d_\gamma * y_\gamma = \delta_0$ and $d_S * y_S = \delta_0$.

Theorem 5. *Let* $S_{\mathfrak{h}/\mathfrak{l}}\subset\Phi(\mathfrak{h}/\mathfrak{l})$ *be a positive system of non compact roots for* $\mathfrak{h}$. *Then*

$$d_{S_{\mathfrak{h}/\mathfrak{l}}} * \sum_{\nu\in i\mathfrak{u}^*_{\mathbf{R}}} m^L(\lambda,\nu)\ \delta_\nu = \pm \sum_{\mu\in i\mathfrak{u}^*_{\mathbf{R}}} m^H(\lambda,\mu)\ \delta_\mu.$$

5. Push forward of Liouville Measure. We now study the semi-classical analogues of multiplicity functions, as in Heckman [5] and Duflo-Heckman-Vergne [2]. Consider an element $\lambda\in i\mathfrak{t}^*$. As usual, we extend λ to a linear functional on $\mathfrak{g}$ which vanishes on each root subspace. The coadjoint orbit $\Omega := G\lambda\subset i\mathfrak{g}^*_{\mathbf{R}}$ is a symplectic manifold. Let β_Ω denote the Kostant-Kirillov-Liouville measure, normalized as in [1], Chapter VII.

From now on, we assume that λ is $\mathfrak{g}$-regular. We denote by $q_{\mathfrak{h}}:\mathfrak{g}^*\to\mathfrak{h}^*$ the restriction map, and we define similarly $q_{\mathfrak{l}}$.

Proposition 4. *The following are equivalent.*

(i) $q_{\mathfrak{l}}$ *is proper on* Ω.

(ii) $q_{\mathfrak{h}}$ *is proper on* Ω.

If moreover λ *is a G-Harish-Chandra parameter, it is also equivalent to*

(iii) π_λ *has an admissible restriction to H.*

From now on, we assume that $q_{\mathfrak{h}}$ is proper on Ω. To describe $q_{\mathfrak{h}}(\Omega)$, it is convenient to choose a positive Weyl Chamber $C_L\subset i\mathfrak{u}^*_{\mathbf{R}}$. The set $q_{\mathfrak{h}}(\Omega)\subset i\mathfrak{h}^*_{\mathbf{R}}$ is H-invariant, and each H-orbit $\omega\subset q_{\mathfrak{h}}(\Omega)$ meets C_L in exactly one point. According to Weinstein [17], $q_{\mathfrak{h}}(\Omega)\cap C_L$ is a convex polyhedron.

This allows to describe the push-forward $(q_{\mathfrak{h}})_*(\beta_\Omega)$ by means of a measure M^H on C_L, called the Duistermaat-Heckman measure. For $\mu\in i\mathfrak{u}^*$, let $\omega_\mu := H\mu\subset i\mathfrak{h}^*$ be its coadjoint orbit, and β_{ω_μ} the corresponding Liouville measure. Then M^H is the positive measure on $q_{\mathfrak{h}}(\Omega)\cap C_L$ such that

$$(q_{\mathfrak{h}})_*(\beta_\Omega) = \int_{C_L} dM^H(\mu)\ \beta_{\omega_\mu}.$$

There is a canonical way to extend M^H to a signed measure on $i\mathfrak{u}^*_{\mathbf{R}}$ which is W_L-skew invariant.

In analogy with section 4, for $\gamma\in i\mathfrak{u}^*_{\mathbf{R}}$ with $\gamma\neq 0$, we define the Heaviside measure Y_γ by $Y_\gamma(\varphi)=\int_0^\infty\varphi(t\gamma)dt$, and for a strict multiset $S=\{\gamma_1,\cdots,\gamma_q\}$ of elements of $i\mathfrak{u}^*_{\mathbf{R}}$, the measure

$$Y_S := Y_{\gamma_1} * \cdots * Y_{\gamma_q}.$$

We also define a measure t_γ by the formula

$$t_\gamma(\varphi) := \int_{-\frac{1}{2}}^{\frac{1}{2}} \varphi(t\gamma)dt,$$

and $t_S := t_{\gamma_1} * \cdots * t_{\gamma_q}$. We have $t_\gamma * y_\gamma = Y_\gamma$ and $t_S * y_S = Y_S$.

Let λ be a G-Harish-Chandra parameter. Assume that Ψ^λ satisfy condition (C) for L, and choose $\Delta(\mathfrak{k}/\mathfrak{l},\mathfrak{u})$ like in definition 1. We use the notations of section 3. We obtain the following analogue of theorem 4.

Theorem 6. *Assume that condition* (C) *holds for* Ψ^λ *and* L. *We have*:

$$M^H = \pm \sum_{w \in W_Z \backslash W_K} \epsilon(w)\ \varpi_{\mathfrak{z}}(w\lambda)\ \delta_{q_{\mathfrak{u}}(w\lambda)} * Y_{S^H_w}.$$

Let S be a positive system for $\Phi(\mathfrak{g}/\mathfrak{h}, \mathfrak{u})$. Since $t_\gamma = t_{-\gamma}$, the measure t_S does not depend on the choice of S, and will be denoted by $r_{\mathfrak{g}/\mathfrak{h}}$. Comparing theorems 4 and 6, we obtain.

Proposition 5.

$$r_{\mathfrak{g}/\mathfrak{h}} * \sum_{\mu \in i\mathfrak{u}^*_{\mathbf{R}}} m^H(\lambda, \mu)\, \delta_\mu = M^H.$$

Theorems about partitions functions due to [15] and others allow to invert this formula.

Acknowledgements. We would like to thank Profs. T. Kobayashi, M. Vergne and D. Vogan for their helpful remarks. This research was supported by CONICET, FONCYT, Agencia Cba Ciencia and SecytUNC (Argentina), Université Paris Diderot-Paris 7 (France) and ICTP (Italy).

References

[1] N. Berline, E. Getzler and M. Vergne, *Heat kernels and Dirac operators*, Corrected reprint of the 1992 original, Springer, Berlin, 2004.

[2] M. Duflo, G. Heckman and M. Vergne, Projection d'orbites, formule de Kirillov et formule de Blattner, Mém. Soc. Math. France (N.S.) No. 15 (1984), 65–128.

[3] B. Gross and N. Wallach, Restriction of small discrete series representations to symmetric subgroups, in *The mathematical legacy of Harish-Chandra (Baltimore, MD, 1998)*, 255–272, Proc. Sympos. Pure Math., 68, Amer. Math. Soc., Providence, RI.

[4] H. Hecht and W. Schmid, A proof of Blattner's conjecture, Invent. Math. **31** (1975), no. 2, 129–154.

[5] G. J. Heckman, Projections of orbits and asymptotic behavior of multiplicities for compact connected Lie groups, Invent. Math. **67** (1982), no. 2, 333–356.

[6] T. Kobayashi, The restriction of $A_q(\lambda)$ to reductive subgroups, Proc. Japan Acad. Ser. A Math. Sci. **69** (1993), no. 7, 262–267.

[7] T. Kobayashi, Discrete decomposability of the restriction of $A_{\mathfrak{q}}(\lambda)$ with respect to reductive subgroups and its applications, Invent. Math. **117** (1994), no. 2, 181–205.

[8] T. Kobayashi, Multiplicity free theorems in branching problems of unitary highest weight modules, Proc. of the Symposium on Repesentation Theory held at Saga, Kyushu (eds. K. Mimachi), (1997), 9–17.

[9] T. Kobayashi, Discrete decomposability of the restriction of $A_{\mathfrak{q}}(\lambda)$ with respect to reductive subgroups. III. Restriction of Harish-Chandra modules and associated varieties, Invent. Math. **131** (1998), no. 2, 229–256.

[10] T. Kobayashi, Discrete decomposability of the restriction of $A_{\mathfrak{q}}(\lambda)$ with respect to reductive subgroups. II. micro local analysis and asymptotic K-support, Annals of Math. **147** (1998), 709–729.

[11] T. Kobayashi, Discrete series representations for the orbit spaces arising from two involutions of real reductive Lie groups, J. Funct. Anal. **152** (1998), no. 1, 100–135.

[12] T. Kobayashi, Branching problems of unitary representations, in *Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002)*, 615–627, Higher Ed. Press, Beijing.

[13] T. Kobayashi, Restrictions of unitary representations of real reductive groups, in *Lie theory*, 139–207, Progr. Math., 229, Birkhäuser, Boston, Boston, MA.

[14] W. Schmid, Die Randwerte holomorpher Funktionen auf hermitesch symmetrischen Räumen, Invent. Math. **9** (1969/1970), 61–80.

[15] A. Szenes and M. Vergne, Residue formulae for vector partitions and Euler-MacLaurin sums, Adv. in Appl. Math. **30** (2003), no. 1–2, 295–342.

[16] J. Vargas, Admissible restriction of holomorphic discrete series for exceptional groups, Rev. Un. Mat. Argentina **50** (2009), no. 1, 67–80.

[17] A. Weinstein, Poisson geometry of discrete series orbits, and momentum convexity for noncompact group actions, Lett. Math. Phys. **56** (2001), no. 1, 17–30.