Note on mod $p$ decompositions of gauge groups

By Daisuke Kishimoto and Akira Kono

Department of Mathematics, Kyoto University, Kitashirakawa-cho, Sakyo-ku, Kyoto 606-8502, Japan


Abstract: We give fibrewise mod $p$ decompositions of the adjoint bundle of a principal $G$-bundle $P$ when the topological group $G$ has mod $p$ decompositions by automorphisms as in [5], which imply mod $p$ decompositions of the gauge group of $P$.

Key words: Gauge group; mod $p$ decomposition.

1. Introduction and statement of the result.

We will always assume that spaces have the homotopy types of CW-complexes.

Let $G$ be a connected topological group, and let $P$ be a principal $G$-bundle over a space $B$. The gauge group of $P$, denoted $G(P)$, is the topological group of $G$-equivariant self-maps of $P$ covering the identity of $B$ with the compact open topology, where the group structure is given by the composite of maps. For an action $\rho$ of $G$ on a space $F$, we denote by $P \times_\rho F$ the fibre bundle associated to $P$ with the action $\rho$. In the special case that $\rho$ is the adjoint action of $G$ onto $G$ itself, we put $\text{ad } P = P \times_\rho G$ and call it the adjoint bundle of $P$. Note that $\text{ad } P$ is a fibrewise topological group in the sense of [3]. Then if we denote the space of sections of a fibrewise space $E \to B$ by $\Gamma(E)$, we have that $\Gamma(\text{ad } P)$ is a topological group. It is shown in [1] that there is a natural isomorphism of topological groups:

$$G(P) \cong \Gamma(\text{ad } P)$$

Thus a fibrewise decomposition of the adjoint bundle $\text{ad } P$ yields a decomposition of the gauge group $G(P)$.

The gauge group $G(P)$, of course, inherits the structures of the topological group $G$. Then if we have a decomposition of $G$, $G(P)$ may have a decomposition. In fact, Theriault [11] showed that mod $p$ decompositions of $G$ induce those of $G(P)$ when the base space $B$ is $S^1$. Other decompositions of gauge groups are discussed in [7] and [8]. The aim of this note is to produce a fibrewise mod $p$ decomposition of the adjoint bundle $\text{ad } P$ for yielding a mod $p$ decomposition of the gauge group $G(P)$ when $G$ has a mod $p$ decomposition by an automorphism as in [5].

In order to state the result, we need some notation. Let $P$ be a set of primes. We denote by $-P$ the localization away from $P$ in the sense of Hilton, Mislin and Roitberg [6]. We also denote by $-P'$ the fibrewise localization away from $P$ in the sense of May [9].

Suppose $G$ has an automorphism $\alpha$ with the subgroup of fixed points $H$. We define a map $\sigma : G/H \to G$ by

$$\sigma(gH) = g\alpha(g)^{-1}$$

for $g \in G$. We also define a map $\theta : H \times G/H \to G$ by

$$\theta(h, gH) = h \cdot \sigma(gH)$$

for $h \in H$ and $g \in G$. Let $\rho$ be the action of $H$ upon $G/H$ defined by

$$\rho(h, gH) = hgH$$

for $h \in H$ and $g \in G$. Now we give the main theorem whose proof will be given in the next section where we also give some examples.

**Theorem 1.1.** Let $G, H, \theta$ and $\rho$ be as above. Suppose that the localized map $\theta_P$ is a homotopy equivalence for some set of primes $P$. Then there is a fibrewise homotopy equivalence:

$$\left(\text{ad } E G\right)_P / \sim_{BH} \simeq \left(\text{ad } EH\right)_P / \times_{BH} \left(EH \times_{\rho} G/H\right)_P$$

Let $E \to B$ be a fibration whose fibre is connected and nilpotent. It follows from the result of Møller [10] that the induced map $\Gamma(E) \to \Gamma(E)_P$ from the fibrewise localization $E \to E_P$ is the localization $\Gamma(E) \to \Gamma(E)_P$. Obviously, we have $\Gamma(E_1 \times_{\rho} E_2) \cong \Gamma(E_1) \times \Gamma(E_2)$ for fibrewise spaces $E_1$ and $E_2$ over $B$. Then we obtain:

**Corollary 1.1.** Let $G, H, \theta$ and $\rho$ be as in Theorem 1.1. Suppose that the localized map $\theta_P$ is a
homotopy equivalence for some set of primes $P$. Then there is a homotopy equivalence:
\[ G(EG|_{BH})_p \simeq G(EH)_p \times \Gamma(EH \times \rho G/H)_p. \]

2. Proof of Theorem 1.1 and examples.
We first give a proof of Theorem 1.1. Let $ad_H$ denote the adjoint action of $H$ onto $G$. Then we have a commutative diagram
\[ \begin{array}{ccc}
H \times G/H & \xrightarrow{\rho} & G/H \\
1 \times \sigma & & \downarrow \sigma \\
H \times G & \xrightarrow{ad_H} & G
\end{array} \]
which induces a fibrewise map $\tau$ fitting into the following commutative diagram of fibre sequences.
\[ \begin{array}{ccc}
H \times G/H & \xrightarrow{\theta} & G \\
\downarrow & & \downarrow \\
\text{ad } EH \times_{BH} (EH \times \rho G/H) & \xrightarrow{\tau} & \text{ad } EG|_{BH} \\
\downarrow & & \downarrow \\
BH & \xrightarrow{\text{id}} & BH
\end{array} \]

Thus Theorem 1.1 follows from Dold’s theorem together with the assumption that the localized map $\theta_p$ is a homotopy equivalence.

Next, we give some examples to which we can apply Theorem 1.1 and Corollary 1.1. The following special gauge groups are of our main interesting. Let $G$ be a connected simple Lie group. Then the principal $G$-bundle over $S^4$ is classified by $\pi_3(G) \cong \mathbb{Z}$.

Definition 2.1. We denote by $G_k(G)$ the gauge group of principal $G$-bundle classified by $k \in \mathbb{Z} \cong \pi_3(G)$.

Example 2.1. Let $G, H, p, d$ and $\alpha$ be as in Table I. Here the matrix $J$ is \( \left( \begin{array}{cc} O & E_n \\ E_n & O \end{array} \right) \). Then each $\alpha$ is an automorphism of $G$ with the subgroup of fixed points $H$. Note that the order of $\alpha$ equals $p$. In [5], Harris showed that the localized map $\theta_p$ is a homotopy equivalence, where $-1$ stands for the localization away from the set of all primes but $p$, that is, inverting $p$. Then we can apply Theorem 1.1 and Corollary 1.1. Moreover, since the inclusion $H \to G$ induces $d$-multiplication in $\pi_3$, we have obtained:

**Proposition 2.1.** Let $G, H, p, d$ and $\rho$ be as above. Then we have a homotopy equivalence
\[ G_k(H) \cong G_k(H) \times (\Gamma(E))_{\rho}, \]
where $E$ is the pullback of $EH \times \rho G/H$ by the map $S^4 \to BH$ representing $k \in \mathbb{Z} \cong \pi_1(BH)$.

Example 2.2. In [2], an involution of $\text{Spin}(2n)$ whose fixed points subgroup is $\text{Spin}(2n-1)$ is constructed. Harris [5] also showed that the localized $\theta_{2}$ is a homotopy equivalence for this involution. Then we can apply Theorem 1.1 and Corollary 1.1. For this example, we can refine Proposition 2.1 a little. Put $n \geq 3$. Let $E$ be the pullback of the bundle $E\text{Spin}(2n-1) \times \rho S^{2n-1}$ by the map $S^4 \to \text{Spin}(2n-1)$ representing $k \in \mathbb{Z} \cong \pi_1(B\text{Spin}(2n-1))$, where $\rho$ is the restriction of the canonical action of $\text{Spin}(2n)$ on $S^{2n-1}$ to $\text{Spin}(2n-1)$. Note that the composite $\pi_3(\text{Spin}(2n-1)) \to \pi_3(\text{Spin}(2n)) \to \pi_{2n+2}(S^{2n-1})$ is the quotient map $\mathbb{Z} \to \mathbb{Z}/24$, where the first arrow is induced from the inclusion $\text{Spin}(2n-1) \to \text{Spin}(2n)$ and the second arrow is the $J$-homomorphism. Now we know that $E$ is fibrewise homotopy equivalent to a fibre space $E_k$ over $S^4$ with fibre $S^{2n-1}$ classified by $[k] \in \mathbb{Z}/24 \cong \pi_{2n+2}(S^{2n-1})$. In particular, if $k$ is a multiple of 3, $E_k^3$ is fibrewise homotopy equivalent to the trivial bundle $S^3 \times S^{2n-1}$. In this case, we have
\[ \Gamma(E_k^3) \cong \text{map}(S^4, S^{2n-1}) \cong S^{2n-1} \times \Omega^1 S^{2n-1}, \]
since $S^{2n-1}$ is an H-space. Thus we have established:

**Proposition 2.2 (cf. [11]).** Let $E_k$ be as above. Then we have a homotopy equivalence
\[ G_k(\text{Spin}(2n))^2 \cong G_k(\text{Spin}(2n-1))^2 \times \Gamma(E_k^3). \]
Moreover, if $k$ is a multiple of 3, we have
\[ \Gamma(E_k^3) \cong S^{2n-1} \times \Omega^1 S^{2n-1}. \]

References


