On the topology of relative orbits for actions of algebraic groups over complete fields

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Abstract: We investigate the problem of equipping a topology on cohomology groups (sets) in its relation with the problem of closedness of (relative) orbits for the action of algebraic groups on affine varieties defined over complete, especially $\mathfrak{p}$-adic fields and give some applications.

Key words: Closed orbits; local fields; algebraic group actions.

Introduction. Let G be a smooth affine algebraic group acting morphically on an affine variety V, all are defined over a field k. Many results of (geometric) invariant theory related to the orbits of the action of G are obtained in the geometric case, i.e., when k is an algebraically closed field. However, since the very beginning of modern geometric invariant theory, as presented in [MFK], there is a need to consider the relative case of the theory. For example, Mumford has considered many aspects of the theory already over sufficiently general base schemes, with arithmetical aim (say, to construct arithmetic moduli of abelian varieties, as in Chap. 3 of [MFK]). Also some questions or conjectures due to Borel [Bo1], Tits [MFK] ... ask for extensions of results obtained to the case of non-algebraically closed fields. As typical examples, we just cite the results by Birkes [Bi], Kempf [Ke], Raghunathan [Ra]... to name a few, which gave the solutions to some of the above mentioned questions or conjectures. Besides, due to the need of number-theoretic applications, the local and global fields k are in the center of such investigation. For example, let an algebraic k-group G act on a k-variety V, $x \in V(k)$. One of the main steps in the proof of the analog of Margulis' super-rigidity theorem in the global function field case (see [Ve,Li,Ma]) was to prove the (locally) closedness of certain sets of the form $G(k).x$, which will be called in the sequel *relative orbits*. In this paper we assume that k is a field which is complete with respect to a non-trivial valuation v of real rank 1 (e.g. $\mathfrak{p}$-adic or real field, i.e., a local field of characteristic 0). Then we can endow $V(k)$ with the (Hausdorff) v-adic topology induced from that of k. Let $x \in V(k)$ be a closed k-point of V. We are interested in a connection between the Zariski-closedness of the orbit $G.x$ of x in V, and Hausdorff closedness of the (relative) orbit $G(k).x$ in $V(k)$. The first result of this type was obtained by Borel and Harish-Chandra [BHC] and then by Birkes [Bi] if $k = \mathbf{R}$, the real field. In fact, it was shown that if G is a reductive $\mathbf{R}$-group, $G.x$ is Zariski closed if and only if $G(\mathbf{R}).x$ is closed in the real topology (see [Bi]). Then this was extended to $\mathfrak{p}$-adic fields in [Bre]. Notice that some proofs previously obtained in [Bi,Bre], ... do not extend to the case of positive characteristic. The aim of this note is to see to what extent the above results still hold for more general class of algebraic groups and fields. In the course of study, it turns out that this question has a close relation with the problem of equipping a topology on cohomology groups (or sets), which has important aspects, say in relation with the duality theory in general (see [Se,Mi]). Some preliminary results on this topic are presented in Section 1. In Section 2 we give some general results on the closedness of (relative) orbits in perfect field case. In Section 3 we consider the general (not-necessarily perfect) case, and also a special class of solvable groups, including commutative groups, in particular tori and unipotent groups over local fields. Details of the proofs will appear elsewhere.

Notations and conventions. By a k-group G we always mean a *smooth affine k-group scheme*

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of finite type (i.e. a linear algebraic k-group, as in [Bo1]). We consider only *closed* points while considering orbits. For flat affine k-group scheme of finite type G, $\mathrm{H}^1_{flat}(k,G)$ stands for the flat cohomology of G.

1. Topology on cohomology sets and groups.

1.1. Ordinary cohomology sets.

1.1.1. Commutative case. Let G be a flat affine *commutative* group scheme of finite type defined over a field k which is complete with respect to a non-trivial valuation v of real rank 1. In many problems related with cohomology, one needs to consider various topologies on the group cohomology, such that all the connecting maps are continuous. As in [Mi], Chap. III, Section 6, one may define a natural topology on the flat cohomology groups of flat commutative group schemes of finite type G, which is in a sense induced from the topology on k and we refer the readers to [Mi] for details. We name this topology as the *canonical topology*. When we are in the category of flat commutative group schemes of finite type, with canonical topology on their flat cohomology groups, all the connecting homomorphisms appearing in any long exact sequence of flat cohomology involving commutative groups are continuous, see loc.cit. In fact, regarding the connecting maps $\mathrm{H}^r_{flat}(k,A) \to \mathrm{H}^r_{flat}(k,B)$, on the level of cocycles, these maps are given by polynomials, induced from the morphism $A \to B$. Thus the induced maps are continuous.

1.1.2. Non-commutative case. H-special topology. Now assume that G is arbitrary and may not be commutative. It seems that not very much is known how to endow canonically a topology on the set $\mathrm{H}^1_{flat}(k,G)$ such that all connecting maps are continuous. First we recall a definition of a topology on $\mathrm{H}^1_{flat}(k,G)$ via embedding of G into special k-groups given in [TT]. Recall that a smooth affine (i.e. linear) algebraic k-group H is called *special* (*over* k) (after Grothendieck and Serre), if the flat (or the same, Galois) cohomology $\mathrm{H}^1_{flat}(K,H)$ is trivial for all extensions K/k. Given an embedding $G \hookrightarrow H$ of G into a special group H, we have the following exact sequence of cohomology

$$1 \to G(k) \to H(k) \to (H/G)(k) \xrightarrow{\delta} \mathrm{H}^1_{flat}(k,G) \to 0.$$

Here H/G is a quasi-projective scheme of finite type defined over k (cf. [DG] or SGA 3). Let k be equipped with Hausdorff topology. Since δ is surjective, by using the natural (Hausdorff) topology on $(H/G)(k)$, induced from that of k, we may endow $\mathrm{H}^1_{flat}(k,G)$ with the strongest topology such that δ is continuous. We call it the *H-special topology*.

1.1.3. Non-commutative case. Canonical topology. Let G be a non-commutative flat affine k-group scheme of finite type. We may also define the *canonical topology* on $\mathrm{H}^1_{flat}(k,G)$ similarly to the commutative case (1.1). We have

1.1.4. Proposition [TT]. *With the above notation and convention, the special topology on* $\mathrm{H}^1(k,G)$ *does not depend on the choice of the embedding into special groups.*

Here we wish to compare the canonical and the special topologies. We have the following

1.1.5. Theorem. *Let k be a field, which is complete with respect to a non-trivial valuation v of real rank* 1. *Then for any smooth affine algebraic k-group G and any special embedding $G \hookrightarrow H$, the H-special topology on* $\mathrm{H}^1(k,G)$ *is stronger than the canonical topology on the cohomology sets* $\mathrm{H}^1(k,G)$, *and when G is commutative, they coincide. Thus if G is smooth, and the canonical topology on* $\mathrm{H}^1(k,G)$ *is discrete, then so is the special topology.*

Remark. Below, while we are discussing a property P related with special topology without mentioning H, it means that there is no need to introduce a special group H, and the statement holds for any special group H.

1.1.6. Theorem. 1) *If a coboundary map between cohomology sets $\delta : C(k) \to \mathrm{H}^1(k,A)$, induced from the exact sequence of k-groups $(*) : 1 \to A \to B \to C \to 1$ is continuous in some H-special topology, then it is so in the canonical topology on* $\mathrm{H}^1(k,A)$.

2) *Any connecting map of cohomology sets in degree ≤ 1 induced from $(*)$ is continuous in the special topology on these sets.*

As a consequence of the proof, we have the following

1.1.7. Proposition. *With the above notation, if k is complete with respect to a non-trivial valuation, then*

1) *Any k-morphism of flat algebraic affine k-group schemes $f : K \to L$ induces a continuous map* $\mathrm{H}^1_{flat}(k,K) \to \mathrm{H}^1_{flat}(k,L)$ *with respect to the H-special topologies for any H.*

2) *For $K \hookrightarrow L$, where K, L are smooth, the induced map* $\mathrm{H}^1(k,K) \to \mathrm{H}^1(k,L)$ *is open in the special topologies on* $\mathrm{H}^1(k,K)$ *and* $\mathrm{H}^1(k,L)$.

The following theorem refines some results proved by various authors, scattered in the literature (see [BT], Section 9, the proof of Lemma 9.2, [Bre, GiMB, Se]).

1.1.8. Theorem. (Compare with [BT], Sec. 9, [Bre], Sec. 5, [GiMB]) *Let k be a field which is complete with respect to a non-trivial valuation of real rank* 1 *and G a smooth affine algebraic group defined over k.*

a) *The subset* $\{1\}$ *is open in the special topology on* $\mathrm{H}^1(k, G)$. *Thus, if G is further commutative then the special* (*or canonical*) *topology on* $\mathrm{H}^1(k, G)$ *is discrete.*

b) *If the characteristic of k is* 0 *then the cohomology set* $\mathrm{H}^1(k, G)$ *is discrete in the special topology. In particular, if k is a local field of characteristic* 0, $\mathrm{H}^1(k, G)$ *is finite and discrete in its special topology. If, moreover, k is non-archimedean and G is commutative, then the same discreteness assertion holds for* $\mathrm{H}^i(k, G)$, $i \geq 1$.

c) *Let a smooth affine algebraic group G act morphically on an affine k-variety V. If $v \in V(k)$ is a closed point such that its stabilizer is smooth* (*e.g., if char.* $k = 0$) *then $G(k).v$ is open in Hausdorff topology of* $(G.v)(k)$.

2. Application to the study of relative orbits over perfect fields.

2.1. In this section we state and prove a property of being closed for orbits of a class $\mathcal{D}$ of algebraic groups, which are close to reductive groups, namely those groups which are direct products of a reductive group and an unipotent group. This result is perhaps the best possible, in the sense that there exists a non-closed orbit for the action of an algebraic group of smallest dimension which does not belong to $\mathcal{D}$. Before going to main results, we need some auxiliary results, some of which are of their independent interest. Below, the terminology "open" or "closed", unless otherwise stated, always means in the sense of Zariski topology.

2.1.1. Lemma. *Let G be an algebraic group acting morphically on a variety V, $v \in V$ a* (*closed*) *point and G° the connected component of G. Then $G.v$ is closed* (*resp. open*) *in V if and only if $G^\circ.v$ is closed* (*resp. open*).

2.1.2. Proposition. *With the notation as in Lemma* 2.1.1 *assume that H is a closed subgroup of G and $v \in V$ is a closed point. Then*

1) *If $G.v$ is closed in V then there is a conjugate H' of H in G such that $H'.v$ is closed in V. In particular, there exists a maximal torus* (*resp. Cartan subgroup*) *and for each standard parabolic subgroup P_θ of type θ of G, there is a parabolic subgroup $P \subset G$, a conjugate of P_θ such that $P.v$ is closed.*

2) *With the above assumption and notation, assume that $G = L \times U$* (*direct product*), *where L is a reductive subgroup of G, and U is a unipotent subgroup of G. Then $G.v$ is closed if and only if so is $L.v$.*

2.2. Next we need an extension of a theorem of Kempf to a class of non-reductive groups.

2.2.1. Theorem. (An extension of a theorem of Kempf) *Let k be a perfect field, $G = L \times U$, where L is a reductive k-group and U is a unipotent k-group. Let G act k-morphically on an affine k-variety V, and let v be a closed point of instability of $V(k)$, i.e., $G.v$ is not closed. Let Y be any closed G-invariant subset of $Cl(G.v) \setminus G.v$. Then there exist a one-parameter subgroup $\lambda : \mathbf{G}_m \to G$, defined over k, and a point $y \in Y \cap V(k)$, such that when $t \to 0$, $\lambda(t).v \to y$.*

Remark. In fact, in the reductive case, the original theorem of Kempf gives more information about the nature of instable orbits and we state here only its simplified version.

2.2.2. Corollary. *Let the notation be as above and $z \in V(k)$ a closed point such that its stabilizer G_z contains all maximal k-split tori of G. Then $G.z$ is closed in V.*

This result complements Corollary 1 of [St, p.70].

2.3. With these preparations we have the following results regarding the topology of the orbits.

2.3.1. Theorem. *Let k be a perfect field, complete with respect to a non-trivial valuation of real rank* 1, *G a smooth affine algebraic k-group acting morphically on an affine k-variety V and $v \in V$ a closed k-point of V.*

1) (*Compare* [Bi, BHC, BT, Bre]) *If $G.v$ is closed and the stabilizer G_v is a smooth k-group, then $G(k).v$ is closed in the Hausdorff topology in $V(k)$.*

2) *Conversely, assume that $G = L \times U$, where L is reductive and U is unipotent, all defined over k. If $G(k).v$ is closed in the Hausdorff topology on $V(k)$, then $G.v$ is also Zariski-closed in V.*

3) *With assumption as in* 2), *$G(k).v$ is closed in $V(k)$ if and only if $G^\circ(k).v$ is closed in $V(k)$.*

Remark. The statement 1) of Theorem 2.3.1 has its origin in Borel and Harish-Chandra [BHC] when $k = \mathbf{R}$, and the converse was proved for reductive groups over the reals by Birkes [Bi]. Then 1) was extended in [Bre], to reductive groups over any local field of characteristic 0. Here we extend their results to the fields which are complete with respect to a non-trivial valuation of real rank 1, for which the general implicit function theorem holds.

2.3.2. Corollary. *Let k, G, V be as in* 2.3.1. *Assume that G_v is a smooth k-group. If G is a smooth nilpotent k-group and T the unique maximal k-torus of G, then the following statements are equivalent.*
a) $G \cdot v$ is closed in Zariski topology;
b) $T \cdot v$ is closed in Zariski topology;
c) $G(k) \cdot v$ is closed in Hausdorff topology;
d) $T(k) \cdot v$ is closed in Hausdorff topology.

2.4. Recall that by a well-known theorem of Mostow, any linear algebraic group G over a field k of characteristic 0 has a decomposition $G = L.U$ into semi-direct product, where U is the largest normal unipotent k-subgroup of G and L is a maximal reductive k-subgroup. The groups which are direct products of a reductive group and a unipotent group are perhaps the best possible for 2.3.1, 2) above to hold. Namely we give below a minimum example among solvable non-nilpotent algebraic groups, for which the assertion 2.3.1, 2) does not hold.

2.4.1. Proposition. *Let B be a smooth solvable affine algebraic group of dimension* 2, *acting morphically on an affine variety X and $x \in X$, a closed point, all defined over a field k of characteristic* 0.
1) *If the stabilizer B_x of x is an infinite subgroup of B, then $B.x$ is always closed.*
2) *Let $G = \mathrm{SL}_2$ and B the Borel subgroup of G, consisting of upper triangular matrices. Consider the standard representation of G by letting G act on the space V_2 of homogeneous polynomials of degree* 2 *with coefficients in* $\mathbf{C}$, *considered as* 3*-dimensional* $\mathbf{C}$*-vector space. Then dim $B = 2$, and for $v = (1, 0, 1)^t \in V_2$, we have*
a) $G.v = \{(x, y, z) \mid 4xz = y^2 + 4\}$ is a closed set in Zariski topology;
b) $B.v = \{(x, y, z) \mid 4xz = y^2 + 4\} \setminus \{z = 0\}$ is a non-closed set in Zariski topology;
c) $B(k).v = \{(a^2 + b^2, 2bd, d^2) \mid ad = 1, a, b, c, d \in k\}$ is a closed set in Hausdorff topology, where k is either $\mathbf{R}$ *or a p-adic field, with $p = 2$ or $p \equiv 3\,(mod. 4)$.*
d) The stabilizer B_v of v in B is finite.

Remark. Also, in the case of solvable groups, in contrast with the nilpotent case (see Corollary 2.3.2), some of the relations between the closedeness of orbits of closed subgroups and that of the ambient groups may not hold, as the following statements show.

2.4.2. Proposition. *Let G be a smooth solvable affine algebraic group defined over k, where k is either* $\mathbf{R}$ *or* $\mathbf{Q}_p$, *T a maximal k-torus of G, $\rho : G \to GL(V)$ a representation of G which is defined over k, and $v \in V(k)$ a closed k-point. We consider the following statements.*
a) $G.v$ is closed in Zariski topology;
b) For any above T, $T.v$ is closed in Zariski topology;
c) $G(k).v$ is closed in Hausdorff topology;
d) For any above T, $T(k).v$ is closed in Hausdorff topology.
Then we have the following logical scheme
b) $\Leftrightarrow$ d), a) $\Rightarrow$ c), a) $\not\Rightarrow$ b), b) $\not\Rightarrow$ a), c) $\not\Rightarrow$ d), d) $\not\Rightarrow$ c), c) $\not\Rightarrow$ a).

3. Relative orbits over non-perfect complete fields.

3.1. In this section we consider the case of a field k which is complete with respect to a non-trivial valuation of real rank 1, (e.g., a local field) of arbitrary characteristic; for example, k can be a local function field, which is one of important cases of non-perfect fields. The first main result of this section is the following Theorem 3.1.1, where, under some mild and natural conditions, we treat the case of reductive and nilpotent groups, and the most satisfactory (i.e. unconditional) results were obtained for commutative and unipotent groups. In 3.2–3.3 we present various results on closedness of orbits under the action of a class of smooth solvable affine algebraic groups, which includes a large class of nilpotent linear groups.

First we recall the notion of strongly separable actions of algebraic groups after [RR]. Let G be a smooth affine algebraic group acting regularly on an affine variety V, all are defined over a field k. Let $v \in V(k)$ be a k-point, G_v the corresponding stabilizer and $Cl(G.v)$ the Zariski closure of $G.v$ in V. The action of G is said to be *strongly separable* (after [RR]) at v if for all $x \in Cl(G.v)$, the stabilizer G_x is smooth, or equivalently, the induced morphism $G \to G/G_x$ is separable. Related with this notion, we call the action *fairly separable* at v, if for all $w \in (G.v)(k)$, the stabilizer G_w is a smooth k-subgroup of G. A priori "strongly separable" implies "fairly separable", and it is quite unlikely that the converse statement is true.

3.1.1. Theorem. *Let k be a field, which is complete with respect to a non-trivial valuation of real rank* 1*, and G a smooth affine algebraic group acting linearly on an affine k-variety V, all defined over k. Let $v \in V(k)$ be a closed k-point and G_v the stabilizer group of v.*
1) *If $G(k).v$ is closed in Hausdorff topology induced from $V(k)$ and either G is nilpotent or G is reductive and the action of G is strongly separable at v in the sense of* [RR]*, then $G.v$ is closed* (*in Zariski topology*) *in V.*
2) *Conversely, with above notation, $G(k).v$ is Hausdorff closed in $V(k)$ if $G.v$ is closed and one of the following conditions holds*:
a) *G_v is smooth and commutative, or G is commutative*;
b) *G_v is a smooth k-group, which is an extension of a smooth unipotent k-group by a diagonalizable k-group*;
c) *k is a local field, and G_v is a smooth connected reductive k-subgroup of G*;
d) *The action at v is fairly separable.*

Remarks. 1) If *char.*$k = 0$, then this theorem is contained in 1.1.8. Thus it is especially interesting in the case of non-perfect fields, e.g. local function fields.
2) The examples similar to 2.4.1 show that if one of the conditions on G in Theorem 3.1.1, 1) (i.e., the nilpotency, or the strong separability of the action), is violated, then the assertion 1) does not hold. For the proof of Part 1), we need the following result due to Birkes, characterizing the so-called Property A in [Bi,Ra].

3.1.2. Theorem. ([Bi], Proposition 9.10) *Let k be an arbitrary field and G a smooth nilpotent k-group acting linearly on a finite dimensional vector space V via a representation $\rho : G \to GL(V)$, all defined over k. If $v \in V(k)$ is a closed point and Y is a non-empty G-stable closed subset of $Cl(G.v) \setminus G.v$, then there exist an element $y \in Y \cap V(k)$, and a one-parameter subgroup $\lambda : \mathbf{G}_m \to G$ defined over k, such that $\lambda(t).v \to y$ while $t \to 0$.*

3.1.3. Corollary. *Let k be a field, complete with respect to a non-trivial valuation of real rank* 1 *and G a smooth unipotent algebraic group defined over k, which acts k-regularly on an affine k-variety V. Let $v \in V(k)$ be a closed point, and assume that the stabilizer group G_v is smooth.*
1) *The trivial cohomology class* $\{1\}$ *is both open and closed in the special topology on* $\mathrm{H}^1(k, G)$*. In particular, $G(k).v$ is always Hausdorff closed in $V(k)$.*
2) *Assume further that V is a finite dimensional k-vector space and G is a smooth unipotent k-subgroup of* $\mathrm{GL}(V)$*. Then for any $v \in V(k)$, with the standard linear action of G on V, $G(k).v$ is closed in Hausdorff topology in $V(k)$.*

3.2. Next we consider the case of connected smooth solvable affine groups which are extensions of unipotent k-groups by diagonalizable k-groups, in particular, the case of connected nilpotent groups. We may assume that G is neither torus, nor unipotent. In the case of connected nilpotent groups G, the maximal diagonalizable subgroup G_s of G is defined over k_s and is stable with respect to Γ. Thus it is defined over k (see [DG], Chap. IV, Sec. 4). Moreover, it is a central k-subgroup of G, which is smooth if G is smooth. The unipotent part of G is not necessarily defined over k, but we still have the following exact sequence $1 \to G_s \to G \xrightarrow{f} U \to 1$, where U is a unipotent k-group, which is called the unipotent quotient of G. By a well-known result of Tits, there is a unique normal, maximal k-split subgroup U_d of U, where U/U_d is k-wound (see [KMT,Oe,Ti]). The inverse image of U_d via f is an affine k-subgroup scheme K of G, containing G_s. It is clear that K is a normal k-subgroup scheme of G.

3.2.1. Proposition. *Let k be a local field, G a connected smooth affine algebraic k-group, which acts k-regularly on an affine k-variety V, and $v \in V(k)$ a closed k-point. Assume that G is an extension of a unipotent k-group by a smooth diagonalizable k-group G_s* (*e.g. a nilpotent linear algebraic group*)*. Let K be as above and assume that K is a smooth k-subgroup of G.*
1) *If $K(k).v$ is closed in $(K.v)(k)$, then so is $G(k).v$ in $(G.v)(k)$.*
2) *The special topology on* $\mathrm{H}^1(k, K)$ *is discrete. In particular, the trivial class* $\{1\}$ *is both open and closed subset there.*

3.2.2. Corollary. *With above notation and assumption, if G is a smooth connected nilpotent affine algebraic k-group and the k-split part U_d of the unipotent quotient G/G_s is commutative, then $G(k).v$ is Hausdorff closed in $V(k)$.*

We have the following general result.

3.2.3. Theorem. *Let notation be as above and let k be a field, complete with respect to a non-trivial* *valuation of real rank* 1*, G a smooth affine algebraic k-group, acting k-regularly on an affine k-variety V and $v \in V(k)$ a closed k-point. Assume that G.v is closed, G_v is an extension of a unipotent k-group by a diagonalizable k-group $G_{s,v}$ and both are smooth k-groups. Then $G(k).v$ is Hausdorff closed in $V(k)$.*

3.2.4. Corollary. *Let k, G, V, v be as in* 3.2.3. *Assume that G.v is closed and G_v is a smooth k-group, which is an extension of a unipotent k-group by a k-split torus $G_{s,v}$. Then $G(k).v$ is Hausdorff closed in $V(k)$.*

3.3. Next we assume that G is a smooth affine nilpotent algebraic k-group, $G = T \times U$, where T is a diagonalizable k-group and U a unipotent k-group. Let $T^{\circ} = T_s.T_a$, where T_s (resp. T_a) is the maximal k-split (resp. k-anisotropic) subtorus of T and the product is almost direct and defined over k.

3.3.1. Proposition. *With above notation and assumption as in* 3.2.1*, let G act k-regularly on an affine k-variety V and $v \in V(k)$ a closed k-point. Assume that G.v is closed in V, $G = T \times U$, where T is a diagonalizable k-group and U is a k-unipotent group. If $(T_s(k) \times U_d(k)).v$ is Hausdorff closed in $((T_s \times U_d).v)(k)$ then $G(k).v$ is Hausdorff closed in $V(k)$.*

3.4. Let k be a local field. By abuse of language, we call a smooth affine algebraic k-group G *compact* if its group of k-rational points $G(k)$ is a compact Hausdorff topological group. Denote by $\mathcal{C}$ the smallest class of linear algebraic k-groups satisfying the following properties.
1) All commutative affine k-groups belong to $\mathcal{C}$;
2) All compact k-groups belong to $\mathcal{C}$;
3) If G is an extension of a compact k-group by a group belong to $\mathcal{C}$, then G also belongs to $\mathcal{C}$.
As a consequence of above consideration, we have

3.4.1. Corollary. *Let k be a local field, G a smooth connected affine algebraic k-group, which acts k-regularly on an affine k-variety V and $v \in V(k)$ a closed k-point. If $G \in \mathcal{C}$ and G.v is closed, then $G(k).v$ is Hausdorff closed in $V(k)$.*

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