Some remarks on symmetric linear functions and pseudotrace maps

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Abstract: Let A be a finite-dimensional associative algebra and ϕ a symmetric linear function on A. In this note, we will show that the pseudotrace maps defined in [6] are obtained as special cases of well-known symmetric linear functions on the endomorphism rings of projective modules. As an application of our approach, we will give proofs of several propositions and theorems in [6] for an arbitrary finite-dimensional associative algebra.

Key words: Symmetric algebras; symmetric linear functions; pseudotrace maps.

1. Introduction. In this note, we work on an algebraically closed field $\mathbf{k}$ of characteristic 0. Let A be a finite-dimensional associative $\mathbf{k}$-algebra. A linear function ϕ on A is said to be *symmetric* if $\phi(ab) = \phi(ba)$ for all $a, b \in A$. We denote the space of symmetric linear functions on A by $\mathrm{SLF}(A)$.

In [6], Miyamoto introduces a notion of a *pseudotrace map* on a basic symmetric $\mathbf{k}$-algebra P in order to construct pseudotrace functions of logarithmic modules of vertex operator algebras satisfying some finiteness condition called C_2-condition. Let ϕ be a symmetric linear function on P which induces a nondegenerate bilinear form $P \times P \to \mathbf{k}$. Then the pseudotrace map tr_W^{ϕ} is a symmetric linear function on the endomorphism ring of a finite-dimensional right P-module W called *interlocked with* ϕ. As it is implicitly mentioned in [6] and it is proved in this note, a finite-dimensional right P-module which is interlocked with ϕ is in fact a direct sum of indecomposable projective modules.

For an arbitrary finite-dimensional $\mathbf{k}$-algebra A, a finitely generated projective right A-module W has an *A-coordinate system of* W, that is, $\{u_i\}_{i=1}^n \subset W$ and $\{\alpha_i\}_{i=1}^n \subset \mathrm{Hom}_A(W, A)$ such that $w = \sum_{i=1}^n u_i \alpha_i(w)$ for all $w \in W$ (see [2]). For any symmetric linear function ϕ on A, we can define a symmetric linear function on $\mathrm{End}_A(W)$ by

$$\phi_W(\alpha) = \phi\left(\sum_{i=1}^n \alpha_i \circ \alpha(u_i)\right)$$

for all $\alpha \in \mathrm{End}_A(W)$ (c.f. [3]). In this note, we show that the symmetric linear function tr_W^{ϕ} coincides with the pseudotrace map when $A = P$ and ϕ induces a nondegenerate symmetric associative bilinear form on P. We also prove that a right P-module W is interlocked with ϕ if and only if W is projective. Then we can prove several propositions and theorems in [6] for *arbitrary finite-dimensional* $\mathbf{k}$*-algebras*.

This note is organized as follows: In section 2, we recall a construction of a symmetric linear function ϕ_W on the endomorphism ring of finitely generated projective modules W from $\phi \in \mathrm{SLF}(A)$. In section 3, we assume that P is indecomposable, basic and symmetric and $\phi \in \mathrm{SLF}(P)$ satisfies some conditions (see section 3). We recall a notion of a right P-module W which is interlocked with ϕ and a notion of a pseudotrace map tr_W^{ϕ} defined in [6]. We show that W is interlocked with ϕ if and only if W is projective. By using this fact, for any indecomposable projective module W, we define ϕ_W and show that ϕ_W coincides with tr_W^{ϕ}. In section 4 and 5, we prove several propositions and theorems for pseudotrace maps in [6] by using ϕ_W for arbitrary finite-dimensional $\mathbf{k}$-algebras.

2. Projective modules and symmetric linear functions. Let A be a finite-dimensional associative $\mathbf{k}$-algebra. We denote a left (resp. right) A-module M by ${}_AM$ (resp. M_A).

In this section, we recall a notion of a symmetric linear function on the endomorphism ring of a finitely generated projective right A-module (c.f. [3]).

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Assume that W_A is finitely generated. Then W_A is projective if and only if there exist subsets $\{u_i\}_{i=1}^n \subset W_A$ and $\{\alpha_i\}_{i=1}^n \subset \mathrm{Hom}_A(W_A, A)$ such that

$$w = \sum_{i=1}^n u_i \alpha_i(w)$$

for all $w \in W_A$ (see [2], chapter II, §2.6, Proposition 12). The set $\{u_i, \alpha_i\}_{i=1}^n$ is called an A-*coordinate system* of W_A.

Assume that W_A is finitely generated and projective. Let $\{u_i, \alpha_i\}_{i=1}^n$ be an A-coordinate system of W_A. Then we define a map

$$T_{W_A} : \mathrm{End}_A(W_A) \to A/[A, A]$$

by $\alpha \mapsto \pi(\sum_{i=1}^n \alpha_i \circ \alpha(u_i))$ where $\pi : A \to A/[A, A]$ is the canonical surjection (c.f. [5,8]). It is known that the map T_{W_A} does not depend on the choice of A-coordinate systems and that $T_{W_A}(\alpha \circ \beta) = T_{W_A}(\beta \circ \alpha)$ for all $\alpha, \beta \in \mathrm{End}_A(W_A)$ (see [5,8]). For $\phi \in \mathrm{SLF}(A)$, we set $\phi_{W_A} = \phi \circ T_{W_A} : \mathrm{End}_A(W_A) \to \mathbf{k}$. Then we have the following

Proposition 2.1. *Assume that W_A is finitely generated and projective and let ϕ be a symmetric linear function on A. Then ϕ_{W_A} is a symmetric linear function on $\mathrm{End}_A(W_A)$.*

3. Miyamoto's psedotrace maps. In this section, we show that the map ϕ_{W_A} coincides with the pseudotrace map defined in [6] if A satisfies extra conditions.

First we recall the definition of a pseudotrace map. Let P be a basic symmetric indecomposable $\mathbf{k}$-algebra We fix a decomposition of the unity 1 by mutually orthogonal primitive idempotents:

$$1 = e_1 + e_2 + \cdots + e_k.$$

We also fix $\phi \in \mathrm{SLF}(P)$ with the condition

$$\text{(3.1)} \qquad \begin{aligned} &\langle a, b\rangle := \phi(ab) \text{ is nondegenerate,} \\ &\phi(e_i) = 0 \text{ for all } 1 \le i \le k. \end{aligned}$$

Note that we have $P/J(P) = \mathbf{k}\bar{e}_1 \oplus \cdots \oplus \mathbf{k}\bar{e}_k$ since P is basic and indecomposable. It is well-known that $\{e_iP | 1 \le i \le k\}$ is the complete list of indecomposable projective right P-modules.

Since $a \in \mathrm{Soc}(P_P)$ if and only if $aJ(P) = 0$ we see that

$$\langle aJ(P), P\rangle = \langle J(P), a\rangle = \langle P, J(P)a\rangle = 0.$$

The same argument for $\mathrm{Soc}({}_PP)$ shows $\mathrm{Soc}(P_P) = \mathrm{Soc}({}_PP)$. Thus $\mathrm{Soc}(P_P) = \mathrm{Soc}({}_PP)$ is a two-sided ideal and we denote it by $\mathrm{Soc}(P)$. Then we have $\langle aJ(P), P\rangle = \langle a, J(P)\rangle$ for any $a \in P$. This identity shows that $\mathrm{Soc}(P) = J(P)^\perp$. Similarly we have $J(P) = \mathrm{Soc}(P)^\perp$. Thus the bilinear form $\langle\ ,\ \rangle$ induces a nondegenerate pairing $\langle\ ,\ \rangle : \mathrm{Soc}(P) \times P/J(P) \to \mathbf{k}$. Let $\{f_1, f_2, \ldots, f_k\}$ be a basis of $\mathrm{Soc}(P)$ which are dual to the basis $\{\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_k\}$ of $P/J(P)$, that is, $\langle f_i, \bar{e}_j\rangle = \langle f_i, e_j\rangle = \delta_{ij}$ for $1 \le i, j \le k$.

Lemma 3.1. *$e_if_j = f_je_i = \delta_{ij}f_j$ for all $1 \le i, j \le k$.*

Proof. Note that $e_if_j \in \mathrm{Soc}(P)$. Thus we have

$$\langle e_if_j, \bar{e}_k\rangle = \delta_{ik}\langle f_j, \bar{e}_k\rangle = \delta_{ik}\delta_{kj}$$

so that $e_if_j = \delta_{ij}f_j$. □

Lemma 3.2. *$\mathrm{Soc}(P) \subseteq J(P)$, in particular, $e_i\mathrm{Soc}(P)e_j \subseteq e_iJ(P)e_j$ for all $1 \le i, j \le k$.*

Proof. Since $P = \bigoplus_{i=1}^k Pe_i$, we see that $\mathrm{Soc}(P) = \bigoplus_{i=1}^k \mathrm{Soc}(Pe_i)$. Then $J(P)e_i$ is the unique maximal submodule of Pe_i. Suppose that $\mathrm{Soc}(Pe_i)$ is not contained in $J(P)$. We have $Pe_i = \mathrm{Soc}(Pe_i) + J(P)e_i$ since $J(P)e_i$ is the unique maximal submodule of Pe_i. Then we conclude $\mathrm{Soc}(Pe_i) = Pe_i$ by Nakayama's lemma. Therefore we can see $e_i \in \mathrm{Soc}(Pe_i)$. By the same argument for $P = \bigoplus_{i=1}^k e_iP$, we obtain $e_i \in \mathrm{Soc}(e_iP)$. Thus we find $J(P)e_i = e_iJ(P) = 0$, which shows that e_i is a central idempotent of P. This contradicts to the assumption that P is indecomposable. □

Since $P = \sum_{i=1}^k \mathbf{k}e_i + J(P)$, we have by Lemma 3.1

$$\text{(3.2)} \qquad e_iPe_j = \begin{cases} \mathbf{k}e_i + e_iJ(P)e_i, & i = j, \\ e_iJ(P)e_j, & i \ne j, \end{cases}$$

and

$$\text{(3.3)} \qquad e_i\mathrm{Soc}(P)e_j = \begin{cases} \mathbf{k}f_i, & i = j, \\ 0, & i \ne j. \end{cases}$$

Set $d_{ij} = \dim_{\mathbf{k}} e_iJ(P)e_j/e_i\mathrm{Soc}(P)e_j$ for all $1 \le i, j \le k$. Then since the pairing

$$\langle\ ,\ \rangle : e_iJ(P)e_j/e_i\mathrm{Soc}(P)e_j \times e_jJ(P)e_i/e_j\mathrm{Soc}(P)e_i \to \mathbf{k}$$

is well-defined and nondegenerate, it follows that $d_{ij} = d_{ji}$ for all $1 \le i, j \le k$.

Also since $e_i\mathrm{Soc}(P)e_j \subseteq e_iJ(P)e_j \subseteq e_iPe_j$, (3.2) and (3.3), we have $\dim_{\mathbf{k}} e_iPe_i = d_{ii} + 2$ and $\dim_{\mathbf{k}} e_iPe_j = d_{ij}$ for $i \ne j$ by (3.2) and (3.3).

Lemma 3.3 ([6], Lemma 3.2)**.** *The algebra P has a basis*

$$\Omega = \{\rho_0^{ii}, \rho_{d_{ii}+1}^{ii}, \rho_{s_{ij}}^{ij} | 1 \le i, j \le k,\ 1 \le s_{ij} \le d_{ij}\}$$

satisfying

(a) $\rho_0^{ii} = e_i$, $\rho_{d_{ii}+1}^{ii} = f_i$,

(b) $e_i \rho_s^{ij} e_j = \rho_s^{ij}$,

(c) $\langle \rho_s^{ij}, \rho_{d_{ab}+1-t}^{ab} \rangle = \delta_{i,b}\delta_{j,b}\delta_{s,t}$,

(d) $\rho_s^{ij}\rho_{d_{ji}+1-s}^{ji} = f_i$,

(e) *the space spanned by* $\{\rho_t^{ij} | t \ge s\}$ *is* $e_i P e_i$-*invariant.*

For $1 \le i, j \le k$, set

$$\Omega_i = \{\rho_s^{ij} | 1 \le j \le k, s\},\ \Omega_{ij} = \{\rho_s^{ij} | s\}.$$

Note that Ω_i is a basis of $e_i P$ for any $1 \le i \le k$ and $\Omega - \{e_1, \dots, e_k\}$ is a basis of $J(P)$. We sometimes denote an element of Ω_{ij} by ρ^{ij}.

Definition 3.4 ([6], Definition 3.6)**.** Assume that W_P is finitely generated. The module W_P is said to be *interlocked with* ϕ if $\ker(f_i) = \{w \in W | wf_i = 0\}$ is equal to $\sum_{\rho \in \Omega - \{e_i\}} W\rho$ for all $1 \le i \le k$.

It is obvious that $\ker(f_i) \supseteq \sum_{\rho \in \Omega - \{e_i\}} W\rho$ since $\rho f_i = 0$ for any $\rho \in \Omega - \{e_i\}$. In [6], the pseudotrace map is defined on the endomorphism ring of a finite-dimensional right P-module which is interlocked with ϕ. The isomorphism stated in [6, p.68] is more precisely understood as follows:

Theorem 3.5. *Let P be a basic symmetric indecomposable algebra. Assume that $\phi \in \mathrm{SLF}(P)$ satisfies the condition* (3.1) *and W_P is finitely generated. Then W_P is interlocked with ϕ if and only if W_P is projective. In particular, if W_P is interlocked with ϕ then the multiplicity of the indecomposable projective module $e_i P$ in W_P is given by* $\dim_{\mathbf{k}} W_P f_i$ *for* $1 \le i \le k$.

In order to prove this theorem, we first show the following lemmas.

Lemma 3.6. *Any indecomposable projective module $e_i P$ for $1 \le i \le k$ is interlocked with ϕ.*

Proof. For $e_i p \in e_i P$, suppose $e_i p f_i = 0$ and express p as $p = \sum_{\rho \in \Omega} a_\rho \rho$ with $a_\rho \in \mathbf{k}$. Then $0 = e_i p f_i = e_i \sum_{\rho \in \Omega} a_\rho \rho f_i = a_{e_i} f_i$. Thus p belongs to the space spanned by $\Omega - \{e_i\}$, which shows $e_i p \in \sum_{\rho \in \Omega - \{e_i\}} e_i P \rho$.

For $i \ne j$, we can see $e_i p f_j = a_{e_j} e_i f_j = 0$ for all $p \in P$. Thus we have $\ker(f_j) \subseteq e_i P = \sum_{\rho \in \Omega - \{e_j\}} e_i P \rho$. □

Lemma 3.7. *The module W_P is interlocked with ϕ if and only if any direct summand of W_P is interlocked with ϕ.*

Proof. Suppose that $W_P = W_1 \oplus W_2$ where W_1 and W_2 are right P-modules. Then we have

$$\sum_{\rho \in \Omega - \{e_i\}} W\rho = \left(\sum_{\rho \in \Omega - \{e_i\}} W_1 \rho \right) \oplus \left(\sum_{\rho \in \Omega - \{e_i\}} W_2 \rho \right). \tag{3.4}$$

By (3.4) and the definition of the module which interlocked with ϕ, we have the lemma. □

Lemma 3.8. *Assume that W_P is interlocked with ϕ. Then*

$$We_i / WJ(P)e_i \cong Wf_i,\ \overline{we_i} \mapsto wf_i$$

for any $1 \le i \le k$.

Proof. The kernel of the map $We_i \to Wf_i$, $we_i \mapsto wf_i$ is equal to $\sum_{\rho \in \Omega - \{e_i\}} W\rho e_i = WJ(P)e_i$ since W_P is interlocked with ϕ. □

Proof of Theorem 3.5. By Lemma 3.6 and Lemma 3.7, any finite direct sum of indecomposable projective modules is interlocked with ϕ.

Conversely, suppose that W_P is interlocked with ϕ. By Lemma 3.8, there exists v^{e_i} such that $v^{e_i} f_i \ne 0$ if $\dim_{\mathbf{k}} Wf_i \ne 0$. Then the map

$$\theta : e_i P \to W,\ e_i p \mapsto v^{e_i} e_i p,$$

is a P-homomorphism. Suppose $\ker(\theta) \ne 0$. Note that $\mathrm{Soc}(e_i P) = \mathbf{k} f_i$ by Lemma 3.1. Since $e_i P$ has the unique simple submodule $\mathrm{Soc}(e_i P)$ (see [4, Proposition 9.9 (ii)]) we have $f_i \in \ker(\theta)$ and $v^{e_i} f_i = 0$. This is a contradiction. Thus θ is injective.

Since P is a symmetric algebra, any projective module is also injective (see [4, Proposition 9.9 (iii)]). Therefore θ is split and then $e_i P$ is a direct summand of W, say, $W \cong e_i P \oplus W'$. By Lemma 3.5, W' is also interlocked with ϕ and $\dim_{\mathbf{k}} W' f_i = \dim_{\mathbf{k}} Wf_i - 1$ since $\dim_{\mathbf{k}} e_i P f_i = 1$. If $Wf_i = 0$ for all $1 \le i \le k$, then $wf_i = 0$ for all $w \in W$ and $1 \le i \le k$. Thus we have

$$W = \bigcap_{i=1}^{k} \left(\sum_{\rho \in \Omega - \{e_i\}} W\rho \right) = WJ(P).$$

By Nakayama's lemma, we have $W = 0$.

Therefore the induction on $\dim_{\mathbf{k}} Wf_i$ proves the theorem. In particular, the multiplicity of $e_i P$ in W is equal to $\dim_{\mathbf{k}} Wf_i$ for all $1 \le i \le k$. □

Assume that W_P is finitely generated and projective. Then W_P is isomorphic to a finite direct sum of indecomposable projective modules:

$$W_P \cong \bigoplus_{i=1}^{k} n_i e_i P, \tag{3.5}$$

where n_i is the multiplicity of e_iP, that is, $n_i = \dim_{\mathbf{k}} Wf_i$. We denote the element of W_P corresponding to e_i by $v_j^{e_i}$ for $1 \le i \le k$ and $1 \le j \le n_i$. Note that W_P has a basis $\{v_j^{e_i}\rho \mid \rho \in \Omega_i,\ 1 \le i \le k,\ 1 \le j \le n_i\}$.

Since $\alpha(v_j^{e_i}) = \alpha(v_j^{e_u} e_i) = \alpha(v_j^{e_i})e_i \in We_i$ for $\alpha \in \mathrm{End}_P(W_P)$ and Lemma 3.3 (b), we have

$$\alpha(v_j^{e_i}) = \sum_{s=1}^{k}\sum_{t=1}^{n_s}\sum_{\rho^{si}\in\Omega_{si}} \alpha_{jt}^{\rho^{si}} v_t^{e_s}\rho^{si} \tag{3.6}$$

for $1 \le i \le k$ and $1 \le j \le n_i$ where $\alpha_{jt}^{\rho^{si}} \in \mathbf{k}$. In [6], the pseudotrace map $\mathrm{tr}_{W_P}^{\phi}$ on $\mathrm{End}_P(W_P)$ is defined by

$$\mathrm{tr}_{W_P}^{\phi}(\alpha) = \sum_{i=1}^{k}\sum_{j=1}^{n_i} \alpha_{jj}^{f_i}. \tag{3.7}$$

In order to show that the pseudotrace map coincides with ϕ_{W_P}, we choose the following P-coordinate system of W_P. Note that ϕ_{W_P} does not depend on the choice of P-coordinate systems.

Set

$$\alpha_j^i(v_t^{e_s}\rho^{sp}) = \begin{cases} \rho^{ip}, & i = s,\ j = t, \\ 0, & \text{otherwise} \end{cases}$$

for $1 \le i \le k$ and $1 \le j \le n_i$. Then α_j^i belongs to $\mathrm{Hom}_P(W_P, P)$ for $1 \le i \le k$ and $1 \le j \le n_i$.

Lemma 3.9. *The set* $\{v_j^{e_i}, \alpha_j^i \mid 1 \le i \le k,\ 1 \le j \le n_i\}$ *is a P-coordinate system of* W_P.

Proof. By the definitions of $v_j^{e_i}$ and α_j^i, we have $v_j^{e_i}\rho^{ip} = v_j^{e_i}\alpha_j^i(v_j^{e_i}\rho^{ip}) = \sum_{s=1}^{k}\sum_{t=1}^{n_s} v_t^{e_s}\alpha_t^s(v_j^{e_i}\rho^{ip})$. Since the elements $v_j^{e_i}\rho^{ip}$ form a basis of W_P, we have shown the lemma. □

Theorem 3.10. *Let P be a basic symmetric algebra. Assume that $\phi \in \mathrm{SLF}(P)$ satisfies the condition* (3.1) *and that W_P is finitely generated and projective. Then $\phi_{W_P} = \mathrm{tr}_{W_P}^{\phi}$.*

Proof. For $\alpha \in \mathrm{End}_P(W_P)$, one has

$$\begin{aligned}
\phi_{W_P}(\alpha) &= \phi\left(\sum_{i=1}^{k}\sum_{j=1}^{n_i} \alpha_j^i \circ \alpha(v_j^{e_i})\right) \\
&= \phi\left(\sum_{i,s=1}^{k}\sum_{j=1}^{n_i}\sum_{t=1}^{n_s}\sum_{\rho^{si}\in\Omega_{si}} \alpha_j^i(\alpha_{jt}^{\rho^{si}} v_t^{e_s}\rho^{si})\right) \\
&= \sum_{i=1}^{k}\sum_{j=1}^{n_i}\sum_{\rho^{ii}\in\Omega_{ii}} \alpha_{jj}^{\rho^{ii}}\phi(\rho^{ii}) \\
&= \sum_{i=1}^{k}\sum_{j=1}^{n_i} \alpha_{jj}^{f_i} = \mathrm{tr}_{W_P}^{\phi}(\alpha)
\end{aligned}$$

since (3.6) and Lemma 3.3 (c). □

4. The center and symmetric linear functions. In this section, we assume that the finite-dimensional $\mathbf{k}$-algebra A contains a nonzero central element ν such that $(\nu - r)^s A = 0$ and $(\nu - r)^{s-1}A \ne 0$ for some $r \in \mathbf{k}$ and $s \in \mathbf{Z}_{>0}$.

Set $\mathcal{K} = \{a \in A \mid (\nu - r)a = 0\}$. Note that $\mathcal{K}$ is a two-sided ideal of A. Let $\alpha : M_A \to N_A$ be an A-module homomorphism. Then $M/M\mathcal{K}$ is an $A/\mathcal{K}$-module and the map $\widehat{\alpha} : M/M\mathcal{K} \to N/N\mathcal{K}$ defined by $\widehat{\alpha}(\overline{m}) = \overline{\alpha(m)}$ is an $A/\mathcal{K}$-module homomorphism where $\overline{m}$ is the image of m under the canonical map $M \to M/M\mathcal{K}$. Assume that W_A is finitely generated and projective and let $\{u_i, \alpha_i\}_{i=1}^n$ be an A-coordinate system of W_A. Then $\{\overline{u}_i, \widehat{\alpha}_i\}_{i=1}^n$ is an $A/\mathcal{K}$-coordinate system of the right $A/\mathcal{K}$-module $W/W\mathcal{K}$.

Let ϕ be a symmetric linear function on A. Then $\phi'(\overline{a}) = \phi((\nu - r)a)$ for any $\overline{a} \in A/\mathcal{K}$ is well-defined and symmetric on $A/\mathcal{K}$.

Proposition 4.1 ([6], Proposition 3.8)**.** *Assume that W_A is finitely generated and projective. Let ϕ be a symmetric linear function on A. Then*

$$\phi_{W_A}(\alpha \circ (\nu - r)) = \phi'_{W/W\mathcal{K}}(\widehat{\alpha})$$

for all $\alpha \in \mathrm{End}_A(W_A)$ where $\nu - r$ is identified as an element of $\mathrm{End}_A(W_A)$.

Proof. Let $\{u_i, \alpha_i\}_{i=1}^n$ be an A-coordinate system of W_A. Then we have

$$\begin{aligned}
\phi'_{W/W\mathcal{K}}(\widehat{\alpha}) &= \phi'\left(\sum_{i=1}^{n} \widehat{\alpha}_i \circ \widehat{\alpha}(\overline{u}_i)\right) \\
&= \phi\left((\nu - r)\sum_{i=1}^{n} \alpha_i \circ \alpha(u_i)\right) \\
&= \phi\left(\sum_{i=1}^{n} \alpha_i \circ \alpha(u_i(\nu - r))\right) \\
&= \phi_{W_A}(\alpha \circ (\nu - r)).
\end{aligned}$$

□

5. Basic algebras and symmetric linear functions. Let

$$1 = \sum_{i=1}^{n}\sum_{j=1}^{n_i} e_{ij} \tag{5.1}$$

be a decomposition of the unity 1 by mutually orthogonal primitive idempotents where $e_{ij}A \cong e_{ik}A$ and $e_{ij}A \not\cong e_{k\ell}A$ for $i \ne k$. Set $e_i = e_{i1}$ for $1 \le i \le n$ and $e = \sum_{i=1}^{n} e_i$. Then $\mathbf{k}$-algebra eAe with the unity e is called a *basic algebra* associated with A. Then Ae is (A, eAe)-bimodule. Let $\ell : A \to \mathrm{End}_{eAe}(Ae_{eAe})$ and $r : eAe \to \mathrm{End}_A({}_AAe)$ be maps defined by $\ell(a)(be) = abe$ for all $a, b \in A$ and $r(eae)(be) = beae$ for all $a, b \in A$.

Lemma 5.1 ([1], Proposition 4.15, Theorem 17.8)**.**
(a) *The map r is an anti-isomorphism of algebras.*
(b) *The map ℓ is an isomorphism of algebras*

By Lemma 5.1, an element $a \in A$ is identified as an element in $\mathrm{End}_{eAe}(Ae)$ and an element $eae \in eAe$ is identified as an element in $\mathrm{End}_A(Ae)$.

Remark 5.2. By Lemma 5.1, we have two linear maps

$$(-)_{Ae_{eAe}} : \mathrm{SLF}(eAe) \to \mathrm{SLF}(A),$$
$$(-)_{{}_AAe} : \mathrm{SLF}(A) \to \mathrm{SLF}((eAe)^{\mathrm{op}}).$$

Since $\mathrm{SLF}(eAe) = \mathrm{SLF}((eAe)^{\mathrm{op}})$, the second map is in fact a map $\mathrm{SLF}(A) \to \mathrm{SLF}(eAe)$.

By (5.1), we have

$$Ae = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{n_i} e_{ij}Ae. \tag{5.2}$$

The following fact is well-known.

Lemma 5.3. *Let e and f be idempotents of A. Then the following assertions are equivalent.*
(a) $Ae \cong Af$.
(b) $eA \cong fA$.
(c) *There exist $p \in eAf$ and $q \in fAe$ such that $pq = e$ and $qp = f$.*

Lemma 5.4. *For $1 \le i \le n$ and $1 \le j \le n_i$, we have $e_iAe \cong e_{ij}Ae$ as right eAe-modules.*

Proof. By Lemma 5.3 and the fact $e_iA \cong e_{ij}A$, there exist $p_{ij} \in e_{ij}Ae_i$ and $q_{ij} \in e_iAe_{ij}$ such that $p_{ij}q_{ij} = e_{ij}$ and $q_{ij}p_{ij} = e_i$. Then the maps $\alpha : e_{ij}Ae \to e_iAe$ defined by $\alpha(e_{ij}ae) = q_{ij}ae$ and $\beta : e_iAe \to e_{ij}Ae$ defined by $\beta(e_iae) = p_{ij}ae$ are eAe-homomorphisms and are inverse each other. Thus we have shown the lemma. $\square$

For any $ae \in Ae$, it is not difficult to check that $ae = \sum_{i=1}^n \alpha_i(a)e_i$ where $\alpha_i(a) = ae_i$. Thus $\{e_i, \alpha_i\}_{i=1}^n$ is an A-coordinate system of ${}_AAe$.

By the proof of Lemma 5.1, we can see that $e_{ij}Ae_{eAe}$ is generated by $p_{ij} \in e_{ij}Ae_i$ such that $p_{ij}q_{ij} = e_{ij}$ and $q_{ij}p_{ij} = e_i$ for some $q_{ij} \in e_iAe_{ij}$. Note that we can choose $p_{i1} = q_{i1} = e_{i1} = e_i$. For any $ae \in Ae$, we set $\beta_{ij}(ea) = q_{ij}ae \in eAe$ for all $1 \le i \le n$ and $1 \le j \le n_i$. Then we have $\beta_{ij} \in \mathrm{Hom}_{eAe}(Ae, eAe)$ and $\sum_{i=1}^n \sum_{j=1}^{n_i} p_{ij}\beta_{ij}(ae) = \sum_{i=1}^n \sum_{j=1}^{n_i} e_{ij}(ae) = ae$ by (5.1). Thus $\{p_{ij}, \beta_{ij} | 1 \le i \le n,\ 1 \le j \le n_i\}$ is an eAe-coordinate system of Ae_{eAe}. In the following, we fix the A-coordinate system $\{e_i, \alpha_i\}_{i=1}^n$ of ${}_AAe$ and the eAe-coordinate system $\{p_{ij}, \beta_{ij} | 1 \le i \le n,\ 1 \le j \le n_i\}$ of Ae_{eAe}.

Lemma 5.5.
(a) *Let ϕ be a symmetric linear function on A. Then $\phi_{{}_AAe}(eae) = \phi(eae)$ for all $eae \in eAe$.*
(b) *Let ψ be a symmetric linear function on eAe. Then $\psi_{Ae_{eAe}}(a) = \psi(\sum_{i=1}^n \sum_{j=1}^{n_i} q_{ij}ap_{ij})$ for all $a \in A$.*

Proof. Since $\phi_{{}_AAe}(eae) = \sum_{i=1}^n \phi(\alpha_i(e_ieae)) = \sum_{i=1}^n \phi(e_iae_i)$ and ϕ is symmetric, we obtain $\phi(e_iAe_j) = \phi(e_je_iAe_j) = 0$ for $i \ne j$, which shows the first assertion.

The second assertion is proved as follows:

$$\begin{aligned}\psi_{Ae_{eAe}}(a) &= \sum_{i=1}^n \sum_{j=1}^{n_i} \psi(\beta_{ij}(ap_{ij})) \\ &= \sum_{i=1}^n \sum_{j=1}^{n_i} \psi(q_{ij}ap_{ij}).\end{aligned}$$

$\square$

Theorem 5.6.
(a) *Let ϕ be a symmetric linear function on A. Then $(\phi_{{}_AAe})_{Ae_{eAe}}(a) = \phi(a)$ for all $a \in A$.*
(b) *Let ψ be a symmetric linear function on eAe. Then we have $(\psi_{Ae_{eAe}})_{{}_AAe}(eae) = \psi(eae)$ for all $eae \in eAe$.*
(c) *The space of symmetric linear functions on A and the one of eAe are isomorphic as vector spaces.*

Proof. By Lemma 5.5, we have

$$\begin{aligned}(\phi_{{}_AAe})_{Ae_{eAe}}(a) &= (\phi)_{{}_AAe}\left(\sum_{i=1}^n \sum_{j=1}^{n_i} q_{ij}ap_{ij}\right) \\ &= \phi\left(\sum_{i=1}^n \sum_{j=1}^{n_i} q_{ij}ap_{ij}\right) \\ &= \phi\left(\sum_{i=1}^n \sum_{j=1}^{n_i} p_{ij}q_{ij}a\right) \\ &= \phi\left(\sum_{i=1}^n \sum_{j=1}^{n_i} e_{ij}a\right) = \phi(a)\end{aligned}$$

which shows (a).

By Lemma 5.5, we have

$$\begin{aligned}(\psi_{Ae_{eAe}})_{{}_AAe}(eae) &= \psi_{Ae_{eAe}}(eae) \\ &= \psi\left(\sum_{i=1}^n \sum_{j=1}^{n_i} q_{ij}eaep_{ij}\right) \\ &= \psi(eae),\end{aligned}$$

since $q_{i1} = p_{i1} = e_i$.

Hence we can see that two linear maps $(-)_{_AAe}$: $\mathrm{SLF}(A) \to \mathrm{SLF}(eAe)$ and $(-)_{Ae_{eAe}} : \mathrm{SLF}(eAe) \to \mathrm{SLF}(A)$ are inverse each other, which shows the last assertion. □

Remark 5.7. The statement (a) of Theorem 5.6 for $a \in \mathrm{Soc}(A)$ is found in [6, Lemma 3.9]. The statement (c) of Theorem 5.6 is well-known (see [7, 6.1]).

For $\phi \in \mathrm{SLF}(A)$, we set $\mathrm{Rad}(\phi) = \{a \in A \mid \phi(Aa) = 0\}$. Then $\mathrm{Rad}(\phi)$ is a two-sided ideal of A and ϕ induces a symmetric linear function on $A/\mathrm{Rad}(\phi)$. Note that $A/\mathrm{Rad}(\phi)$ is a symmetric algebra since ϕ is well-defined on $A/\mathrm{Rad}(\phi)$ and induces a nondegenerate symmetric associative bilinear form on $A/\mathrm{Rad}(\phi)$.

Let $A = A_1 \oplus A_2 \oplus \cdots \oplus A_\ell$ be a decomposition into two-sided ideals of A. For any $\phi \in \mathrm{SLF}(A)$, we have $\phi = \phi_1 + \phi_2 + \cdots + \phi_\ell$ where $\phi_i = \phi|_{A_i}$. Note that $\phi_i \in \mathrm{SLF}(A_i)$. If $\phi(aA) = 0$ for some $a \in A$, then we can see that $\phi_i(aA_i) \subseteq \phi(aA) = 0$.

Theorem 5.8. *Let ϕ be a symmetric linear function on A and ν a central element of A. Assume that there exists $s \in \mathbf{Z}_{>0}$ such that $\phi((\nu - r)^s a) = 0$ for any $a \in A$ and that $A = A_1 \oplus A_2 \oplus \cdots \oplus A_\ell$ is a decomposition of A into two-sided ideals. Then there exist symmetric linear functions $\phi_i \in \mathrm{SLF}(A_i)$, basic symmetric algebras P_i of $B_i = A/\mathrm{Rad}(\phi_i)$ and (A, P_i)-bimodules M_i satisfying $(\nu - r)^s M_i = 0$. Moreover,*

$$\phi(b) = \sum_{i=1}^{\ell} ((\phi_i)_{_{B_i}M_i})_{(M_i)_{P_i}}(b)$$

for all $b \in A$ where b in the right hand side is viewed as a linear map defined by the left action of $b \in A$ on each (A, P_i)-bimodule M_i.

Proof. Set $B_i = A/\mathrm{Rad}(\phi_i)$. Since $\mathrm{Rad}(\phi_i) \supseteq A_j$ for $j \neq i$, we can see that $B_i = A_i/\mathrm{Rad}(\phi_i)$. We first note that the symmetric linear function ϕ_i on B_i is well-defined and that B_i is naturally a left A-module. Let $P_i = \overline{e}_i(A/\mathrm{Rad}(\phi_i))\overline{e}_i$ be the basic algebra of $A/\mathrm{Rad}(\phi_i)$ where $\overline{e}_i$ is an idempotent of B_i. The basic algebra P_i is a symmetric algebra by [7, 10.1]. Then we set $M_i = (A/\mathrm{Rad}(\phi_i))\overline{e}_i$ which is an (A, P_i)-bimodule. By the argument before the statement of this theorem, we can see that $(\nu - r)^s \in \mathrm{Rad}(\phi_i)$ and thus $(\nu - r)^s M_i = 0$. Note that the left action of $a \in A$ defines a right P_i-module endomorphism of M_i. By Lemma 5.5, we have $\phi_i(b) = \phi_i(\overline{b}) = ((\phi_i)_{_{B_i}M_i})_{(M_i)_{P_i}}(\overline{b}) = ((\phi_i)_{_{B_i}M_i})_{(M_i)_{P_i}}(b)$ for all $b \in A_i$, which shows the theorem. □

Remark 5.9. This theorem is found in [6, Theorem 3.10]. In the proof of [6, Theorem 3.10], it is shown that a symmetric linear function on A may be written as a sum of pseudotrace maps even if A is indecomposable by using the fact $(\phi_{_AAe})_{Ae_{eAe}}(a) = \phi(a)$ for all $a \in \mathrm{Soc}(A)$ (see [6, Lemma 3.9]) in our notation. However, since $(\phi_{_AAe})_{Ae_{eAe}}(a) = \phi(a)$ for all $a \in A$, any symmetric linear function can be written by only one symmetric linear function on the endomorphism ring of the (A, P)-bimodule if A is indecomposable.

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