The best constant of Sobolev inequality corresponding to clamped-free boundary value problem for $(-1)^M(d/dx)^{2M}$

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Abstract: Green function of the clamped-free boundary value problem for $(-1)^M(d/dx)^{2M}$ on the interval $(-1,1)$ is obtained. Its Green function is a reproducing kernel for a suitable set of Hilbert space and an inner product. By using the fact, the best constant of Sobolev inequality corresponding to this boundary value problem is obtained as a function of M. The best constant is the maximal value of the diagonal value $G(y,y)$ of Green function $G(x,y)$.

Key words: Sobolev inequality; best constant; Green function; reproducing kernel; LU decomposition.

1. Conclusion. For $M = 1, 2, 3, \cdots$, we introduce Sobolev space

$$H = H(M) = \{u(x) | u(x), u^{(M)}(x) \in L^2(-1,1),$$
$$u^{(i)}(-1) = 0 \ (0 \le i \le M-1)\}$$

Sobolev inner product

$$(u,v)_M = \int_{-1}^{1} u^{(M)}(x)\overline{v}^{(M)}(x)dx$$

and Sobolev energy

$$\|u\|_M^2 = \int_{-1}^{1} \left|u^{(M)}(x)\right|^2 dx.$$

$(\cdot,\cdot)_M$ is proved to be an inner product of H in section 4. H is Hilbert space with the inner product $(\cdot,\cdot)_M$.

Our conclusion is as follows:

Theorem 1. *For any function $u(x) \in H$, there exists a positive constant C which is independent of $u(x)$ such that the following Sobolev inequality holds.*

$$(1) \qquad \left(\sup_{|y|\le 1} |u(y)|\right)^2 \le C \int_{-1}^{1} \left|u^{(M)}(x)\right|^2 dx$$

Among such C the best constant C_0 is given by

$$C_0 = C(M) = \max_{|y|\le 1} G(y,y)$$
$$= G(1,1) = \frac{2^{2M-1}}{(2M-1)((M-1)!)^2}.$$

If we replace C by C_0 in (1)*, the equality holds for*

$$u(x) = c\,G(x,1) \quad (-1 < x < 1)$$

for every complex number c.

Up to now, the best constant of Sobolev inequality has been studied by G. Bliss [1] and G. Talenti [2]. They mainly used the functional analysis technique. Papers [7] and [8] are the early studies that paid attention to a property as a reproducing kernel of Green function. The technique is quite different from that in the precedent studies. We calculated the best constant of Sobolev inequality by using Green function of the differential equation.

The engineering meaning of this result is that the square of the maximum bending of a string $(M = 1)$ or a beam $(M = 2)$ is estimated from above by the constant multiple of the potential energy [3]. We obtained the best constant of Sobolev inequality which corresponds to Dirichlet, free and periodic boundary value problems for $(-1)^M(d/dx)^{2M}$ [4–6]. The purpose of this paper is to derive Sobolev inequality which corresponds to clamped-free boundary value problem, and to obtain the best constant by using the property as the reproducing kernel of Green function.

This paper is organized as follows: In section 2, we consider a boundary value problem for $(-1)^M(d/dx)^{2M}$ with clamped-free boundary condition. In section 3, it is clarified that Green function $G(x,y)$ is the reproducing kernel for H and $(\cdot,\cdot)_M$. Finally, section 4 is devoted to the proof of Theorem 1 by using the LU decomposition method.

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2. Clamped-Free boundary value problem. We consider the following clamped-free boundary value problem.

$$\mathrm{BVP}(M)\quad \begin{cases} (-1)^M u^{(2M)} = f(x) \ (-1<x<1) \\ u^{(i)}(-1)=0 \\ u^{(M+i)}(1)=0 \ (0\le i\le M-1) \end{cases}$$

For later convenience sake, we introduce the monomials $\{K_j(x)\}$.

$$K_j(x)=K_j(M;x) = \begin{cases} x^{2M-1-j}/(2M-1-j)! & (0\le j\le 2M-1) \\ 0 & (2M\le j) \end{cases}$$

Concerning the uniqueness and existence of the solution to $\mathrm{BVP}(M)$, we have the following theorem.

Theorem 2. *For any bounded continuous function $f(x)$ on an interval $-1<x<1$, $\mathrm{BVP}(M)$ has a unique classical solution $u(x)$ expressed as follows*:

$$u(x)=\int_{-1}^{1} G(x,y)f(y)dy \ (-1<x<1).$$

Green function $G(x,y)=G(M;x,y)$ is given by the following two equivalent expressions.

$$\text{(i)}\quad G(x,y)=\frac{(-1)^M}{2}\Big[K_0(|x-y|) - (\cdots K_j \cdots)(x+1)\Big(K_{M+i+j}\Big)^{-1}(2)\begin{pmatrix}\vdots\\ K_{M+i}\\ \vdots\end{pmatrix}(1-y) - (\cdots K_j \cdots)(y+1)\Big(K_{M+i+j}\Big)^{-1}(2)\begin{pmatrix}\vdots\\ K_{M+i}\\ \vdots\end{pmatrix}(1-x)\Big]$$

$(K_{M+i+j})^{-1}(2)$ $(0\le i,j\le M-1)$ is the inverse of the $M\times M$ matrix $(K_{M+i+j})(2)$.

$$\text{(ii)}\quad G(x,y)=\frac{(-1)^M}{2}\Bigg[K_0(|x-y|) + \kappa^{-1}\Bigg\{\begin{vmatrix} K_{M+i+j}(2) & K_{M+i}(1-y) \\ K_j(x+1) & 0\end{vmatrix} + \begin{vmatrix} K_{M+i+j}(2) & K_{M+i}(1-x) \\ K_j(y+1) & 0\end{vmatrix}\Bigg\}\Bigg] \quad (-1<x,y<1)$$

where $\kappa=|K_{M+i+j}|(2)=(-1)^{M(M-1)/2}$. Two terms on the right-hand side are determinants of $(M+1)\times(M+1)$ matrices.

Proof of Theorem 2. The equivalence between (i) and (ii) of Theorem 2 follows from the following well-known lemma.

Lemma 1. *For any $N\times N$ regular matrix $\boldsymbol{A}$ and $N\times 1$ matrices $\boldsymbol{b}$ and $\boldsymbol{c}$ the following formula holds.*

$$ {}^t\boldsymbol{b}\boldsymbol{A}^{-1}\boldsymbol{c} = -\begin{vmatrix} \boldsymbol{A} & \boldsymbol{c} \\ {}^t\boldsymbol{b} & 0 \end{vmatrix} \Big/ \begin{vmatrix}\boldsymbol{A}\end{vmatrix}$$

Now we proceed to prove Theorem 2 (i). We suppose that $\mathrm{BVP}(M)$ has a classical solution $u(x)$. We introduce new functions

$$\boldsymbol{u}={}^t(u_0,\cdots,u_{2M-1})$$
$$u_i=u^{(i)} \ (0\le i\le 2M-1)$$

and $2M\times 2M$ nilpotent matrix

$$\boldsymbol{N}=(\delta_{i,j-1})=\begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0\end{pmatrix},$$

where $\delta_{i,j}$ is Kronecker delta symbol defined by

$$\delta_{i,j}=1 \quad (i=j), \quad 0 \quad (i\ne j).$$

$\mathrm{BVP}(M)$ is rewritten as follows:

$$\text{(2)}\qquad \boldsymbol{u}'=\boldsymbol{N}\boldsymbol{u}+{}^t(0,\cdots,0,1)(-1)^M f(x) \quad (-1<x<1).$$

The fundamental solution $\boldsymbol{E}(x)$ to an initial value problem:

$$\begin{cases} \boldsymbol{E}'(x)-\boldsymbol{N}\boldsymbol{E}(x) \\ \boldsymbol{E}(0)=(\delta_{i,j})\end{cases}$$

is expressed as

$$\boldsymbol{E}(x)=\boldsymbol{K}(x)\boldsymbol{K}(0)^{-1},$$

where

$$\boldsymbol{K}(x)=\Big(K_{i+j}\Big)(x) \ (0\le i,j\le 2M-1)$$

$$\boldsymbol{K}(0)=\begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}=\boldsymbol{K}(0)^{-1}.$$

Solving (2), we have

$$\boldsymbol{u}(x) = \boldsymbol{E}(x+1)\boldsymbol{u}(-1) + \int_{-1}^{x} \boldsymbol{E}(x-y)^t(0,\cdots,0,1)(-1)^M f(y)dy$$

$$\boldsymbol{u}(x) = \boldsymbol{E}(x-1)\boldsymbol{u}(1) - \int_{x}^{1} \boldsymbol{E}(x-y)^t(0,\cdots,0,1)(-1)^M f(y)dy$$

or equivalently, for $0 \le i \le 2M-1$,

$$u_i(x) = \sum_{j=0}^{2M-1} K_{i+j}(x+1)u_{2M-1-j}(-1) + \int_{-1}^{x}(-1)^M K_i(x-y)f(y)dy$$

$$u_i(x) = \sum_{j=0}^{2M-1} K_{i+j}(x-1)u_{2M-1-j}(1) - \int_{x}^{1}(-1)^M K_i(x-y)f(y)dy.$$

Employing the boundary conditions $u_i(-1) = 0$, $u_{M+i}(1) = 0$ $(0 \le i \le M-1)$, we have

$$u_i(x) = \sum_{j=0}^{M-1} K_{i+j}(x+1)u_{2M-1-j}(-1) + \int_{-1}^{x}(-1)^M K_i(x-y)f(y)dy$$

$$u_i(x) = \sum_{j=0}^{M-1} K_{M+i+j}(x-1)u_{M-1-j}(1) - \int_{x}^{1}(-1)^M K_i(x-y)f(y)dy$$

for $0 \le i \le M-1$. In particular if $i = 0$, we have

$$u_0(x) = \sum_{j=0}^{M-1} K_j(x+1)u_{2M-1-j}(-1) + \int_{-1}^{x}(-1)^M K_0(x-y)f(y)dy \tag{3}$$

$$u_0(x) = \sum_{j=0}^{M-1} K_{M+j}(x-1)u_{M-1-j}(1) - \int_{x}^{1}(-1)^M K_0(x-y)f(y)dy. \tag{4}$$

Using the boundary conditions $u_i(-1) = 0$, $u_{M+i}(1) = 0$ $(0 \le i \le M-1)$ again, we have

$$0 = u_i(-1) = \sum_{j=0}^{M-1} K_{M+i+j}(-2)u_{M-1-j}(1) - \int_{-1}^{1}(-1)^M K_i(-1-y)f(y)dy$$

$$0 = u_{M+i}(1) = \sum_{j=0}^{M-1} K_{M+i+j}(2)u_{2M-1-j}(1) + \int_{-1}^{1}(-1)^M K_{M+i}(1-y)f(y)dy.$$

Solving the above linear system of equations with respect to $u_{M-1-i}(1)$, $u_{2M-1-i}(-1)$ $(0 \le i \le M-1)$, we have

$$(u_{M-1-i})(1) = \int_{-1}^{1}(-1)^M \begin{pmatrix} K_{M+i+j} \end{pmatrix}^{-1}(-2) \begin{pmatrix} \vdots \\ K_i \\ \vdots \end{pmatrix}(-1-y)f(y)dy \tag{5}$$

$$(u_{2M-1-i})(-1) = -\int_{-1}^{1}(-1)^M \begin{pmatrix} K_{M+i+j} \end{pmatrix}^{-1}(2) \begin{pmatrix} \vdots \\ K_{M+i} \\ \vdots \end{pmatrix}(1-y)f(y)dy. \tag{6}$$

Substituting (6) and (5) into (3) and (4), we have

$$u_0(x) = -\int_{-1}^{1}(-1)^M(\cdots K_j \cdots)(x+1) \begin{pmatrix} K_{M+i+j} \end{pmatrix}^{-1}(2) \begin{pmatrix} \vdots \\ K_{M+i} \\ \vdots \end{pmatrix}(1-y)f(y)dy + \int_{-1}^{x}(-1)^M K_0(|x-y|)f(y)dy$$

$$u_0(x) = \int_{-1}^{1}(-1)^M(\cdots K_{M+j} \cdots)(x-1) \begin{pmatrix} K_{M+i+j} \end{pmatrix}^{-1}(-2) \begin{pmatrix} \vdots \\ K_i \\ \vdots \end{pmatrix}(-1-y)f(y)dy + \int_{x}^{1}(-1)^M K_0(|x-y|)f(y)dy.$$

Taking an average of the above two expressions, we have obtained the expression for a solution $u(x) = u_0(x)$ to BVP(M).

$$u(x) = \int_{-1}^{1} G(x,y)f(y)dy \ (-1 < x < 1) \tag{7}$$

where

$$G(x,y)=\frac{(-1)^M}{2}\Bigg[K_0(|x-y|)$$
$$-\big(K_0,\cdots,K_{M-1}\big)(x+1)$$
$$\bigg(K_{M+i+j}\bigg)^{-1}(2)\begin{pmatrix}K_M\\ \vdots\\ K_{2M-1}\end{pmatrix}(1-y)$$
$$+\big(K_M,\cdots,K_{2M-1}\big)(x-1)$$
$$\bigg(K_{M+i+j}\bigg)^{-1}(-2)\begin{pmatrix}K_0\\ \vdots\\ K_{M-1}\end{pmatrix}(-1-y)\Bigg]$$
$$(-1<x,y<1).$$

Theorem 2 (i) follows immediately from the relation $K_i(-x)=(-1)^{i+1}K_i(x)$ $(0\le i\le M-1)$.

Since the right-hand side of (7) includes only a data function $f(x)$, the solution to BVP(M) is unique. Using the properties (i), (ii) and (iii) of the following Theorem 3, we can show that $u(x)$ defined by (1) satisfies BVP(M), which guarantees the existence of the solution. □

Theorem 3. *Green function $G(x,y)$ satisfies the following conditions.*

(i) $\partial_x^{2M}G(x,y)=0$ $(-1<x,y<1,x\neq y)$

(ii) $\partial_x^{i}G(x,y)|_{x=-1}=0,\ \partial_x^{M+i}G(x,y)|_{x=1}=0$ $(0\le i\le M-1,\ -1<y<1)$

(iii) $\partial_x^{i}G(x,y)|_{y=x-0}-\partial_x^{i}G(x,y)|_{y=x+0}$
$$=\begin{cases}0 & (0\le i\le 2M-2)\\ (-1)^M & (i=2M-1)\end{cases}\quad(-1<x<1)$$

(iv) $\partial_x^{i}G(x,y)|_{x=y+0}-\partial_x^{i}G(x,y)|_{x=y-0}$
$$=\begin{cases}0 & (0\le i\le 2M-2)\\ (-1)^M & (i=2M-1)\end{cases}\quad(-1<y<1).$$

Proof of Theorem 3. By rewriting Green function $G(x,y)$ in the form Theorem 2 (ii), it is easy to show that $G(x,y)$ satisfies properties (i) $\sim$ (iv) through direct calculation. □

3. Reproducing kernel. In this section, it is shown that Green function $G(x,y)$ is a reproducing kernel for a set of function space H and its inner product $(\cdot,\cdot)_M$ introduced in section 1.

Theorem 4. *For any $u(x)\in H$, we have the following reproducing relation.*

(8) $$u(y)=(u(\cdot),G(\cdot,y))_M\ (-1\le y\le 1)$$

(9) $$G(y,y)=\|G(\cdot,y)\|_M^2\ (-1\le y\le 1).$$

Proof of Theorem 4. For functions $u=u(x)$ and $v=v(x)=G(x,y)$ with y arbitrarily fixed in $-1\le y\le 1$, we have

$$u^{(M)}v^{(M)}-u(-1)^Mv^{(2M)}=\left(\sum_{j=0}^{M-1}(-1)^{M-1-j}u^{(j)}v^{(2M-1-j)}\right)'.$$

Integrating this with respect to x on intervals $-1<x<y$ and $y<x<1$, we have

$$\int_{-1}^{1}u^{(M)}(x)v^{(M)}(x)dx-\int_{-1}^{1}u(x)(-1)^Mv^{(2M)}(x)dx$$
$$=\left[\sum_{j=0}^{M-1}(-1)^{M-1-j}u^{(j)}(x)v^{(2M-1-j)}(x)\right]\left\{\Big|_{x=-1}^{x=y-0}+\Big|_{x=y+0}^{x=1}\right\}$$
$$=\sum_{j=0}^{M-1}(-1)^{M-1-j}\Big[u^{(j)}(1)v^{(2M-1-j)}(1)-u^{(j)}(-1)v^{(2M-1-j)}(-1)\Big]$$
$$+\sum_{j=0}^{M-1}(-1)^{M-1-j}u^{(j)}(y)\Big[v^{(2M-1-j)}(y-0)-v^{(2M-1-j)}(y+0)\Big].$$

Using (i), (ii) and (iv) in Theorem 3, we have (8). (9) follows from (1) by $u(x)=G(x,y)$. We have proved Theorem 4. □

4. Sobolev inequality. In this section, we give a proof of Theorem 1. Applying Schwarz inequality to (8) and using (9), we have

$$|u(y)|^2\le\|G(\cdot,y)\|_M^2\|u\|_M^2=G(y,y)\|u\|_M^2.$$

Noting that

$$C_0=\max_{|y|\le 1}G(y,y)=G(y_0,y_0),$$

we have following Sobolev inequality.

(10) $$\left(\sup_{|y|\le 1}|u(y)|\right)^2\le C_0\|u\|_M^2$$

It should be noted that it requires Schwartz inequality but does not require “positive definiteness” of the inner product in order to prove (10).

In the second place, we apply this inequality to $u(x) = G(x, y_0) \in H$ and have

$$\left(\sup_{|y|\le 1} |G(y, y_0)|\right)^2 \le C_0 \|G(\cdot, y_0)\|_M^2 = C_0^2.$$

Combining this and trivial inequality

$$C_0^2 = (G(y_0, y_0))^2 \le \left(\sup_{|y|\le 1} |G(y, y_0)|\right)^2,$$

we have

$$C_0^2 \le \left(\sup_{|y|\le 1} |G(y, y_0)|\right)^2 \le C_0 \|G(\cdot, y_0)\|_M^2 = C_0^2.$$

That is to say

$$\left(\sup_{|y|\le 1} |G(y, y_0)|\right)^2 = C_0 \|G(\cdot, y_0)\|_M^2.$$

Lemma 2. *Function space H is Hilbert space with the inner product $(\cdot,\cdot)_M$.*

Proof of Lemma 2. From Sobolev inequality (10), we have $(u, u)_M \ge 0$ and $(u, u)_M = 0$ holds if and only if $\sup_{|y|\le 1} |u(y)| = 0$, that is $u(y) \equiv 0$ $(-1 \le y \le 1)$. □

The best constant of Sobolev inequality can be calculated by using the following theorem.

Theorem 5. *For any $y \in \mathbf{R}$, the equality holds.*

$$(-1)^M \left|\begin{array}{c|c} K_{M+i+j}(2) & K_{M+i}(1-y) \\ \hline K_j(1+y) & 0 \end{array}\right| + \Big/ |K_{M+i+j}|\ (2)$$

$$= \binom{2M-2}{M-1} K_0(1+y) \quad (0 \le i, j \le M-1).$$

Proof of Theorem 5. Numerator of L.H.S. $= (-1)^M$

$$\left|\begin{array}{c|c} 2^{M-i-j-1}\binom{M-i-1}{j}\frac{j!}{(M-i-1)!} & \frac{(1-y)^{M-i-1}}{(M-i-1)!} \\ \hline (1+y)^{2M-1-j}/(2M-1-j)! & 0 \end{array}\right|$$

$$= \frac{(-1)^M}{(2M-1)!} \prod_{k=0}^{M-1} \frac{1}{(M-k-1)!}$$

$$\left|\begin{array}{c|c} 2^{M-i-j-1}\binom{M-i-1}{j} j! & (1-y)^{M-i-1} \\ \hline (2M-j)_j (!+y)^{2M-1-j} & 0 \end{array}\right|,$$

where $(a)_j$ is Pochhammer's symbol defined by

$$(a)_j = \frac{\Gamma(a+j)}{\Gamma(a)} \ \ (j \ne 0, -1, -2, \cdots).$$

Replacing $(M-1-i)$-th line with i-th line $(0 \le i \le [(M-1)/2]$, we have

$$= \frac{(-1)^{\frac{(M-1)M}{2}+M}}{(2M-1)!} \prod_{k=0}^{M-1} \frac{1}{(M-k-1)!}$$

$$\left|\begin{array}{c|c} 2^{i-j}\binom{i}{j} j! & (1-y)^i \\ \hline (2M-j)_j (1+y)^{2M-1-j} & 0 \end{array}\right|. \tag{11}$$

We use the following Lemma here.

Lemma 3. *$(M+1) \times (M+1)$ matrix A defined by*

$$A = \left(\begin{array}{c|c} 2^{i-j}\binom{i}{j} j! & (1-y)^i \\ \hline a_j(y) & 0 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 0! & & \mathbf{0} & 1 \\ & \ddots & & \vdots \\ 2^{M-1} & & (M-1)! & (1-y)^{M-1} \\ \hline & a_0(y) \cdots a_{M-1}(y) & & 0 \end{array}\right)$$

where

$$a_j(y) = (2M-j)_j (1+y)^{2M-1-j}$$
$$(0 \le i, j \le M-1)$$

has the following LU decomposition

$$A = LU$$

$$L = \left(\begin{array}{c|c} & 0 \\ 2^{i-j}\binom{i}{j} & \vdots \\ & 0 \\ \hline a_j(y)/j! & 1 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 1 & & \mathbf{0} & 0 \\ & \ddots & & \vdots \\ 2^{M-1} & & 1 & 0 \\ \hline & a_0(y) \cdots \frac{a_{M-1}(y)}{(M-1)!} & & 1 \end{array}\right)$$

$$U = \left(\begin{array}{c|c} i!\delta_{i,j} & (-1)^i (1+y)^i \\ \hline 0 \cdots 0 & b(y) \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 0! & & \mathbf{0} & 1 \\ & \ddots & & \vdots \\ \mathbf{0} & & (M-1)! & (-1)^{M-1}(1+y)^{M-1} \\ \hline & 0 \cdots 0 & & b(y) \end{array}\right)$$

where

$$b(y) = (-1)^M \binom{2M-2}{M-1}(1+y)^{2M-1}.$$

Proof of Lemma 3. We can easily decompose A to the product of L and U by using the binomial expansion:

$$(1-y)^i = \sum_{j=0}^{i} (-1)^j 2^{i-j} \binom{i}{j} (1+y)^j$$
$$(0 \leq i \leq M-1)$$

and the relation

$$b(y) = -(1+y)^{2M-1} \sum_{j=0}^{M-1} \frac{(-1)^j (2M-j)_j}{j!}$$
$$= (-1)^M \binom{2M-2}{M-1}(1+y)^{2M-1}$$

where the last equality follows from the next formula; see Knuth [9, p. 165]

$$\sum_{k=0}^{m} \binom{r}{k}(-1)^k = (-1)^m \binom{r-1}{m}. \qquad \square$$

Using Lemma 3, we rewritten (11) as follows:

$$= \frac{(-1)^{\frac{(M-1)M}{2}+M}}{(2M-1)!} \prod_{k=0}^{M-1} \frac{1}{(M-k-1)!} |L||U|.$$

Since $|L| = 1$, the equality can be rewritten as follows:

$$= \frac{(-1)^{\frac{(M-1)M}{2}+M}}{(2M-1)!} \prod_{k=0}^{M-1} \frac{k!}{(M-k-1)!} b(y)$$
$$= (-1)^{\frac{(M-1)M}{2}} \binom{2M-2}{M-1} \frac{(1+y)^{2M-1}}{(2M-1)!}.$$

Since $|K_{M+i+j}|(2) = (-1)^{\frac{(M-1)M}{2}}$ and $K_0(1+y) = (1+y)^{2M-1}/(2M-1)!$, we obtain Theorem 5. $\square$

From the Lemma 3, we can calculate the maximum value of $G(y,y)$ in $-1 \leq y \leq 1$.

$$\max_{|y|\leq 1} G(y,y)$$
$$= \binom{2M-2}{M-1} \frac{1}{(2M-1)!} \max_{|y|\leq 1} (1+y)^{2M-1}$$
$$= \binom{2M-2}{M-1} \frac{2^{2M-1}}{(2M-1)!} = \frac{2^{2M-1}}{(2M-1)\Gamma(M)^2}.$$

This completes the proof of Theorem 1. $\square$

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