A remark on uniqueness theorems in an angular domain

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Abstract: In this paper, we deal with the problem of uniqueness for meromorphic functions in the whole complex plane **C** under some shared-value/set conditions in an angular domain instead of the whole plane. Results are obtained extending some results by Lin, Mori and Tohge [W. C. Lin, S. Mori and K. Tohge, Uniqueness theorems in an angular domain, Tohoku Math. J., **58** (2006), 509–527].

Key words: Uniqueness of meromorphic function; shared-set; angular domain.

1. Introduction and main results. In this paper, unless otherwise stated, we mean a meromorphic function that is defined and meromorphic in the whole complex plane **C**. We use the standard notation of Nevanlinna's value distribution theory and assume that the reader is familiar with the basic results of Nevanlinna's value distribution theory (see e.g. [6,12]). Meanwhile, the order λ, lower order μ and hyper order λ_2 of a meromorphic function $f(z)$ are defined as follows:

$$\mu := \mu(f) = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r},$$

$$\lambda := \lambda(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}$$

and

$$\lambda_2 := \lambda_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}.$$

For the sake of convenience, we use the following notations (see e.g. [9]). Let S be a nonempty subset of distinct elements in $\mathbf{C}_\infty := \mathbf{C}\cup\{\infty\}$ and $X \subseteq \mathbf{C}$. Define $E_X(S,f) = \cup_{a\in S}\{z \in X | f_a(z) = 0,$ counting multiplicities$\}$ and $\overline{E}_X(S,f) = \cup_{a\in S}\{z\in X | f_a(z) = 0,$ ignoring multiplicities$\}$, where $f_a(z) = f(z) - a$ if $a \in \mathbf{C}$ and $f_\infty(z) = \frac{1}{f(z)}$. Let f and g be two nonconstant meromorphic functions in **C**. If $E_X(S,f) = E_X(S,g)$, we say f and g share the set S CM (counting multiplicities) in X. If $\overline{E}_X(S,f) = \overline{E}_X(S,g)$, we say f and g share the set S IM (ignoring multiplicities) in X. In particular, when $S = \{a\}$, where $a \in \mathbf{C}_\infty$, we also say f and g share the value a CM in X if $E_X(S,f) = E_X(S,g)$, and we say f and g share the value a IM in X if $\overline{E}_X(S,f) = \overline{E}_X(S,g)$. When $X = \mathbf{C}$, we give the simple notations as before, $E(S,f), \overline{E}(S,f)$ and so on (see [12]). Throughout this paper, we set $S_j (j = 1,2,3)$ as $S_1 = \{0\}, S_2 = \{\infty\}$ and $S_3 = \{w | w^n(w+a) - b = 0\}$, where $n \in \mathbf{N}$, and the algebraic equation $w^n(w+a) - b = 0$ has no multiple roots.

Since R.Nevanlinna proved his four-CM and five-IM theorems, there have been many results on the uniqueness of meromorphic functions in the complex plane (see e.g. [12]). Upon the problem of uniqueness for meromorphic functions in the whole complex plane **C** under some shared-value/set conditions in the whole plane, Gross [5] posed the following question.

Question A. *Can one find two finite sets $S_i (i = 1,2)$ such that any two entire functions f and g satisfying $E(S_i,f) = E(S_i,g) (i = 1,2)$ must be identical?*

It seems that H. X. Yi first has drew the affirmative answer to above Question A completely (see [13]). In 1998, H. X. Yi [14] gave many examples that answer the above Question A and proved the following

Theorem A. *Let $n \in \mathbf{N}$ and $n \geq 2$. If f and g are two entire functions satisfying $E(S_j,f) = E(S_j,g), j = 1,3,$ then $f \equiv g$.*

For two meromorphic functions f and g satisfying $E(S_2,f) = E(S_2,g)$, H. X. Yi and W. C. Lin [15] have proved the following

Theorem B. *Let $n \in \mathbf{N}$ and $n \geq 3$. If f and g are two meromorphic functions satisfying $E(S_j,f) = E(S_j,g)$ for $j = 1,2,3$ and $\Theta(\infty,f) > 0$, then $f \equiv g$.*

In [17], J. H. Zheng firstly took into account the

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uniqueness dealing with five shared values in some angular domains of $\mathbf{C}$. After that, J. H. Zheng [16] investigated the uniqueness of transcendental meromorphic functions dealing with shared values in an angular domain instead of the whole complex plane. Following Zheng [16,17], W. C. Lin, S. Mori and K. Tohge [9] posed the following question.

Question B. *Does there exist an angular domain* $X = X(\alpha,\beta) = \{z : \alpha \le \arg z \le \beta\}, 0 < \beta - \alpha < 2\pi$ *such that* $f \equiv g$ *is always the case when* f *and* g *are two entire functions satisfying* $E_X(S_i, f) = E_X(S_i, g)(i = 1,3)$*?*

In response to Question B, Lin, Mori and Tohge [9] dealt with Theorem B under certain value/set-sharing condition in a sector instead of the whole plane $\mathbf{C}$ and proved the following theorems.

Theorem C. *Let* $n \in \mathbf{N}$ *and* $n \ge 3$. *Assume that* f *is a meromorphic function of lower order* $\mu(f) \in (\frac{1}{2}, \infty)$ *in* $\mathbf{C}$ *and* $\delta := \delta(\iota, f) > 0$ *for some* $\iota \in \mathbf{C}_\infty - \{0, -a\}$. *Then for each* $\sigma < \infty$ *with* $\mu(f) \le \sigma \le \lambda(f)$ *there exists an angular domain* $X = X(\alpha,\beta)$ *with* $0 \le \alpha < \beta$ *and*

$$\beta - \alpha > \max\left\{\frac{\pi}{\sigma}, 2\pi - \arcsin\sqrt{\frac{\delta}{2}}\right\} \tag{1}$$

such that if the conditions $E(S_1, f) = E(S_1, g)$ *and* $E_X(S_j, f) = E_X(S_j, g)(j = 2,3)$ *hold for a meromorphic function* g *in* $\mathbf{C}$ *of finite order or more generally with the growth satisfying either* $\log T(r,f) = O(\log T(r,g))$ *or*

$$\lim_{r\to\infty, r\notin E_1} \frac{\log\log T(r,g)}{\min\{\log r, \log T(r,f)\}} = 0, \tag{2}$$

where E_1 *is a set of finite linear measure, then* $f \equiv g$.

Under the condition that $\lambda(f) = \infty$, W. C. Lin, S. Mori and K. Tohge [9] obtained the following theorem.

Theorem D. *Let* $n \in \mathbf{N}$ *and* $n \ge 3$. *Assume that* f *is a meromorphic function of infinite order but* $\lambda_2(f) < \infty$ *and assume further that* $\delta := \delta(\iota, f) > 0$ *for some* $\iota \in \mathbf{C}_\infty - \{0, -a\}$. *Then there exists a direction* $\arg z = \theta$ *such that for any* $\varepsilon(0 < \varepsilon < \frac{\pi}{2})$, *if a meromorphic function* g *satisfying the growth condition* $\log T(r,g) = O(r^\tau \log rT(r,f))$, $r \notin E$ *for a constant* $\tau > 0$ *and a set* E *of finite linear measure, and* $E(S_1, f) = E(S_1, g)$ *and* $E_X(S_j, f) = E_X(S_j, g)(j = 2,3)$ *in the angular domain* $X = X(\theta - \varepsilon, \theta + \varepsilon)$, *then* $f \equiv g$.

In this paper, we also investigate Question B. We relax the growth condition of f in Theorems C, D and prove the following results.

Theorem 1. *Let* $n \in \mathbf{N}$ *and* $n \ge 3$. *Assume that* f *is a meromorphic function of order* $\lambda := \lambda(f) > \frac{1}{2}$ *in* $\mathbf{C}$ *and* $\delta := \delta(\iota, f) > 0$ *for some* $\iota \in \mathbf{C}_\infty - \{0, -a\}$. *Then there exists an angular domain* $X = X(\alpha,\beta)$ *such that if the condition* $E(S_1, f) = E(S_1, g)$ *and* $E_X(S_j, f) = E_X(S_j, g)(j = 2,3)$ *hold for a meromorphic function* g *of order* λ, *then* $f \equiv g$.

Under the condition that $\lambda(f) = \infty$, we also relax the growth condition of f in Theorem D using the following concept of a proximate order as introduced in [1,7,8].

Lemma 1. *Let* $B(r)$ *be a positive and continuous function in* $[0, +\infty)$ *which satisfies* $\limsup_{r\to\infty} \frac{\log B(r)}{\log r} = \infty$, *then there exists a continuously differentiable function* $\rho(r)$, *which satisfies the following conditions.*

(*i*) $\rho(r)$ *is continuous and nondecreasing for* $r \ge r_0 (r_0 > 0)$ *and tends to* $+\infty$ *as* $r \to +\infty$.

(*ii*) *The function* $U(r) = r^{\rho(r)} (r \ge r_0)$ *satisfies the condition*

$$\lim_{r\to+\infty} \frac{\log U(R)}{\log U(r)} = 1, \quad R = r + \frac{r}{\log U(r)}.$$

(*iii*) $\limsup_{r\to+\infty} \frac{\log B(r)}{\log U(r)} = 1.$

Lemma 1 is due to K. L. Hiong [7]. A simple proof of the existence of $\rho(r)$ was given by Chuang [3].

Definition 1. We define $\rho(r)$ and $U(r)$ in Lemma 1 by the proximate order and type function of $B(r)$ respectively. For a transcendental meromorphic function $f(z)$ of infinite order, we define its proximate order and type function as the proximate order and type function of $T(r,f)$. We denote $M(\rho(r))$ by the set of all meromorphic functions $f(z)$ in $\mathbf{C}$ such that $\limsup_{r\to+\infty} \frac{\log T(r,f)}{\log U(r)} = 1$.

We now state the second theorem of this paper.

Theorem 2. *Let* $f, g \in M(\rho(r))$, *and assume further that* $\delta(\iota, f) > 0$ *for some* $\iota \in \mathbf{C}_\infty - \{0, -a\}$. *Then there exists a direction* $\arg z = \theta$ *such that for any* $\varepsilon(0 < \varepsilon < \frac{\pi}{2})$, *if* $E(S_1, f) = E(S_1, g)$ *and* $E_X(S_j, f) = E_X(S_j, g)(j = 2,3)$ *in the angular domain* $X = X(\theta - \varepsilon, \theta + \varepsilon)$, *then* $f \equiv g$.

2. Some Lemmas. Our proof requires the Nevanlinna theory of meromorphic functions defined in an angular domain (see [10]). For the sake of convenience, we recall some notation and definitions. Let $f(z)$ be a meromorphic function on the closed angular domain $\overline{X} := \overline{X}(\alpha,\beta) = \{z : \alpha \le \arg z \le \beta\} \cup S_1 \cup S_2$, where $0 < \beta - \alpha \le 2\pi$. Nevanlinna defined the following notation (also see [1,4]).

$$A_{\alpha\beta}(r,f) := \frac{k}{\pi}\int_1^r \left(\frac{1}{t^k} - \frac{t^k}{r^{2k}}\right)\{\log^+|f(te^{i\alpha})| + \log^+|f(te^{i\beta})|\}\frac{dt}{t},$$

$$B_{\alpha\beta}(r,f) := \frac{2k}{\pi r^k}\int_\alpha^\beta \log^+|f(te^{i\alpha})|\sin k(\theta-\alpha)d\theta,$$

$$C_{\alpha\beta}(r,f) := 2\sum_{b\in\triangle}\left(\frac{1}{|b_v|^k} - \frac{|b_v|^k}{r^{2k}}\right)\sin k(\beta_v - \alpha),$$

where $k = \frac{\pi}{\beta-\alpha}$, $1 \le r < \infty$ and the summation $\sum_{b\in\triangle}$ is taken over all poles $b = |b|e^{i\theta}$ of the function $f(z)$ in the sector $\triangle := \{z : 1 < |z| < r, \alpha < \arg z < \beta\}$, counting multiplicities. The corresponding notation $\overline{C}_{\alpha\beta}(r,f)$ then applies to distinct poles. The notation $C_{2,\alpha\beta}(r,f)$ is the counting function of a simple pole is counted once and a multiple pole is counted twice. Furthermore, for $r > 1$, we define

$$D_{\alpha\beta}(r,f) = A_{\alpha\beta}(r,f) + B_{\alpha\beta}(r,f),$$
$$S_{\alpha\beta}(r,f) = C_{\alpha\beta}(r,f) + D_{\alpha\beta}(r,f).$$

For sake of simplicity, we omit the subscript in all notations and use $A(r,f)$, $B(r,f)$, $C(r,f)$, $D(r,f)$ and $S(r,f)$ instead of $A_{\alpha\beta}(r,f)$, $B_{\alpha\beta}(r,f)$, $C_{\alpha\beta}(r,f)$, $D_{\alpha\beta}(r,f)$ and $S_{\alpha\beta}(r,f)$, respectively. We shall give some properties of $S(r,f)$ as follows:

Lemma 2. [4] *Let f be a nonconstant meromorphic function in $\mathbf{C}$ and $X = X(\alpha,\beta)$ be an angular domain, where $0 < \beta-\alpha \le 2\pi$. Then,*
(i) For any value $a \in \mathbf{C}$, we have

$$S\left(r, \frac{1}{f-a}\right) = S(r,f) + O(1).$$

(ii) If f is of finite order, then $Q(r,f) = A(r,\frac{f'}{f}) + B(r,\frac{f'}{f}) = O(1)$.

If $f \in M(\rho(r))$, then (see e.g. [8,11]) $Q(r,f) = A(r,\frac{f'}{f}) + B(r,\frac{f'}{f}) = O(\log U(r))$.

Lemma 3. [4] *Let P be a polynomial of degree $d > 0$, and f be a nonconstant meromorphic function in $\overline{X} = \overline{X}(\alpha,\beta)$. Then $S(r,P(f)) = dS(r,f) + O(1)$.*

Lemma 4. [9] *Let f and g be two nonconstant meromorphic functions in $\mathbf{C}$ such that $f(z)$ and $g(z)$ share $1,\infty$ CM in $X = X(\alpha,\beta)$. Then, one of the following three cases holds:*

(i) $S(r) \le C_2(r,\frac{1}{f}) + C_2(r,\frac{1}{g}) + 2\overline{C}(r,f) + Q(r,f) + Q(r,g)$;

(ii) $f \equiv g$;

(iii) $fg \equiv 1$, where $S(r) = \max\{S(r,f), S(r,g)\}$, $Q(r,f)$ and $Q(r,g)$ as defined in Lemma 2.

Lemma 5. [8] *Let f be a meromorphic function in $\overline{X} = \overline{X}(\alpha,\beta)$, and $0 \le \alpha < \beta \le 2\pi$. Then*

$$(q-2)S(r,f) \le \sum_{i=1}^q \overline{C}\left(r, \frac{1}{f-a_i}\right) + Q(r,f),$$

where $Q(r,f)$ as defined in Lemma 2.

Lemma 6. [9] *Let f and g be two nonconstant meromorphic functions in $\mathbf{C}$ and $X = X(\alpha,\beta)$ be an angular domain, where $0 < \beta-\alpha \le 2\pi$. Assume that $\overline{E}_X(S_1,f) = \overline{E}_X(S_1,g)$, $E_X(S_j,f) = E_X(S_j,g) (j = 2,3)$ and $f^n(f+a) \ne g^n(g+a) (n \ge 2)$, then*

(i) $\overline{C}(r,\frac{1}{f}) = \overline{C}(r,\frac{1}{9}) = Q(r,f) + Q(r,g)$,

(ii) $C(r,f) = C(r,g) \le \frac{1}{n}(S(r,f) + S(r,g)) + Q(r,f) + Q(r,g)$.

Lemma 7. [9] *Suppose that $\overline{E}(S_1,f) = \overline{E}(S_1,g)$ and $\delta(\iota,f) > 0$ for some $\iota \in \mathbf{C}_\infty - \{0,-a\}$. If $f^n(f+a) \equiv g^n(g+a) (n \ge 2)$, then $f \equiv g$.*

Moreover, we need the following definition of a Borel direction of a function of infinite order (see [1]).

Definition 2. Assume that $f \in M(\rho(r))$. A direction $\arg z = \theta (0 \le \theta < 2\pi)$ from the origin is called a Borel direction of order $\rho(r)$, if for arbitrary $\varepsilon > 0$, we have

$$\limsup_{r\to+\infty} \frac{\log n(r, X_{\theta,\varepsilon}, f=a)}{\log r^{\rho(r)}} = 1,$$

for all but at most two $a \in \mathbf{C}_\infty$, where $n(r, X_{\theta,\varepsilon}, f = a)$ is the number of the roots of $f(z) = a$ in $\{|z| < r\} \cap X_{\theta,\varepsilon}$ and $X_{\theta,\varepsilon} := X(\theta-\varepsilon, \theta+\varepsilon)$.

The following Lemma was proved by Chuang Chi-tai [2].

Lemma 8. *Assume that $f \in M(\rho(r))$. A direction $\arg z = \theta (0 \le \theta < 2\pi)$ is a Borel direction of order $\rho(r)$, if and only if for arbitrary $\varepsilon > 0$, in the angular domain $\overline{X}_\varepsilon$, we have*

$$\limsup_{r\to+\infty} \frac{\log S(r,f)}{\log r^{\rho(r)}} = 1.$$

2.1. Proof of Theorem 2.

Proof. It is well known that a meromorphic function $f \in M(\rho(r))$ has at least one Borel direction $\arg z = \theta (0 \le \theta < 2\pi)$ of order $\rho(r)$. In the following, we prove that the direction $\arg z = \theta$ satisfies Theorem 2. For any $\varepsilon (0 < \varepsilon < \frac{\pi}{2})$, let $X := X_{\theta,\varepsilon}$, then by Lemma 8 we get that

$$\limsup_{r\to+\infty}\frac{\log S(r,f)}{\log r^{\rho(r)}}=1 \tag{3}$$

holds in the angular domain $\overline{X}$. Let

$$F=\frac{f^n(f+a)}{b},\quad G=\frac{g^n(g+a)}{b},n\geq 3.$$

Then F and G share 1 and ∞ CM in X. Assume that $FG\equiv 1$. Then

$$f^n(f+a)g^n(g+a)\equiv b^2$$

which implies that $0,-a$ and ∞ are all Picard exceptional values of f in X. This contradicts with $\arg z=\theta$ is a Borel direction of $f(z)$.

Suppose that $F\not\equiv G$. Then Lemma 6 implies that

$$\overline{C}\left(r,\frac{1}{f}\right)=\overline{C}\left(r,\frac{1}{g}\right)=Q(r,f)+Q(r,g). \tag{4}$$

Therefore, by the expression of F and G and (4) we have

$$\begin{aligned}&C_2(r,\tfrac{1}{F})+C_2(r,\tfrac{1}{G})+2\overline{C}(r,F)\\&\quad\leq C(r,\tfrac{1}{f+a})+C(r,\tfrac{1}{g+a})\\&\quad+2\overline{C}(r,f)+Q(r,f)+Q(r,g).\end{aligned} \tag{5}$$

Set $S_1(r):=\max\{S(r,f),S(r,g)\}$. Then, from the expression of F and G and Lemma 3, we have

$$S(r)=(n+1)S_1(r)+O(1), \tag{6}$$

where $S(r):=\max\{S(r,F),S(r,G)\}$. By (5), (6), Lemmas 2 and 3, we get

$$\begin{aligned}&C_2(r,\tfrac{1}{F})+C_2(r,\tfrac{1}{G})+2\overline{C}(r,F)\\&\quad\leq(2+\tfrac{4}{n})S_1(r)+Q(r,f)+Q(r,g)\\&\quad\leq\tfrac{2+\frac{4}{n}}{n+1}S(r)+Q(r,F)+Q(r,G).\end{aligned} \tag{7}$$

Since $n\geq 3$, then $\frac{2+\frac{4}{n}}{n+1}<1$. We can see from (7) and Lemma 4 that $S_1(r)\leq Q(r,f)+Q(r,g)$. By Lemma 2 (ii), we have

$$S(r,f)=O(\log U(r)), \tag{8}$$

which leads to a contradiction for (3). Hence $F\equiv G$ and the theorem follows from Lemma 7. This completes the proof of the Theorem 2. □

2.2. Proof of Theorem 1.

Proof. We distinguish two cases.

Case I. $\lambda(g)=\lambda(f)=\infty$. By Lemma 1, there exits $\rho(r)$ such that $f(z),g(z)\in M(\rho(r))$. By Theorem 2, we can see that Theorem 1 holds in this case.

Case II. $\lambda(g)=\lambda(f)\in(\frac{1}{2},\infty)$. Put $\sigma:\frac{1}{2}<\sigma<\lambda(f)$. For given angular domain $X=X(\alpha,\beta)$, $\beta-\alpha=\frac{\pi}{\sigma}$, we have $\omega=\frac{\pi}{\beta-\alpha}=\sigma<\lambda(f)$. Without loss of generality, we may assume that $f(z)$ has at least one Borel direction in the angular domain $X(\alpha+\varepsilon,\beta-\varepsilon)(0<\varepsilon<\frac{\pi}{2})$. Hence, there exists a finite complex number a such that

$$\limsup_{r\to\infty}\frac{\log n(r,X(\alpha+\varepsilon,\beta-\varepsilon),f=a)}{\log r}>\omega. \tag{9}$$

Let F and G be defined as in the proof of Theorem 2. If $FG\equiv 1$, then

$$f^n(f+a)g^n(g+a)\equiv b^2,$$

which implies that $0,-a$ and ∞ are all Picard exceptional values of f in X. By Lemmas 2 and 5 we get

$$S(r,f)=O(1). \tag{10}$$

For any $a\in\mathbf{C}$, let $b_v=|b_v|e^{i\beta_v}$ $(v=1,2,\ldots)$ be the roots of $f=a$ in the angular domain $X(\alpha+\varepsilon,\beta-\varepsilon)$, counting multiplicities. Put $n(r)=n(r,X(\alpha+\varepsilon,\beta-\varepsilon),f=a)$. From the Lemma 2 (i), it follows that

$$\begin{aligned}S(2r,f)&\geq C(2r,a)+O(1)\\&=2\sum_{1<|b_v|<2r,\alpha<\beta_v<\beta}\left(\tfrac{1}{|b_v|^k}-\tfrac{|b_v|^k}{(2r)^{2k}}\right)\sin k(\beta_v-\alpha)\\&\quad+O(1)\\&\quad\geq 2\sin(k\varepsilon)\sum_{1<|b_v|<2r,\alpha+\varepsilon<\beta_v<\beta-\varepsilon}\left(\tfrac{1}{|b_v|^k}-\tfrac{|b_v|^k}{(2r)^{2k}}\right)\\&\quad+O(1)\\&\quad\geq 2(1-4^{-k})\sin(k\varepsilon)\tfrac{n(r)}{r^k}+O(1),\end{aligned}$$

where $k=\frac{\pi}{\varepsilon}=\omega$. Then on combining (10), for any $a\in\mathbf{C}$ we have

$$n(r,X(\alpha+\varepsilon,\beta-\varepsilon),f=a)=O(r^k)=O(r^\omega), \tag{11}$$

when r is sufficiently large. This contradicts with (9) and hence $FG\not\equiv 1$. Suppose that $F\not\equiv G$, as we did in the proof of Theorem 2, we can see that

$$S(r,f)=O(1). \tag{12}$$

By a similar argument as above, (12) yields a contradiction. Hence $F\equiv G$ and the theorem follows from Lemma 7 in this case. This completes the proof of the Theorem 1. □

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