A note on norm estimates of the numerical radius

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Abstract: For a bounded linear operator A on a Hilbert space $\mathcal{H}$, let $\|A\|$ denote the operator norm and $w(A)$ the numerical radius. It is well-known that

$$\frac{1}{2}\|A\| \leqq w(A) \leqq \|A\|.$$

For equalities, we consider linear operators A with $A^2 = 0$ and normaloid matrices.

Key words: Numerical radius; normaloid matrix.

1. Introduction. Let $\mathcal{H}$ be a complex Hilbert space. Let $B(\mathcal{H})$ denote the set of bounded linear operators on $\mathcal{H}$. For $A \in B(\mathcal{H})$, we denote the operator norm by $\|A\|$, the spectral radius by $r(A)$, and the numerical radius by $w(A)$:

$$r(A) := \sup\{|\lambda| : \lambda \in \sigma(A)\},$$

where $\sigma(A)$ is the spectrum of A, and

$$w(A) := \sup\{|\langle Ax, x\rangle| : x \in \mathcal{H}, \|x\| = 1\}.$$

It is known that

$$\frac{1}{2}\|A\| \leqq w(A) \leqq \|A\|, \quad r(A) \leqq w(A).$$

(See [2], for instance.)

The purpose of this note is to study equalities in these inequalities and related topics; although most of the results are known to specialists, we include them for this note to be self-contained.

In section 2, we study operators $A \in B(\mathcal{H})$ with $A^2 = 0$ and equality $\|A\| = 2w(A)$. In section 3, we recall Pták's theorem and observe normaloid matrices and equality $\|A\| = w(A)$. We also show that for $A \in M_2(\mathbf{C})$

$$\|A^2\| = \|A\|^2 \Longleftrightarrow A \textit{ is normal.}$$

2. Operator A with $A^2 = 0$. In this section, we prove:

Theorem 2.1. *For $A \in B(\mathcal{H})$*

$$A^2 = 0 \Longrightarrow \|A\| = 2w(A).$$

Bouldin [1, Theorem 1] gives the estimation of $w(A)$ in terms of the angle between the ranges of A and A^*, and as a corollary [1, Corollary 2] he has Theorem 2.1 from the equivalent condition that the ranges are orthogonal. See also [2, Theorem 1.3-4]. Kittaneh [5, Corollary 1] shows Theorem 2.1 as a corollary of his norm inequality

$$w(A) \leqq \frac{1}{2}(\|A\| + \|A^2\|).$$

Haagerup and de la Harpe [3] observe nilpotent operators A with $A^n = 0$ and show

$$w(A) \leqq \|A\| \cos\frac{\pi}{n+1}.$$

In particular, when $A^2 = 0$, Theorem 2.1 follows.

We give an alternate proof using a block matrix description of A:

Proof. Let $\mathcal{M} := \overline{\operatorname{ran}(A^*)}$. On the orthogonal decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$, let us consider a block matrix representation of A:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}.$$

Then A^* is of the form

$$A^* = \begin{bmatrix} A_{11}^* & A_{21}^* \\ A_{12}^* & A_{22}^* \end{bmatrix}.$$

Since $\mathcal{M}^\perp = \ker(A)$, $A_{12} = A_{22} = 0$. By assumption, $(A^*)^2 = 0$; hence, $A^* = 0$ on $\mathcal{M}$. This means that $A_{11}^* = A_{12}^* = 0$. Therefore, we have

$$A = \begin{bmatrix} 0 & 0 \\ A_{21} & 0 \end{bmatrix}.$$

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This representation implies that

$$\|A\| = \|A_{21}\| = \sup\{|\langle A_{21}x, y\rangle| : x \in \mathcal{M}, y \in \mathcal{M}^{\perp}, \|x\| = 1, \|y\| = 1\}.$$

Hence, we have

$$\begin{aligned} w(A) &= \sup\left\{\frac{|\langle A(x\oplus y), x\oplus y\rangle|}{\|x\oplus y\|^2} : x \in \mathcal{M}, y \in \mathcal{M}^{\perp}, x\oplus y \neq 0\right\} \\ &= \sup\left\{\frac{|\langle A_{21}x, y\rangle|}{\|x\|^2 + \|y\|^2} : x \in \mathcal{M}, y \in \mathcal{M}^{\perp}, \|x\|^2 + \|y\|^2 \neq 0\right\} \\ &= \sup\left\{\frac{|\langle A_{21}x, y\rangle|}{\|x\|^2 + \|y\|^2} : x \in \mathcal{M}, y \in \mathcal{M}^{\perp}, \|x\| = \|y\| \neq 0\right\} \\ &= \sup\left\{\frac{|\langle A_{21}x, y\rangle|}{2} : x \in \mathcal{M}, y \in \mathcal{M}^{\perp}, \|x\| = \|y\| = 1\right\} \\ &= \frac{\|A\|}{2}. \end{aligned}$$

Here, the first equality follows from the definition of $w(A)$, the third one from the arithmetic-geometric inequality, and the others from preceding arguments. Therefore, the proof is complete. □

3. Normaloid matrices. An operator $A \in B(\mathcal{H})$ is said to be *normaloid* if $\|A\| = r(A)$. We recall the well-known fact (see [2,4]):

Proposition 3.1. *For $A \in B(\mathcal{H})$, the following are equivalent*:
(i) $\|A\| = r(A)$;
(ii) $\|A^n\| = \|A\|^n$ $(\forall n \in \mathbf{N})$;
(iii) $\|A\| = w(A)$.

In this section, we assume that $\dim \mathcal{H} < \infty$ so that we present statements in terms of matrices; let $M_n(\mathbf{C})$ denote the set of n-square complex matrices.

We have a characterization of normaloid matrices:

Proposition 3.2. *For $A \in M_n(\mathbf{C})$ with $\|A\| = 1$, A is normaloid if and only if there is a reducing subspace $\mathcal{K}$ $(\subseteqq \mathbf{C}^n)$ such that $A_{|\mathcal{K}}$ is unitary.*

Proof. Sufficiency is clear. Necessity: by assumption, we have a unit eigenvector $x \in \mathbf{C}^n$ for an eigenvalue λ $(|\lambda| = 1)$. Since

$$\begin{aligned} \|A^*x - \overline{\lambda}x\|^2 &= \|A^*x\|^2 - 2\,\mathrm{Re}\langle A^*x, \overline{\lambda}x\rangle + \|x\|^2 \\ &\leqq \|x\|^2 - 2\,\mathrm{Re}\langle x, \overline{\lambda}Ax\rangle + \|x\|^2 = 0, \end{aligned}$$

$A^*x = \overline{\lambda}x$: that is, x is a normal eigenvector. Hence, the subspace $\mathcal{K} := \mathbf{C}x$ reduces A, and the restriction of A to $\mathcal{K}$ is unitary. Therefore, we get the conclusion. □

Corollary 3.3. *For $A \in M_2(\mathbf{C})$,*

$$A \textit{ is normal} \Longleftrightarrow A \textit{ is normaloid.}$$

Note that this result is generalized: in fact,

$$\textit{spectraloid} \Longleftrightarrow \textit{normal}$$

on $M_2(\mathbf{C})$ and a short proof using Schur's lemma can be seen in [2, Theorem 6.5-1]. See [2] for related results on $M_3(\mathbf{C})$ and $M_4(\mathbf{C})$.

Proof. We assume that $\|A\| = 1$. We show the implication $\Leftarrow$. In Proposition 3.2, if $\mathcal{K}$ is of dimension 2, A itself is unitary, hence A is normal. If $\mathcal{K}$ is of dimension 1, then so is the orthocomplement $\mathcal{K}^{\perp}$. Therefore, $A = A_{\mathcal{K}} \oplus A_{\mathcal{K}^{\perp}}$ is normal. □

For $A \in M_n(\mathbf{C})$, we recall Pták's theorem without proof:

Theorem 3.4. (Pták [6])**.** *For $A \in M_n(\mathbf{C})$*

$$\|A^n\| = \|A\|^n \Longleftrightarrow A \textit{ is normaloid.}$$

Combining this with Corollary 3.3, we have

Corollary 3.5. *For $A \in M_2(\mathbf{C})$*

$$\|A^2\| = \|A\|^2 \Longleftrightarrow A \textit{ is normal.}$$

Halmos [4, p.110] says that the implication $\Rightarrow$ follows from "an unpleasant computation", and its proof is omitted. Here we present an alternate proof which seems to be simpler.

Proof. Assume that $\|A\| = \|A^2\| = 1$. Then we have a unit vector $x \in \mathbf{C}^2$ such that

$$\|A^2x\| = \|x\| = 1,$$

from which it follows that $\|Ax\| = 1$. Since

$$\langle(1 - A^*A)x, x\rangle = 0, \quad \langle(1 - A^*A)Ax, Ax\rangle = 0,$$

and

$$1 - A^*A \geqq 0,$$

we have

$$x, Ax \in \ker(1 - A^*A).$$

If x and Ax are linearly independent, $\ker(1 - A^*A) = \mathbf{C}^2$ or $A^*A = 1$: A is an isometry (and hence unitary) so that A is normal.

Suppose that x and Ax are linearly dependent: $Ax = \lambda x$ for some $\lambda \in \mathbf{C}$. Since $\|Ax\| = \|x\| = 1$, $|\lambda| = 1$. It follows as in the proof of Proposition 3.2 that $A^*x = \overline{\lambda}x$ or x is a normal eigenvector for A.

Hence, $\mathcal{K} := \mathbf{C}x$ reduces A and $A_{|\mathcal{K}}$ is unitary. Applying Proposition 3.2 and Corollary 3.3, we get the conclusion. □

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