Generating functions and generalized Euler numbers

By Guodong Liu

Department of Mathematics, Huizhou University, Huizhou, Guangdong, 516015, P. R. China

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Abstract: In this paper we shall give an explicit formula for the coefficient of the expansion of a given generating function raised to an arbitrary power, when that function has an appropriate form. One of the many examples is the generalized Euler numbers and we shall clarify the situation surrounding the congruence $E_{\frac{p-1}{2}} \not\equiv 0 (\mathrm{mod}\ p), p \equiv 1 (\mathrm{mod}\ 4)$, a prime, and restore the priority. At the same time we shall state the true meaning of such a congruence.

Key words: Generating functions; the generalized Euler numbers; explicit formula; congruences.

1. Generating functions. First we state notation and terminology. If $f(z)$ is a generating function (a power series) for a sequence $\{A_n\}$ (where $\{n\}$ may be a subsequence of natural numbers), we denote the sequence of coefficients of the expansion of $f(z)^x$ (with the power function taking the principal value) by $A_n^{(x)}$, where x is a fixed real number $\neq 0$:

$$f(z) = \sum_{n=0}^{\infty} \frac{A_n}{n!} z^n, \tag{1.1}$$
$$(f(z))^x = \sum_{n=0}^{\infty} \frac{A_n^{(x)}}{n!} z^n$$

absolutely convergent in a neighborhood of the origin. We shall suppress the last proviso and conduct formally-looking, legitimate manipulations in what follows.

Suppose $f(z)$ has a subsidiary generating function $g(z)$ so that

$$f(z) = (1 + g(z))^{-1} \quad \text{and} \quad |g(z)| < 1. \tag{1.2}$$

We assume further that the expansion of $g(z)^n$ starts from some terms onwards:

$$g(z)^n = \sum_{m=M(n)}^{\infty} \frac{a_m^{(n)}}{m!} z^m, \tag{1.3}$$

where $M(n)$ is a non-negative integer. Note that we tacitly assume $g(z) = \sum_{m=0}^{\infty} \frac{a_m}{m!} z^m$.

We introduce the Stirling numbers $s(n, k)$ of the first kind generated by

$$z(z-1)\cdots(z-n+1) = \sum_{k=1}^{n} s(n,k) z^k, \tag{1.4}$$

where we omit the case $n = 0$ (cf. e.g. [16]).

We may now state our Theorem.

Theorem. *Let*

$$\alpha(m,k) = (-1)^k \sum_{n=k}^{M^{-1}(m)} \frac{1}{n!} s(n,k) a_m^{(n)}, \tag{1.5}$$

where $M^{-1}(m)$ indicates the inverse function of M (in most cases, it is simply $M^{-1}(m) = m$). Then

$$A_m^{(x)} = \sum_{k=1}^{M^{-1}(m)} \alpha(m,k) x^k, \quad m \geq 1. \tag{1.6}$$

Proof. The lines of proof is quite simple: binomial expansion plus base change (1.4).

Indeed, we have first of all,

$$f(z)^x = (1 + g(z))^{-x} = \sum_{n=0}^{\infty} \binom{-x}{n} g(z)^n. \tag{1.7}$$

Using (1.3), we obtain

$$\begin{aligned} f(z)^x &= (1 + g(z))^{-x} \\ &= \sum_{n=0}^{\infty} \binom{-x}{n} \sum_{m=M(n)}^{\infty} \frac{a_m^{(n)}}{m!} z^m \\ &= \sum_{m=0}^{\infty} \frac{z^m}{m!} \sum_{n=0}^{M^{-1}(m)} \binom{-x}{n} a_m^{(n)}. \end{aligned} \tag{1.8}$$

Comparing (1.1) and (1.8), we conclude that

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$$A_m^{(x)} = \sum_{n=0}^{M^{-1}(m)} \binom{-x}{n} a_m^{(n)}. \tag{1.9}$$

Hereafter, we exclude the case $M(x) = 0$ and so the summation in (1.9) is over $n \geq 1$. Then by (1.4), we may change the base

$$\binom{-x}{n} = \frac{1}{n!} \sum_{k=1}^{n} s(n,k)(-x)^k. \tag{1.10}$$

Substituting (1.10) in (1.9), we obtain

$$A_m^{(x)} = \sum_{n=0}^{M^{-1}(m)} \sum_{k=1}^{n} \frac{(-1)^k}{n!} s(n,k) a_m^{(n)} x^k, \tag{1.11}$$

whence, by changing the order of summation, we see that the coefficient of x^k is exactly $\alpha(m,k)$, and the proof is complete. □

If we assume further that the expansion

$$\log f(z) = \sum_{n=1}^{\infty} \frac{\widetilde{A_n}}{n!} z^n, \tag{1.12}$$

holds in a neighborhood of the origin, we get the following Corollary.

Corollary. *In the case $M^{-1}(m) = m$, $\alpha(m,k)$ has another representation*

$$\alpha(m,k) = \frac{m!}{k!} \sum_{\substack{v_1,\cdots,v_k \in \mathbf{N} \\ v_1+\cdots+v_k=m}} \frac{\widetilde{A_{v_1}} \cdots \widetilde{A_{v_k}}}{v_1! \cdots v_k!}. \tag{1.13}$$

Proof. As was done in my previous papers, we note that by (1.6)

$$k!\alpha(m,k) = \frac{d^k}{dx^k} A_m^{(x)}|_{x=0}.$$

Hence, differentiating the second expression in (1.1) k-times in the form

$$e^{x \log f(z)} = \sum_{m=0}^{\infty} \frac{A_m^{(x)}}{m!} z^m$$

and putting $x = 0$, we deduce that

$$k! \sum_{m=k}^{\infty} \alpha(m,k) \frac{z^m}{m!} = (\log f(z))^k. \tag{1.14}$$

Substituting (1.12) in (1.14) and using multinomial expansion, we conclude (1.13). □

Example 1. The Nörlund polynomials $B_n^{(x)}$ and $b_n^{(x)}$ are generated by

$$\left(\frac{z}{e^z - 1}\right)^x = \sum_{n=0}^{\infty} B_n^{(x)} \frac{z^n}{n!} \tag{1.15}$$

and

$$\left(\frac{z}{\log(1+z)}\right)^x = \sum_{n=0}^{\infty} b_n^{(x)} z^n \tag{1.16}$$

(cf. [4,12] etc.) respectively.

In the case of (1.15), $f(z) = \frac{z}{e^z-1}$ and $g(z) = \frac{1}{z}(e^z - 1 - z)$. Then $g(z)^n = z^{-n}(e^z - 1 - z)^n$, and for the associated Stirling numbers $b(n,k)$ we have an expansion [12, (1.12)]:

$$(e^z - 1 - z)^k = k! \sum_{m=2k}^{\infty} b(m,k) \frac{z^m}{m!}. \tag{1.17}$$

Hence

$$g(z)^n = \sum_{m=2n}^{\infty} n! b(m,n) \frac{z^{m-n}}{m!} = \sum_{m=n}^{\infty} \frac{m!n!}{(m+n)!} b(m+n,n) \frac{z^m}{m!},$$

whence $M(n) = n$ and $a_m^{(n)} = \frac{1}{\binom{m+n}{n}} b(m+n,n)$.

Theorem gives

$$B_n^{(x)} = \sum_{k=1}^{n} \sigma(n,k) x^k, \tag{1.18}$$

where

$$\sigma(n,k) = (-1)^k \sum_{j=k}^{n} s(j,k) \frac{1}{j!\binom{n+j}{j}} b(n+j,j), \tag{1.19}$$

which is Theorem 1 of [12].

Using Carlitz's result [3], we get

$$\log f(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} B_n}{n} \frac{z^n}{n!}, \tag{1.20}$$

where B_n significs the n-th Bernoulli number.

Corollary now gives [12, Theorem 2]:

$$\sigma(n,k) = (-1)^{n-k} \frac{n!}{k!} \sum_{\substack{v_1,\cdots,v_k \in \mathbf{N} \\ v_1+\cdots+v_k=n}} \frac{B_{v_1} \cdots B_{v_k}}{(v_1 \cdots v_k) v_1! \cdots v_k!} \tag{1.21}$$

for $n \geq k$.

In the case of (1.16), we have

$$f(z) = \frac{z}{\log(1+z)} \quad \text{and} \quad g(z) = \frac{1}{z}(\log(1+z) - z).$$

Using the associated Stirling numbers $d(n,k)$ (cf. [4,12] etc.),

$$(1.22)\qquad (\log(1+z)-z)^k = k!\sum_{n=2k}^{\infty}(-1)^{n-k}d(n,k)\,\frac{z^n}{n!},$$

similarly as above we have

$$b_n^{(x)} = \sum_{k=1}^{n}\tau(n,k)x^k$$

and

$$\tau(n,k) = (-1)^{n-k}\sum_{j=k}^{n}\frac{s(j,k)d(n+j,j)}{(n+j)!},$$

which is Theorem 3 of [12]. Similarly, an expansion for $\tau(n,k)$ corresponding to (1.21) is obtained, which is Theorem 4 of [12], thus covering all the results.

Example 2 ([9,13]**).** The generalized Euler numbers $E_{2n}^{(x)}$ are defined by

$$(1.23)\qquad \left(\frac{2}{e^z+e^{-z}}\right)^x = \sum_{n=0}^{\infty}E_{2n}^{(x)}\,\frac{z^{2n}}{(2n)!}.$$

Hence $f(z)=\frac{2}{e^z+e^{-z}}=\frac{1}{\sinh z}$ and $g(z)=\frac{1}{2}(e^z+e^{-z}-2)$. Since the central factorial numbers $T(n,k)$ (cf. [14]) are generated by

$$(1.24)\qquad (e^z+e^{-z}-2)^k = (2k)!\sum_{n=k}^{\infty}T(n,k)\,\frac{z^{2n}}{(2n)!}.$$

We have

$$g(z)^n = 2^{-n}(2n)!\sum_{m=n}^{\infty}T(m,n)\,\frac{z^{2m}}{(2m)!}.$$

Hence $M(n)=n$ and

$$a_m^{(n)} = 2^{-n}(2n)!T(m,n),$$

so that by Theorem

$$(1.25)\qquad E_{2m}^{(x)} = \sum_{k=1}^{n}\rho(m,k)x^k,$$

and

$$(1.26)\qquad \rho(m,k) = (-1)^k\sum_{n=k}^{m}s(n,k)\,\frac{2^{-n}(2n)!}{n!}\,T(m,n),$$

which is Theorem 2.1 of [13].

To deduce [13, Theorem 2.3] we use another generating function

$$f_1(z)=\sec z\qquad\left(|z|<\frac{\pi}{2}\right):$$

$$(1.27)\qquad f_1(z)^x = (\sec z)^x = \sum_{n=0}^{\infty}(-1)^nE_{2n}^{(x)}\,\frac{z^{2n}}{(2n)!}.$$

Now in view of the expansion

$$(1.28)\qquad \log\sec z = \sum_{n=0}^{\infty}(-1)^{n-1}E_{2n-2}^{(2)}\,\frac{z^{2n}}{(2n)!}$$

([13, (2.11)] etc.) Corollary gives [13, Theorem 2.3]

$$(1.29)\qquad \rho(n,k) = (-1)^k\,\frac{(2n)!}{k!}\sum_{\substack{v_1,\cdots,v_k\in\mathbf{N}\\ v_1+\cdots+v_k=n}}\frac{E_{2v_1-2}^{(2)}\cdots E_{2v_k-2}^{(2)}}{(2v_1)!\cdots(2v_k)!}$$

for $n\ge k$.

We may unify many existing explicit formulas under our Theorem and Corollary and then apply them to generalized Bernoulli and Euler polynomials, which we shall carry out elsewhere.

In §2 we give a clarification of the congruence (2.1) as an application of Example 2, which has been done in [13], our treatment, however, being thorough and enlightening.

2. Location of the congruence on Euler numbers. First suppose p is a prime such that $p\equiv 1 \pmod 4$. Then there is a conjecture about Euler numbers that was posed in 1980:

$$(2.1)\qquad E_{\frac{p-1}{2}}\not\equiv 0 \pmod p.$$

In [11] we stated that Liu [10] proved (2.1) for $p\equiv 5 \pmod 8$ and Yuan [19] proved the general case using the result of [10] and the class number formula for the imaginary quadratic field $Q(\sqrt{-4p})$. As pointed out by Professors Tanigawa and Kanemitsu, this statement is historically incorrect. We would like to clarify the situation and restore the priority of Professor Ernvall [6]. According to [6], (2.1) in the case $p\equiv 5 \pmod 8$ was proved by E. Lehmer [8] prior to [10] by 66 years, and also by Ernvall [6] in an elementary way. (2.1) in the general case was proved by Ernvall [6] prior to Yuan [19] by 22 years. The proof is only 4 lines and depends on the result [5] that

$$(2.2)\qquad E_{\frac{p-1}{2}}\equiv 4\sum_{a=1}^{\frac{p-1}{4}}a^{\frac{p-1}{2}}. \pmod p.$$

If we take the following reasoning and Dirichlet's result for granted, we are to say that (2.1) were almost proved by Ernvall in 1979. Indeed by Euler's criterion, we have $a^{\frac{p-1}{2}} \equiv \left(\frac{a}{p}\right) \pmod p$, where the latter indicates the Legendre symbol. Hence the right-hand side of (2.2) amounts to the 1/4 interval character sum:

$$E_{\frac{p-1}{2}} \equiv 4 \sum_{a=1}^{\frac{p-1}{4}} \left(\frac{a}{p}\right) \pmod p. \tag{2.3}$$

Then it was already known to Dirichlet (although Ernvall refers to more recent Berndt [1]) that the 1/4 interval character sum can be expressed as $\frac{4p}{\pi} L(1, \chi'\chi_4)$, where $\chi'(a) = \left(\frac{a}{p}\right)$ and χ_4 is a non-principal Dirichlet character mod 4. This in turn reduces to $2h(-4p)$, where $h(-4p)$ designates the class number of $Q(\sqrt{-4p})$, which is a natural number. This latter fact allowed Dirichlet to prove $L(1,\chi) \neq 0$, which assures the infinitude of primes in an arithmetic progression mod p.

On the other hand, 4 times the 1/4 interval character sum implies that the right-hand side is $\leq 4\,\frac{p-1}{4} = p-1$. Hence $E_{\frac{p-1}{2}}$ is congruent to the least positive residue mod p, and *a fortiori* (2.1) must hold.

Thus, we have revealed that the deeper and proper understanding of (2.1) is that $E_{\frac{p-1}{2}}$ is congruent to 4 times the 1/4 interval character sum, which via the Dirichlet L-function, amounts to the class number of $Q(\sqrt{-4p})$.

It was also pointed out by Professor Kanemitsu that our statement about our Lemma [13]

$$\sum_{a=1}^{\frac{p-1}{4}} \left(\frac{a}{p}\right) = \frac{1}{2}\,\frac{\sqrt{|-4p|}}{\pi}\,L(1,\chi'\chi_4) = \frac{\sqrt{p}}{\pi}\,L(1,\chi'\chi_4)$$

that we gave a simple and direct elementary proof is not correct because we applied to (3.24):

$$\frac{1}{4p}\sum_{a=1}^{4p} a\chi'\chi_4(a) = -\frac{1}{\pi}\sqrt{|-4p|}L(1,\chi'\chi_4),$$

which is one of many finite expressions for $L(1,\chi)$ and of the class number of imaginary quadratic fields.

He also pointed out that Professor Berndt's important work [1] was completed by Yamamoto [18] whose method allows to express any short interval character sum in terms of a linear combination of $L(1,\chi)$ and *a fortiori*, of a linear combination of class numbers.

We note the following relation between Euler numbers and generalized Bernoulli numbers [15]. If $B_n(x)$ is the n-th Bernoulli polynomial, then the Euler polynomial is $E_n(x) = \frac{2^{n+1}}{n+1}\left(B_{n+1}(\frac{x+1}{2}) - B_{n+1}(\frac{x}{2})\right)$. Hence

$$E_n = 2^n \frac{2^{n+1}}{n+1}\left(B_{n+1}\left(\frac{3}{4}\right) - B_{n+1}\left(\frac{1}{4}\right)\right).$$

On the other hand, the generalized Bernoulli number $B_{n,\chi}$ with a primitive Dirichlet character χ mod f is defined by

$$B_{n,\chi} = f^{n-1}\sum_{a=1}^{f}\chi(a)B_n\left(\frac{a}{f}\right).$$

For $f = 4, \chi = \chi_4$, we have

$$B_{n+1,\chi_4} = 4^n\left(B_{n+1}\left(\frac{1}{4}\right) - B_{n+1}\left(\frac{3}{4}\right)\right).$$

Hence, for $n = \frac{p-1}{2}$, we have

$$\begin{aligned} E_{\frac{p-1}{2}} &\\ &= -\frac{4}{p+1}B_{\frac{p+1}{2},\chi_4} = -2L\left(-\frac{p-1}{2},\chi_4\right). \end{aligned} \tag{2.4}$$

Thus, we see that the quantity has much deeper meaning than stated by (2.1), and we are to study its property through algebraic number theoretic as well as p-adic theoretic point of view. This will be conducted elsewhere.

We now turn to the proof of (2.3) by the results on $E_{2n}^{(k)}$ in §1. The point is to use the negative exponents, say $-m, m \in \mathbf{N}$. Then by Example 2,

$$\begin{aligned} \sum_{n=0}^{\infty} E_{2n}^{(-m)}\frac{z^{2n}}{(2n)!} &= f(z)^{-m} = 2^{-m}(e^z + e^{-z})^m \\ &= 2^{-m}\sum_{j=0}^{m}\binom{m}{j}e^{(2j-m)z} \\ &= 2^{-m}\sum_{j=0}^{m}\binom{m}{j}\sum_{k=0}^{\infty}\frac{1}{k!}(2j-m)^k z^k \\ &= 2^{-m}\sum_{k=0}^{\infty}\frac{z^k}{k!}\sum_{j=0}^{m}\binom{m}{j}(2j-m)^k. \end{aligned}$$

Comparing both sides, we see that k can take only even integer values, and therefore

$$\begin{aligned} E_{2n}^{(-m)} &= 2^{-m}\sum_{j=0}^{m}\binom{m}{j}(m-2j)^{2n} \\ &= 2^{1-m}\sum_{j=0}^{[m/2]}\binom{m}{j}(m-2j)^{2n}. \end{aligned} \tag{2.5}$$

Recall from (1.26)

$$E_{2n}^{(-m)} = \sum_{j=1}^{n} \rho(n,j)(-m)^j.$$

Choosing $m = p - r$ (p an odd prime $\equiv 1 (\mathrm{mod}\ 4)$), we have on one hand, $1 \le r < p$

$$E_{2n}^{(r-p)} = \sum_{j=1}^{n} \rho(n,j)(r-p)^j \equiv \sum_{j=1}^{n} \rho(n,j) r^j = E_{2n}^{(r)} \pmod{p}. \tag{2.6}$$

On the other hand,

$$\begin{aligned} E_{2n}^{(r-p)} &\equiv 2^{p-1} E_{2n}^{(r-p)} \\ &= 2^r \sum_{j=0}^{[\frac{p-r}{2}]} \binom{p-r}{j} (p-r-2j)^{2n} \pmod{p} \end{aligned}$$

by (2.5) and Fermat's little theorem, which, in conjunction with (2.6), gives

$$E_{2n}^{(r)} \equiv 2^r \sum_{j=0}^{[\frac{p-r}{2}]} \binom{p-r}{j} (p-r-2j)^{2n} \pmod{p}. \tag{2.7}$$

Choosing $r = 1$, we obtain, on noting $\binom{p-1}{j} \equiv \binom{-1}{j} \equiv (-1)^j \pmod{p}$,

$$E_{2n} = E_{2n}^{(1)} \equiv 2 \sum_{j=0}^{\frac{p-1}{2}} (-1)^j (p-1-2j)^{2n} \pmod{p}$$

(cf. also Sun [17]).

By the change of variable $a = \frac{p-1}{2} - j$, we have

$$E_{2n} \equiv 2 \sum_{a=1}^{\frac{p-1}{2}} (-1)^a (2a)^{2n} \pmod{p},$$

and in particular, for $2n = \frac{p-1}{2}$,

$$E_{\frac{p-1}{2}} \equiv 2 \sum_{a=1}^{\frac{p-1}{2}} (-1)^a (2a)^{\frac{p-1}{2}} \pmod{p},$$

which is the first congruence of (3.26) of [13], whence we may derive (2.3) in the same way.

We now note that as is implicit in (2.6)

$$E_{2n}^{(k+m)} \equiv E_{2n}^{(k)} \pmod{m}. \tag{2.8}$$

With (2.8), we contend that (2.7) in the form

$$E_{2n}^{(r)} \equiv 2^{r-1} \sum_{j=0}^{p-r} \binom{p-r}{j} (p-r-2j)^{2n} \pmod{p} \tag{2.9}$$

for an odd prime p, is a shortcut to a proof of other results of ours. We illustrate by Theorem 1 (see [11]). Suppose $r \equiv 2k+1 \pmod{p}, 1 \le k \le \frac{p-1}{2}$. Then

$$\begin{aligned} \sum_{i=1}^{\frac{p-1}{2}} E_{2n+2i}^{(r)} &\equiv \sum_{i=1}^{\frac{p-1}{2}} E_{2n+2i}^{(2k+1)} \\ &= 2^{2k} \sum_{j=0}^{p-2k-1} \binom{p-2k-1}{j} (p-r-2j)^{2n+1} \\ &\quad \times \sum_{i=1}^{\frac{p-1}{2}} (p-2k-1-2j)^{2i} \pmod{p}. \end{aligned}$$

The inner geometric series sum to 0 and we have [11, (1.7)]:

$$\sum_{i=1}^{\frac{p-1}{2}} E_{2n+2i}^{(r)} \equiv 0 \pmod{p}.$$

Finally, suppose p is a prime $\equiv 3 \pmod 4$ and let us locate [13, (3.15)]:

$$E_{\frac{p-3}{2}}^{(2)} \equiv \frac{2\sqrt{p}}{\pi} \left(\left(\frac{2}{p}\right) - 2 \right) L(1,\chi') \pmod{p}. \tag{2.10}$$

On the other hand, by the well-known relation

$$2nE_{2n-2}^{(2)} = 2^{2n}(2^{2n}-1)B_{2n},$$

with $2n = (p+1)/2$, we deduce that

$$\begin{aligned} E_{\frac{p-3}{2}}^{(2)} &\equiv (p+1)E_{\frac{p-3}{2}}^{(2)} = 2\left(2^{p+1} - 2^{\frac{p+1}{2}}\right) B_{\frac{p+1}{2}} \\ &= 4\left(2 - \left(\frac{2}{p}\right)\right) B_{\frac{p+1}{2}} \pmod{p}. \end{aligned} \tag{2.11}$$

Hence, comparing (2.10) and (2.11), we deduce that

$$h(\sqrt{-p}) = \frac{\sqrt{p}}{\pi} L(1,\chi') \equiv -2B_{\frac{p+1}{2}} \pmod{p},$$

which is known to Voronoï and stated in Borevich and Shafarevich [2] and later reincorporated in Ireland-Rosen [7].

Remark. In a recent paper of W.-P. Zhang and Z.-F. Xu [20], they refer to the papers of myself (p. 284) and P. Yuan and attribute the priority to us, which statement is to be corrected according to section 2 above.

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