On the rotation angles of a finite subgroup of a mapping class group

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Abstract: Let G be a finite subgroup of the mapping class group of genus σ, which acts on a compact Riemann surface of genus σ. In this paper, we introduce a new method to determine the rotation angle of an element $g \in G$ around the fixed points of g. Our main result is Theorem 3.2.

Key words: Riemann surface; mapping class group; finite group; elliptic operator.

1. Introduction. Let Σ^σ be a compact Riemann surface of genus $\sigma \geq 2$ and G a subgroup of the mapping class group of genus σ. We can assume that the action of G on Σ^σ is effective and biholomorphic (see [4]). Let p be an odd prime number which divides the order $|G|$ of G. Then it follows from the Cauchy's theorem that there exists an element $g \in G$ of order p. Let $q_1, \cdots, q_b$ be the fixed point set of g and suppose that g acts on the tangent space $T_{q_i}M$ via multiplication by α^{t_i} $(1 \leq t_i < p)$ where α is the primitive p-th root of unity. Then g^s acts on the tangent space $T_{q_i}M$ via multiplication by α^{st_i}. We call $\{t_1, \cdots, t_b\}$ the rotation angle of g. Two rotation angles $\{t_1, \cdots, t_b\}$, $\{t'_1, \cdots, t'_b\}$ are defined to be equivalent iff there exists an integer s such that a permutation of $\{st'_1, \cdots, st'_b\}$ is equivalent to $\{t_1, \cdots, t_b\}$ mod.p. For example, since $3(1, 2) \equiv (3, 1) \pmod 5$, $\{1, 3\}$ is equivalent to $\{1, 2\}$ when $p = 5$. Let $\mathbf{Z}_p$ be the cyclic group generated by g and suppose that the genus of $\Sigma^\sigma/\mathbf{Z}_p$ is τ. Then it follows from the Riemann-Hurwitz equation that

$$2\sigma - 2 = p(2\tau - 2) + b(p - 1). \tag{1}$$

Set $\Sigma_0^\sigma = \Sigma^\sigma \setminus \{q_1, \cdots, q_b\}$ and $\Sigma_0^\tau = \Sigma_0^\sigma/\mathbf{Z}_p$. Then there exists an exact sequence

$$\pi_1(\Sigma_0^\sigma) \xrightarrow{\pi_*} \pi_1(\Sigma_0^\tau) \xrightarrow{\partial} \mathbf{Z}_p \quad (\pi : \Sigma_0^\sigma \longrightarrow \Sigma_0^\tau).$$

Let x_i be an element of $\pi_1(\Sigma_0^\tau)$ represented by a counterclockwise loop around $\pi(q_i)$ and $\bar{t}$ denote the mod.p-inverse of t. Then the equality $\partial(x_i) = \bar{t}_i \in \mathbf{Z}_p$ holds and we have

$$\sum_{i=1}^{b} \bar{t}_i = 0 \in \mathbf{Z}_p. \tag{2}$$

Conversely if $\tau, b, t_1, \cdots, t_b$ satisfy the conditions (1), (2), then $\mathbf{Z}_p$ acts on Σ^σ with b fixed points and the rotation angle $\{t_1, \cdots, t_b\}$ (see [2,3]). In this paper, a rotation angle $\{t_1, \cdots, t_b\}$ is called possible when $\{t_1, \cdots, t_b\}$ satisfies the conditions (1), (2).

Let $L = \otimes^\ell T\Sigma^\sigma$ be the tensor product of ℓ $T\Sigma^\sigma$'s, which is a complex G-line bundle over Σ^σ and D_ℓ the L-valued Dirac (Dolbeault) operator on Σ^σ. Then in [5] an additive group homomorphism $I_{D_\ell} : G \longrightarrow \mathbf{R}/\mathbf{Z}$ is defined by using the equivariant determinant of D_ℓ and a calculation formula for $I_{D_\ell}(g)$ is given by using the rotation angle of g. Using the formula, we can obtain a condition for rotation angle of g.

2. Admissible rotation angle. Let $g \in G$ be an element of odd prime order p and $\{t_1, \cdots, t_b\}$ the rotation angle of g.

Definition 2.1. *For integers z, ℓ such that $1 \leq z, \ell < p$, we set*

$$\Psi_p(z, \ell, t_1, \cdots, t_b)$$
$$= \frac{(p-1)(1-\sigma)(2\ell+1)}{2p}$$
$$+ \frac{1}{12p} \sum_{i=1}^{b} \Bigg\{ \{(p-1)(7p-11)zt_i$$

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$$+ 6 \sum_{j=[\frac{(\ell+1)zt_i}{p}]+1}^{[\frac{(\ell+p+1)zt_i}{p}]} f_p\left(\left[\frac{jp-1}{zt_i}\right] - \ell - 1\right)\Bigg\}$$

where $f_p(x) = x^2 - (p-2)x - (p-1)^2$ *and* $[\ \ ]$ *is the Gauss' symbol. In this paper, a rotation angle* $\{t_1, \cdots, t_b\}$ *is called admissible when* $\{t_1, \cdots, t_b\}$ *is possible and* $\Psi_p(z, \ell, t_1, \cdots, t_b)$ *is an integer for any* $1 \leq z, \ell < p$.

Note that $\Psi_p(z, \ell, t_1, \cdots, t_b) \equiv I_{D_\ell}(g^z) \pmod{\mathbf{Z}}$ (see [5, Proposition 3.2]).

Example 2.2. *Set* $p = 7, \sigma = 9$. *Then direct computation shows that a possible rotation angle is equivalent to one of* $\{1,1,1,1,5\}$, $\{1,1,1,3,6\}$, $\{1,1,1,4,4\}$, $\{1,1,2,3,5\}$, $\{1,1,2,4,6\}$, $\{1,1,3,3,4\}$, *in which only* $\{1,1,2,4,6\}$ *is an admissible angle.*

Example 2.3. *Set* $\sigma = p - 1$ *and let* $f(p)$, $g(p)$ *be the numbers of equivalence classes of possible rotation angles and admissible rotation angles respectively. Then direct computation shows that*

$$g(3)/f(3) = 1/1,\ g(5)/f(5) = 2/3,$$
$$g(7)/f(7) = 2/4,\ g(11)/f(11) = 3/8.$$

3. Main Results.

Let G be a finite subgroup of the mapping class group of genus σ and g an element of G of prime order p.

Definition 3.1. *In this paper,* $h \in G$ *is called a free ordering of* $g \in G$ *if, for some* n, *there exist* $\gamma_1, \gamma_2, \cdots, \gamma_n \in G$ *such that* $g = \gamma_1\gamma_2\cdots\gamma_n$ *and* $h = \gamma_{\rho(1)}\gamma_{\rho(2)}\cdots\gamma_{\rho(n)}$ *for some permutation* ρ *on* n *letters. If* h *is a free ordering of* g, *it is denoted by* $g \xrightarrow{\text{FO}} h$.

For example, $\gamma_3\gamma_2\gamma_1^2$ is a free ordering of $\gamma_1\gamma_2\gamma_3\gamma_1$ and denoted by $\gamma_1\gamma_2\gamma_3\gamma_1 \xrightarrow{\text{FO}} \gamma_3\gamma_2\gamma_1^2$. Then we have the next theorem.

Theorem 3.2. *Assume that* $\gamma_1\cdots\gamma_n = 1$ *for* $\gamma_1, \cdots, \gamma_n \in G$ *and that a free ordering of* $\gamma_1\cdots\gamma_n$ *is equal to* g^q *for a natural number* q *which is not a multiple of* p. *Then the rotation angle* $\{t_1, \cdots, t_b\}$ *of* g *is admissible.*

Proof. Since I_{D_ℓ} is an additive group homomorphism, it follows from the assumption that $qI_{D_\ell}(g) = I_{D_\ell}(g^q) = I_{D_\ell}(\gamma_1\cdots\gamma_n) = I_{D_\ell}(1) = 0 \in \mathbf{R}/\mathbf{Z}$. Moreover since $pI_{D_\ell}(g) = I_{D_\ell}(g^p) = I_{D_\ell}(1) = 0 \in \mathbf{R}/\mathbf{Z}$ and q is not a multiple of p, it follows that $I_{D_\ell}(g) = 0 \in \mathbf{R}/\mathbf{Z}$. Now the result of the theorem follows from Proposition 3.2 in [5]. □

Corollary 3.3. *Assume that* g^q *is contained in the commutator subgroup* $[G, G]$ *for a natural number* q *which is not a multiple of* p. *Then the rotation angle* $\{t_1, \cdots, t_b\}$ *of* g *is admissible.*

Proof. It follows from the assumption that there exists elements $\gamma_1, \gamma_2 \in G$ such that $\gamma_1^{-1}\gamma_2^{-1}\gamma_1\gamma_2 = g^q$. Since $\gamma_1^{-1}\gamma_1\gamma_2^{-1}\gamma_2 = 1$, the result of the corollary immediately follows from the theorem above. □

Example 3.4. *Let* G *be a perfect group whose order is divided by an odd prime number* p *and* $g \in G$ *an element of order* p. *Then it follows from the corollary above that the rotation angle of* g *is admissible.*

Example 3.5. *Let* D_n *be the dihedral group generated by* γ, τ *with the relation* $\gamma^n = \tau^2 = 1, \tau^{-1}\gamma\tau = \gamma^{-1}$. *Let* p *be an odd prime number which divides* n *and set* $m = n/p$. *Then the order of* $g = \gamma^m$ *is* p *and we have*

$$1 = (\tau^{-1}\gamma\tau)^m\gamma^m = \tau^{-1}g\tau g \xrightarrow{\text{FO}} \tau^{-1}\tau g^2 = g^2.$$

Hence the rotation angle of g *is admissible.*

Remark 3.6. *It follows from Corollary* 2.5 *in* [1] *that the dihedral group* D_p *with odd prime* p *acts on* Σ^{p-1}. *(See Example* 2.3.*)*

Example 3.7. *Let* S_n *be the symmetric group of* $n \geq 3$ *letters* $1, 2, \cdots, n$. *Then* S_n *is generated by transpositions and the order* $|S_n|$ *of* S_n *is* $n!$. *Let* p *be an odd prime number which is less than or equal to* n *and* $g \in S_n$ *an element of order* p. *Suppose that* $g = \tau_1\cdots\tau_m$ *for transpositions* $\tau_1, \cdots, \tau_m$. *Then we have*

$$1 = \tau_1^2\cdots\tau_m^2 \xrightarrow{\text{FO}} \tau_1\cdots\tau_m\tau_1\cdots\tau_m = g^2.$$

Hence the rotation angle of g *is admissible.*

References

[1] E. Bujalance et al., On compact Riemann surfaces with dihedral groups of automorphisms, Math. Proc. Cambridge Philos. Soc. **134** (2003), no. 3, 465–477.

[2] H. Glover and G. Mislin, Torsion in the mapping class group and its cohomology, J. Pure Appl. Algebra **44** (1987), no. 1–3, 177–189.

[3] W. J. Harvey, Cyclic groups of automorphisms of a compact Riemann surface, Quart. J. Math. Oxford Ser. (2) **17** (1966), 86–97.

[4] S. P. Kerckhoff, The Nielsen realization problem, Ann. of Math. (2) **117** (1983), no. 2, 235–265.

[5] K. Tsuboi, The finite group action and the equivariant determinant of elliptic operators, J. Math. Soc. Japan **57** (2005), no. 1, 95–113.