A homological characterization of surface coverings

By Masaharu Tanabe

Tokyo Institute of Technology, Department of Mathematics, Ohokayama, Meguro, Tokyo 152-8551, Japan

(Communicated by Heisuke Hironaka, m.j.a., Oct. 14, 2008)

Abstract: Let $f : X \to Y$ be a covering of closed oriented surfaces. Then f induces a homomorphism $f_* : H_1(X, \mathbf{Z}) \to H_1(Y, \mathbf{Z})$ of the first homology groups. We consider the converse and characterize—in terms of matrices—the abstractly given homomorphisms of the first homology groups which can be induced by coverings of prime degree. We also classify the induced homomorphisms in these cases.

Key words: Homology groups; Riemann surfaces; surface coverings.

1. Introduction. Throughout this paper, all of the surfaces will be closed, oriented and of gerera ≥ 1. Let X and Y be surfaces of genera g and γ, respectively. Let $f : X \to Y$ be a covering, possibly ramified. Then f induces a homomorphism $f_* : H_1(X, \mathbf{Z}) \to H_1(Y, \mathbf{Z})$ of the first homology groups. It is natural to ask for a characterization of the abstract homomorphisms of the first homology groups induced by coverings. This question was originally raised by Hopf [3] and Martens [4] named it a problem of Hopf. Let E_n be the $n \times n$ identity matrix. Let $J_g = \begin{pmatrix} 0 & E_g \\ -E_g & 0 \end{pmatrix}$.

Any basis for $H_1(X, \mathbf{Z})$ (say $\{\chi_1, \ldots, \chi_{2g}\}$), with intersection matrix (that is a matrix whose (k, j)-entry is given by the intersection number $\chi_k \cdot \chi_j$) J_g will be called a canonical homology basis. Hopf gave the following necessary condition.

Lemma 1 (Condition of Hopf)**.** *Let M be a $2\gamma \times 2g$ matrix of integers representing the induced homology homomorphism, with respect to canonical homology bases, of a covering $f : X \to Y$. Then M satisfies*

$$MJ_g{}^tM = dJ_\gamma,$$

where d is the degree of the covering.

Poincaré discovered the following so called normal form lemma in his work on the reduction of abelian integrals (cf. [6,7] and volume III of his Collected Works) with a rather sketchy proof. A complete proof was given by Martens [5].

Lemma 2 (Normal form lemma)**.** *Let $1 \leq \gamma < g$, and let M be a $2\gamma \times 2g$ matrix with integer entries such that $MJ_g{}^tM$ is non-singular. Then $M = SNT$ where S is a $2\gamma \times 2\gamma$ non-singular matrix of integers, T is a $2g \times 2g$ symplectic unimodular matrix, and N is a $2\gamma \times 2g$ matrix of integers in block form*

$$N = \begin{pmatrix} E_\gamma & 0 & 0 & 0 \\ 0 & A & \Delta & 0 \end{pmatrix}$$

where Δ is a diagonal matrix of integers each diagonal entry $d_{j,j}$ is a multiple of the subsequent diagonal entry $d_{j+1,j+1}$, and A is a $\gamma \times (g - \gamma)$ matrix where $a_{jj} = 1$ for all $j \leq r$ for some r with $0 \leq r \leq min.(\gamma, g - \gamma)$, and the remaining entries are zero.

We will call N the *Poincaré normal form* for M. Martens [4] studied the problem of Hopf and he pointed out that the following necessary condition holds for coverings of prime degrees.

Lemma 3. *If d is prime and M is a $2\gamma \times 2g$ matrix with integer entries satisfying the equation $MJ_g{}^tM = dJ_\gamma$, then*

$$M = T_1 \begin{pmatrix} E_\gamma & 0 & 0 & 0 \\ 0 & A & dE_\gamma & 0 \end{pmatrix} T_2,$$

where the T_j are symplectic unimodular, and the A of the middle factor is the same as in Lemma 2.

Remark. By Lemma 3, for a covering of prime degree, the Poincaré normal form is uniquely determined by the covering since the greatest common divisor of $2\gamma \times 2\gamma$ subdeterminants of M is preserved under multiplication on the right or the left by a unimodular matrix as Martens pointed out [5, p.121]. We also note that the lemma shows that if we choose canonical homology bases properly

2000 Mathematics Subject Classification. Primary 14H30; Secondary 30F30.

then the representation with respect to these bases is a normal form since T_j are symplectic unimodular.

From Lemma 3, as Martens pointed out, we see that for a matrix satisfying the Hopf condition of prime degree there are at most γ possibilities for A. He also constructed examples of coverings represented by normal forms for the case where A has γ non-zero columns and for the case where A has only $\gamma-1$ non-zero columns (cf. [4]). Martens ended his paper by describing that his observation should suggest that something may be gained from a closer study of the Poincaré normal form.

Here, we use both algebraic and complex analytic methods to show that only two normal forms can represent coverings when the degree is a prime number.

Theorem 1. *Let X and Y be surfaces of genera g and γ $(g>\gamma)$, respectively. Let $f: X \to Y$ be a covering of prime degree d.*

If $\gamma > 1$, then there are only two possible normal forms for the covering f, namely, the entries a_{ij} of A in a normal form

$$\begin{pmatrix} E_\gamma & 0 & 0 & 0 \\ 0 & A & dE_\gamma & 0 \end{pmatrix}$$

satisfy

(I) $a_{jj}=1$ *for all* $0 \le j \le min.(\gamma, g-\gamma)$ *and the remaining entries are zero, or*

(II) $a_{jj}=1$ *for all* $0 \le j \le min.(\gamma, g-\gamma)-1$ *and the remaining entries are zero.*

Moreover, case (II) *occurs if and only if the covering is normal and unramified.*

If $\gamma = 1$, then there are only one possible normal form, namely, the entries a_{1j} of A in a normal form satisfy

(I) $a_{11}=1$ *and the remaining entries are zero.*

By Theorem 1, we see that, except the cases where Martens has given examples, a normal form cannot be a representation of a covering.

2. Preliminaries. It is well-known that if $f: X \to Y$ is a surface covering and Y has a conformal structure Φ_2, then there exists a unique conformal structure Φ_1 on X which makes f a holomorphic map of (X, Φ_1) into (Y, Φ_2) (cf. [2,8]). Thus, in the following, we think of X and Y as compact Riemann surfaces of genera g and γ, respectively, and view f as a holomorphic map.

We denote by $H^1(X, \mathbf{R})$ the first de Rham cohomology group of X. For a canonical homology basis $\{\chi_1, \ldots, \chi_{2g}\}$ for $H_1(X, \mathbf{Z})$, there is a unique dual basis $\{\alpha_1, \ldots, \alpha_{2g}\}$ for $H^1(X, \mathbf{R})$, namely $\langle \alpha_k, \chi_j \rangle = \int_{\chi_j} \alpha_k = \delta_{jk}$ $(j, k = 1, \ldots, 2g)$. Let $\{\chi'_1, \ldots, \chi'_{2\gamma}\}$ be a canonical homology basis for $H_1(Y, \mathbf{Z})$ and let $\{\alpha'_1, \ldots, \alpha'_{2\gamma}\}$ be its dual basis for $H^1(Y, \mathbf{R})$.

Let $f: X \to Y$ be a holomorphic map. Let $f_*(\chi_j) = \sum_{k=1}^{2\gamma} m_{kj} \chi'_k$ and put $M = (m_{kj}) \in M(2\gamma, 2g; \mathbf{Z})$. (We denote by $M(m, n; K)$ the set of $m \times n$ matrices with K-coefficients.) We will call M the matrix representation of f_* or f (or simply the representation of f) with respect to the canonical homology bases. There is another interpretation of M. Denote by $f^*\alpha'_k$ the pull back of α'_k by f. Considering the relation $\langle f^*\alpha'_k, \chi_j \rangle = \langle \alpha'_k, f_*(\chi_j) \rangle$, we may write $f^*\alpha'_k = \sum_{j=1}^{2g} m_{kj} \alpha_j$. Thus the induced map $f^*: H^1(Y, \mathbf{R}) \to H^1(X, \mathbf{R})$ is represented by the transpose ${}^tM$. We denote by $\mathbf{e}_k$ the $2g$-tuple column vector whose k-th entry is 1 and others are 0, as usual.

Proposition 1. *Let $f: X \to Y$ be a covering of degree d. Suppose that there exist canonical homology bases $\chi_1, \ldots, \chi_{2g}$ and $\chi'_1, \ldots, \chi'_{2\gamma}$ on X and Y, respectively, so that the matrix representation M of f with respect to these bases is of the form*

$$M = \begin{pmatrix} * & \cdots & * & * & * & \cdots & * \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & d & 0 & \cdots & 0 \end{pmatrix},$$

namely, the last row of M is $d\,{}^t\mathbf{e}_k$ for some $k \in \{1, 2, \ldots, 2g\}$. Then the covering is normal and unramified. Furthermore, the group of covering transformations is cyclic of order d.

Proof. Regarding ${}^tM$ as the representation of f^* the induced homomorphism on the de Rham cohomology groups, we have $f^* \frac{\alpha'_{2\gamma}}{d} = \alpha_k$. Let $c' \subset Y$ be a closed curve meeting no branch point of f and homologous to $\chi'_{2\gamma}$. Denote by p'_0 the base point of c'. Let $p_0 \in f^{-1}(p'_0)$ and lift the curve c' via f to a curve c_0 with initial point p_0. Inductively, we put c_j the lift of c' with initial point p_j and put p_{j+1} the end point of c_j for $j = 0, 1, 2, \ldots, d-2$. We see the points $p_0, p_1, \ldots, p_{d-1}$ are distinct from one another. Indeed, if $p_i = p_j$ $(i \ne j)$, then $c := c_0 + c_1 + \ldots + c_{i-1} - c_{j-1} - c_{j-2} - \ldots - c_0$ is a closed curve and so $\int_c \alpha_k \in \mathbf{Z}$. On the other hand,

$$\int_{c_m} \alpha_k = \int_{c'} \frac{\alpha'_{2\gamma}}{d} = \frac{1}{d} \quad (m = 0, 1, \ldots, d-2),$$

and we have $\int_c \alpha_k = \frac{i-j}{d}$, a contradiction. Thus, $\{p_0, p_1, \ldots, p_{d-1}\} = f^{-1}(p'_0)$. Let $q' \in Y$ be an unbranched point. We draw a path l' connecting q' to p'_0 avoiding branch points. Then, $l' + c' - l'$ is a closed curve meeting no branch point of f and homologous to $\chi'_{2\gamma}$. By the same method as above, we obtain $\{q_0, q_1, \ldots, q_{d-1}\} = f^{-1}(q')$.

Now $f : X \to Y$ is unramified, for if there were a branch point, say $p' \in Y$, then we could deform c' (defined above) to get a closed curve τ' passing through p' via a homotopy having no branch point of f other than p'. Then there exists a lift, say τ, of τ' starting from p_0 ending at p_j which is different from p_1 the end point of c_0. Then $\tau - c_{j-1} - c_{j-2} - \ldots - c_0$ is a closed curve so that the same considerations as above lead us to a contradiction.

Define a map T by $T(q_j) = q_{j+1}$ for $j = 0, 1, 2, \ldots, d-2$ and $T(q_{d-1}) = q_0$. So $T(q_j)$ is the end point of the lift of $l' + c' - l'$ starting from q_j. Although there is ambiguity in the numbering process of $f^{-1}(q')$, the map T is well-defined (independent of the choice of the path l'). We see this as follows: Let l'_i $(i = 1, 2)$ be two paths connecting q' to p'_0. Then, $l'_i + c' - l'_i$ $(i = 1, 2)$ are closed curves homologous to $\chi'_{2\gamma}$. Let $q_0 \in f^{-1}(q')$ and lift the curve $l'_i + c' - l'_i$ via f to a curve l_i $(i = 1, 2)$ with initial point q_0. Denote by $q_{i,1}$ the end point of l_i $(i = 1, 2)$ and suppose that $q_{1,1} \neq q_{2,1}$. Then as above we may think of $q_{2,1}$ as the end point of ml_1 the lift of $m(l'_1 + c' - l'_1)$ starting from q_0, for some $m \in \{2, 3, \ldots, d-1\}$. Thus $ml_1 - l_2$ is a closed curve and this implies $\int_{ml_1 - l_2} \alpha_k \in \mathbf{Z}$. On the other hand, seeing

$$\int_{l_i} \alpha_k = \int_{l'_i + c' - l'_i} \frac{\alpha'_{2\gamma}}{d} = \frac{1}{d} \quad (i = 1, 2),$$

we have $\int_{ml_1 - l_2} \alpha_k = \frac{m-1}{d}$, a contradiction. It is easy to see that T is conformal and $Y = X/ < T >$. $\square$

The following lemma (cf. [1]) is a key tool in the proof of Theorem 1.

Lemma 4 (Accola)**.** *Let T be an automorphism of a closed Riemann surface of genus greater than one. Suppose that there exist four independent cycles $\chi_1, \chi_2, \chi_3, \chi_4$ so that $\chi_1 \cdot \chi_3 = 1$, $\chi_2 \cdot \chi_4 = 1$, otherwise $\chi_i \cdot \chi_j = 0$ and that $T(\chi_i) = \chi_i$ for $i = 1, 2, 3, 4$. Then T is the identity.*

3. Proof of Theorem 1. First, recall that a normal form for a covering of prime degree is of the form $\begin{pmatrix} E_\gamma & 0 & 0 & 0 \\ 0 & A & dE_\gamma & 0 \end{pmatrix}$ where A is a $\gamma \times (g - \gamma)$ matrix whose entries satisfy $a_{jj} = 1$ for all $j \leq r$ for some r with $0 \leq r \leq min.(\gamma, g - \gamma)$, and the remaining entries are zero. Suppose $\gamma > 1$. We show that there are only two possible normal forms, case (I) and case (II). Suppose that the normal form for a covering f of prime degree d is neither of (I) nor of (II). Then, we see the $(2\gamma - 1)$-th row of the normal form is $d\,{}^t\mathbf{e}_{g+\gamma-1}$, and the 2γ-th row of the normal form is $d\,{}^t\mathbf{e}_{g+\gamma}$. We choose canonical homology bases $\chi_1, \ldots, \chi_{2g}$ and $\chi'_1, \ldots, \chi'_{2\gamma}$ on X and Y, respectively, so that the matrix representation with respect to these bases is such a normal form. By Proposition 1, the covering is normal, unramified and $Y = X/ < T >$ for some $T \in \mathrm{Aut}(X)$. From the form of the last two rows of the matrix representation and viewing the matrix as the representation of the induced homomorphism on the spaces of differentials again, we have $f^* \frac{\alpha'_{2\gamma-1}}{d} = \alpha_{g+\gamma-1}$ and $f^* \frac{\alpha'_{2\gamma}}{d} = \alpha_{g+\gamma}$. Further, we have $f^* \alpha'_{\gamma-1} = \alpha_{\gamma-1}$ and $f^* \alpha'_\gamma = \alpha_\gamma$. By $f = f \circ T$, $f^* = T^* \circ f^*$ holds. This implies that $T^* \alpha_{g+\gamma-1} = \alpha_{g+\gamma-1}$, $T^* \alpha_{g+\gamma} = \alpha_{g+\gamma}$, $T^* \alpha_{\gamma-1} = \alpha_{\gamma-1}$, and $T^* \alpha_\gamma = \alpha_\gamma$. Denote by L the matrix representation of T with respect to the canonical homology basis $\chi_1, \ldots, \chi_{2g}$. Then the j-th row of L is ${}^t\mathbf{e}_j$ for $j = \gamma - 1,\ \gamma,\ g + \gamma - 1,\ g + \gamma$. Since L is symplectic, writing $L = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in $g \times g$ blocks, we have $L^{-1} = \begin{pmatrix} {}^tD & -{}^tB \\ -{}^tC & {}^tA \end{pmatrix}$. Hence the j-th column of L^{-1} is $\mathbf{e}_j$ for $j = \gamma - 1,\ \gamma,\ g + \gamma - 1,\ g + \gamma$. By Lemma 4, T^{-1} and hence T is the identity. Thus $Y = X/ < T >= X$. But this case is impossible since we have assumed $g > \gamma$. Thus an arbitrary normal form must be of type (I) or type (II).

Next, we will show that case (II) occurs if and only if the covering is normal and unramified. By Proposition 1, the covering is normal and unramified in case (II).

Conversely, suppose that the covering f is normal and unramified. We show that if we choose canonical homology bases on X and Y properly, the matrix representation of f with respect to these bases is of the form

$$M = \begin{pmatrix} * & \ldots & * & * & * & \ldots & * \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & d & 0 & \ldots & 0 \end{pmatrix},$$

where the $(2\gamma, g + \gamma)$-entry is d and the $(2\gamma, j)$-entry is 0 for $j \neq g + \gamma$. Let $\chi'_{2\gamma}$ be a simple closed curve on

Y which is not homologus to 0 and satisfies that $f^{-1}(\chi'_{2\gamma})$ is a simple closed curve which covers $\chi'_{2\gamma}$ d times. (The existence of such a curve $\chi'_{2\gamma}$ can be shown as follows: If such a curve does not exist, we can choose a canonical homology basis $\chi'_1, \ldots, \chi'_{2\gamma}$ on Y such that each cycle in the basis is a simple closed curve and such that $f^{-1}(\chi'_k)$ consists of d distinct simple closed curves for each $k = 1, \ldots, 2\gamma$. But this contradicts the Riemann-Hurwitz relation $2(g-1) = 2d(\gamma-1)$.) We put $\chi_{g+\gamma} = f^{-1}(\chi'_{2\gamma})$. Then there exists a cycle $\chi_\gamma \neq 0$ which is a simple closed curve and satisfies $\chi_\gamma \cdot \chi_{g+\gamma} = 1$. Put $\chi'_\gamma = f(\chi_\gamma)$ and denote by T the covering transformation. Then χ_γ, $T(\chi_\gamma), \ldots, T^{d-1}(\chi_\gamma)$ are d distinct simple closed curves which divide X into d copies of $Y - \chi'_\gamma$. Choosing cycles $\chi'_1, \ldots, \chi'_{\gamma-1}$ and $\chi'_{\gamma+1}, \ldots, \chi'_{2\gamma-1}$ on $Y - \chi'_\gamma$ properly and combining these with χ'_γ and $\chi'_{2\gamma}$, we get a canonical homology basis $\chi'_1, \ldots, \chi'_{2\gamma}$ on Y. Obviously, $f^{-1}(\chi'_k)$ consists of d distinct simple closed curves for $k = 1, \ldots, \gamma-1, \gamma+1, \ldots, 2\gamma-1$. Ordering these cycles on X properly and combining these with χ_γ and $\chi_{g+\gamma}$, we get a canonical homology basis $\chi_1, \ldots, \chi_{2g}$ on X such that the matrix representation of f with respect to $\chi_1, \ldots, \chi_{2g}$ and $\chi'_1, \ldots, \chi'_{2\gamma}$ is of the form M above. Recalling the remark following Lemma 3, we see that the normal form is not of type (I). Consequently, it must be of type (II). This completes the proof for $\gamma > 1$. Suppose next $\gamma = 1$. In this case, A in a normal form is a $(g-1)$-tuple row vector. Suppose that the first entry a_{11} of A is 0. Then the covering is normal and unramified by Proposition 1. If the covering is unramified, genus of the source surface must be 1 by the Riemann-Hurwitz relation and it contradicts the assumption $g > \gamma$. Consequently, a_{11} must be 1. □

Acknowledgements. It is a pleasure to thank the referee for his careful reading of this manuscript and for his valuable comments which improve the contents of the paper.

Partially supported by Grant-in-Aid for Scientific Research (No. 14540155, (C)(2)), Japan Society for the Promotion of Science.

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