Modular relation interpretation of the series involving the Riemann zeta values

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Abstract: We shall locate Katsurada's results, in our framework of modular relations, on two series involving the values of the Riemann zeta-function, which are decisive generalizations of earleir results of Chowla and Hawkins and of Buschman and Srivastava *et al.* We shall elucidate these results as an improper or a proper modular relation according as the involved parameter ν exerts effects on the series or not, eventually indicating that they are disguised form of modular relations as given by Theorem 4 in §3.

Key words: Riemann zeta-function; Fox H-function; modular relation; hypergeometric function.

1. Introduction and statement of results. Let ν denote a complex parameter and for $x > 0$, let

$$H_\nu(x) = \sum_{n > \operatorname{Re}\nu+1} (-1)^n \binom{x}{n} \zeta(n-\nu), \tag{1}$$

where $\zeta(s)$ is the Riemann zeta-function defined by (11) below, $\binom{x}{n}$ is the binomial coefficient $\frac{(x)_n}{n!}$, with $(x)_n$ indicating the falling factorial $(x)_n = x(x-1)\cdots(x-n+1)$ and the sum is extended over $n \in \mathbf{N}\cup\{0\}$, $n > \operatorname{Re}\nu + 1$. Hence if $\operatorname{Re}\nu + 1 < 0$, the parameter has no effect on the summation, which we call the proper case.

Chowla and Hawkins [2] considered the terminating case of (1) for $x \in \mathbf{N}$ and obtained an asymptotic formula, which was later sharpened by Verma [12].

Katsurada [6] (cf. also [5]) has taken the decisive step of generalizing all the previous investigations from the point of view of the Mellin-Barnes type of integrals (for a general theory cf. [7]) and obtained a confluent hypergeometric series expansion, which in turn gives rise to an asymptotic formula for $H_\nu(x)$.

In this note we shall locate Katsurada's results in the modular relation framework, our main results being Theorems 3 and 4 in §3 below. However, to make the process more accessible, we shall elucidate two theorems of Katsurada from the standpoint of the proper and improper modular relations, proving eventually that Katsurada's theorem [6, Theorem 4.1] properly formulated is really a modular relation as furnished by Theorem 3 below.

Theorem 1. *For all* $x \geq |\operatorname{Re}\nu| + 2$ *and* $\nu \notin \mathbf{Z}$,

$$\sum_{n=0}^{\infty} (-1)^n \binom{x}{n} \zeta(n-\nu) = \frac{\Gamma(x+1)\,\Gamma(-\nu-1)}{\Gamma(x-\nu)} + \mathcal{H}_\nu(x), \tag{2}$$

where

$$\mathcal{H}_\nu(x) = \Gamma(x+1) \sum_{n=1}^{\infty} \Big(\Psi(x+1, \nu+2; -2\pi i n) + \Psi(x+1, \nu+2; 2\pi i n) \Big) \tag{3}$$

and where $\Psi(a, c; z)$ *is the confluent hypergeometric function defined by*

$$\Psi(a, c; z) = \frac{1}{\Gamma(a)} \int_0^{\infty e^{i\phi}} e^{-zt} t^{a-1} (1+t)^{c-a-1} dt \tag{4}$$

for $\operatorname{Re} a > 0$, $-\pi < \phi < \pi$ *and* $-\frac{\pi}{2} < \phi + \arg z < \frac{\pi}{2}$ [3, p. 256, 6.5 (3)].

In the case $\nu \in \mathbf{Z}$, $\nu \geq -1$, *the first term on the right of* (2) *is to be replaced by*

$$(-1)^\nu \binom{x}{\nu+1} \left(\frac{\Gamma'}{\Gamma}(x-\nu) + 2\gamma - H_{\nu+1} \right), \tag{5}$$

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Dedicated to Professor Doctor Imre Kátai on his 70th birthday *where γ is Euler's constant and $H_{\nu+1}$ is the $(\nu+1)$-th harmonic number*

$$H_{\nu+1} = \sum_{n=1}^{\nu+1} \frac{1}{n}.$$

Remark 1. In the case $\mathrm{Re}\,\nu + 1 \geq 0$, the parameter ν affects the series and we are to shift the first $[\mathrm{Re}\,\nu+1]$ partial sum to the right so as to indicate that these $\zeta(n-\nu)$'s are not the series (11) but its analytic continuation:

$$(6)\quad H_\nu(x) = \frac{\Gamma(x+1)\,\Gamma(-\nu-1)}{\Gamma(x-\nu)} - \sum_{n=0}^{[\mathrm{Re}\,\nu+1]} (-1)^n \binom{x}{n} \zeta(n-\nu) + \mathcal{H}_\nu(x)$$

for $\nu \notin \mathbf{Z}$; for $\nu \in \mathbf{Z}$ we are to replace the first term by (5). These are the results stated by Katsurada [6, Theorem 4.1], which we refer to as the improper modular relation (cf. §3 for more elucidation). They do not look like at a first glance a modular relation as developed in [4,11] but even in the case $\mathrm{Re}\,\nu + 1 > 0$, stated in the form of our Theorem 1, it is a modular relation as furnished by Theorem 3 below.

Katsurada [6] also obtained a K-Bessel series expansion for the power series

$$(7)\quad G_\nu(x) = \sum_{n > \mathrm{Re}\,\nu+1} \frac{(-1)^n}{n!}\, x^n \zeta(n-\nu),$$

whose special cases and asymptotic formulas were studied first by Chowla and Hawkins, and later by Buschman and Srivastava [1] and [10, p. 141] among others (for related references, see [6, p. 16, l. 8]).

We may elucidate Katsurada's results [6, Theorems 3.1 and 3.2] in much the same light as with Theorem 1 above.

Theorem 2. *For $x \geq 2$ and $\nu \neq -1, 0, 1, 2\cdots$, we have*

$$(8)\quad \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\, x^n \zeta(n-\nu) = \Gamma(-\nu-1)\, x^{\nu+1} + \mathcal{G}_\nu(x),$$

where

$$(9)\quad \mathcal{G}_\nu(x) = 2\sum_{n=1}^{\infty} \left(\left(\frac{x}{2\pi n}\right)^{\frac{1}{2}(\nu+1)} e^{-\frac{1}{4}(\nu+1)\pi i} K_{\nu+1}\left(2e^{\frac{1}{4}\pi i}\sqrt{2\pi n x}\right) + \left(\frac{x}{2\pi n}\right)^{\frac{1}{2}(\nu+1)} e^{\frac{1}{4}(\nu+1)\pi i} K_{\nu+1}\left(2e^{-\frac{1}{4}\pi i}\sqrt{2\pi n x}\right)\right).$$

2. Proofs.

Proof of Theorem 1. Our proof depends on evaluating the integral

$$(10)\quad \tilde{H}_\nu(x) = \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(x+1)\,\Gamma(-s)}{\Gamma(x+1-s)}\,\zeta(s-\nu)\,ds, \qquad \mathrm{Re}\,\nu < c \leq [\mathrm{Re}\,\nu+2].$$

First consider the proper case, i.e. $\mathrm{Re}\,\nu + 1 < 0$ and suppose that $\nu \notin \mathbf{Z}$ and $\mathrm{Re}\,\nu + 1 < c < 0$. Then we may use the absolutely convergent series

$$(11)\quad \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \sigma > 1$$

to obtain

$$(12)\quad \begin{aligned} \tilde{H}_\nu(x) &= \Gamma(x+1)\sum_{n=1}^{\infty} n^\nu \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(-s)}{\Gamma(x+1-s)}\, n^{-s} ds \\ &= \Gamma(x+1)\sum_{n=1}^{\infty} n^\nu G_{1,1}^{0,1}\left(n \,\middle|\, \begin{matrix} 1-0 \\ 1-(x+1) \end{matrix}\right), \end{aligned}$$

where $G_{p,q}^{m,n}$ is the Meijer G-function (cf. [3, p. 202–222] and [9]). By the well-known properties of the G-function,

$$(13)\quad \begin{aligned} G_{1,1}^{0,1}\left(n \,\middle|\, \begin{matrix} 1-0 \\ 1-(x+1) \end{matrix}\right) &= G_{1,1}^{1,0}\left(n^{-1} \,\middle|\, \begin{matrix} x+1 \\ 0 \end{matrix}\right) \\ &= \frac{1}{\Gamma(x+1)}\left(1-\frac{1}{n}\right)^x \\ &= \frac{1}{\Gamma(x+1)}\sum_{k=0}^{\infty}(-1)^k \binom{x}{k} n^{-k}. \end{aligned}$$

Hence

$$\tilde{H}_\nu(x) = \sum_{n=1}^{\infty}\sum_{k=0}^{\infty}(-1)^k \binom{x}{k} \frac{1}{n^{k-\nu}},$$

whence by changing the order of summation, we deduce that

$$(14)\quad \tilde{H}_\nu(x) = \sum_{n=0}^{\infty}(-1)^n \binom{x}{n} \zeta(n-\nu).$$

Now we shift the line of integration to $\sigma = \sigma_0, -\frac{1}{2} < \sigma_0 < \mathrm{Re}\,\nu + 1 < c$, encountering a simple pole at $s=\nu+1$ with residue

$$\frac{\Gamma(x+1)\,\Gamma(-\nu-1)}{\Gamma(x-\nu)}$$

to obtain

$$\tilde{H}_\nu(x)=\frac{\Gamma(x+1)\,\Gamma(-\nu-1)}{\Gamma(x-\nu)}+\tilde{\mathcal{H}}_\nu(x), \tag{15}$$

with

$$\tilde{\mathcal{H}}_\nu(x) = \Gamma(x+1)\frac{1}{2\pi i}\int_{(\sigma_0)}\frac{\Gamma(-s)}{\Gamma(x+1-s)}\zeta(s-\nu)\,ds. \tag{16}$$

Recalling the functional equation

$$\zeta(z)=\pi^{z-\frac{1}{2}}\frac{\Gamma(\frac{1-z}{2})}{\Gamma(\frac{z}{2})}\zeta(1-z) \tag{17}$$

and transforming the gamma factor slightly as

$$\frac{\Gamma(\frac{1-z}{2})}{\Gamma(\frac{z}{2})}=\frac{\Gamma(\frac{1-z}{2})\Gamma(\frac{1}{2}+\frac{1-z}{2})}{\Gamma(\frac{z}{2})\Gamma(1-\frac{z}{2})} = \sqrt{\pi}2^{1-(1-z)}\Gamma(1-z)\frac{\sin\frac{\pi}{2}z}{\pi}.$$

Where we used the reciprocity relation and the duplication formula, we obtain

$$\frac{\Gamma(\frac{1-z}{2})}{\Gamma(\frac{z}{2})}=\frac{1}{\sqrt{\pi}}2^z\,\Gamma(1-z)\frac{e^{\frac{\pi}{2}iz}-e^{-\frac{\pi}{2}iz}}{2i}$$

by Euler's formula. Hence it follows that

$$\begin{aligned}&\frac{1}{2\pi i}\int_{(\sigma_0)}\frac{\Gamma(\nu+1-s)}{\Gamma(x+1-s)}\left(\frac{1}{2\pi in}\right)^{-s}ds\\ &\quad-(-1)^\nu\frac{1}{2\pi i}\int_{(\sigma_0)}\frac{\Gamma(\nu+1-s)}{\Gamma(x+1-s)}\left(-\frac{1}{2\pi in}\right)^{-s}ds\\ &=\Gamma(x+1)\sum_{n=1}^\infty\frac{1}{(2\pi n)^{\nu+1}}\\ &\left(e^{-\frac{1}{2}(\nu+1)\pi i}G^{0,2}_{2,1}\left(\frac{1}{2\pi in}\middle|\begin{matrix}1-0,1-(\nu+1)\\1-(x+1)\end{matrix}\right)\right.\\ &\quad\left.+e^{\frac{1}{2}(\nu+1)\pi i}G^{0,2}_{2,1}\left(-\frac{1}{2\pi in}\middle|\begin{matrix}1-0,1-(\nu+1)\\1-(x+1)\end{matrix}\right)\right).\end{aligned}$$

Now the G-function part becomes

$$\begin{aligned}&e^{-\frac{1}{2}(\nu+1)\pi i}G^{2,0}_{1,2}\left(2\pi in\middle|\begin{matrix}x+1\\0,\nu+1\end{matrix}\right)\\ &\quad+e^{\frac{1}{2}(\nu+1)\pi i}G^{2,0}_{1,2}\left(-2\pi in\middle|\begin{matrix}x+1\\0,\nu+1\end{matrix}\right)\\ &=e^{-2\pi in}(2\pi n)^{\nu+1}\Psi(x+1,\nu+2;2\pi in)\\ &\quad+e^{2\pi in}(2\pi n)^{\nu+1}\Psi(x+1,\nu+2;-2\pi in)\end{aligned} \tag{18}$$

by Lemma 1 below, whence

$$\tilde{\mathcal{H}}_\nu(x)=\mathcal{H}_\nu(x), \tag{19}$$

where $\mathcal{H}_\nu(x)$ is defined in (3).

We turn to the improper case, where $\mathrm{Re}\,\nu+1\ge 0$, in which case, the integral in (12) is not a G-function. First assume that $\nu\notin\mathbf{Z}$. We are to shift the path to $\sigma=\sigma_0$, $-\frac{1}{2}<\sigma_0<0$. In doing so, we encounter simple poles at $s=0,\cdots,[\mathrm{Re}\,\nu+1]$, so that

$$\begin{aligned}&\frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(-s)}{\Gamma(x+1-s)}\,n^{-s}ds\\ &\quad=-\sum_{k=0}^{[\mathrm{Re}\,\nu+1]}\frac{(-1)^k}{k!}\frac{1}{\Gamma(x+1-k)}\,n^{-k}\\ &\qquad+G^{0,1}_{1,1}\left(n\middle|\begin{matrix}1-0\\1-(x+1)\end{matrix}\right).\end{aligned} \tag{20}$$

The first term on the right of (20) becomes

$$\frac{1}{\Gamma(x+1)}\sum_{k=0}^{[\mathrm{Re}\,\nu+1]}(-1)^k\binom{x}{k}n^{-k},$$

while the second term is calculated in (13). Hence

$$\begin{aligned}&\frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(-s)}{\Gamma(x+1-s)}\,n^{-s}ds\\ &\quad=\frac{1}{\Gamma(x+1)}\sum_{k>[\mathrm{Re}\,\nu+1]}(-1)^k\binom{x}{k}n^{-k}.\end{aligned} \tag{21}$$

Substituting (21) in (12), we deduce that

$$\begin{aligned}\tilde{H}_\nu(x)=H_\nu(x)&=\sum_{k>[\mathrm{Re}\,\nu+1]}(-1)^k\binom{x}{k}\sum_{n=0}^\infty n^{\nu-k}\\ &=\sum_{k>[\mathrm{Re}\,\nu+1]}(-1)^k\binom{x}{k}\zeta(k-\nu),\end{aligned}$$

which is [6, (4.4)].

As in the proper case, we shift the line of integration to $\sigma=\sigma_0,-\frac{1}{2}<\sigma_0<0$, encountering simple poles at $s=0,\cdots,[\mathrm{Re}\,\nu+1]$ and $\nu+1$. The sum of residues at these poles is

$$\sum_{n=0}^{[\mathrm{Re}\,\nu+1]}(-1)^k\binom{x}{k}\zeta(k-\nu)+\frac{\Gamma(x+1)\,\Gamma(-\nu-1)}{\Gamma(x-\nu)}.$$

Hence, in place of (15), we obtain

$$H_\nu(x) = \frac{\Gamma(x+1)\,\Gamma(-\nu-1)}{\Gamma(x-\nu)} - \sum_{n=0}^{[\mathrm{Re}\,\nu+1]} (-1)^k \binom{x}{k} \zeta(k-\nu) + \tilde{\mathcal{H}}_\nu(x), \tag{22}$$

where $\tilde{\mathcal{H}}_\nu(x)$ is defined in the first instance by (16) is equal to $\mathcal{H}_\nu(x)$ by (19), and (22) is the same as (2).

In the case $\nu = -1, 0, 1, \cdots$, the treatment remains the same except for the evaluation of the residues at the double pole $s = \nu + 1$, whose residue is given by (5). This completes the proof of Theorem 1.

We have appealed in the proof to

Lemma 1. *We have*

$$G_{1,2}^{2,0}\left(z \,\middle|\, \begin{matrix} a \\ b, c \end{matrix}\right) = e^{-z} z^a \Psi(a-c, b-c+1; z) \tag{23}$$

and in particular

$$G_{1,2}^{2,0}\left(z \,\middle|\, \begin{matrix} 1+a-b \\ 0, 1-b \end{matrix}\right) = e^{-z} \Psi(a, b; z). \tag{24}$$

Proof. We may prove (23) via the Whittaker function $W_{\kappa,\mu}$ [3, p. 264]. [3, p. 216, (6)] reads

$$G_{1,2}^{2,0}\left(z \,\middle|\, \begin{matrix} a \\ b, c \end{matrix}\right) = z^{\frac{1}{2}(b+c-1)} e^{-\frac{1}{2}z} W_{\frac{1}{2}(1+b+c)-a, \frac{1}{2}(b-c)}(z), \tag{25}$$

while [3, (2), p. 264] reads

$$W_{\kappa,\mu}(z) = e^{-\frac{1}{2}z} z^{\frac{1}{2}\gamma} \Psi(\alpha, \gamma; z), \tag{26}$$

where

$$\alpha = \frac{1}{2} - \kappa + \mu,\ \gamma = 2\mu + 1.$$

Choosing $\alpha = a - c, \gamma = b - c + l$, we conclude (23).

Remark 2. Katsurada gave an independent proof of (18) using (4), but a more general form (23) is already available in Erdélyi as indicated above. Indeed, (18) can be found in Prudnikov [8, p. 716, 8.4, 46.7].

Proof of Theorem 2. The proof goes on the same lines as those of the proof of Theorem 1, and we shall only point out the substantial ingredients.

Correspondingly to (10), we consider the integral

$$\tilde{G}_\nu(x) = \frac{1}{2\pi i} \int_{(c)} \Gamma(-s)\,\zeta(s-\nu)\,x^s ds, \tag{27}$$

where $x > 0$, and $\mathrm{Re}\,\nu + 1 < c < 0$ or $\mathrm{Re}\,\nu + 1 < c \le [\mathrm{Re}\,\nu + 2]$, according to the proper and improper case. The formula corresponding to (14),

$$\tilde{G}_\nu(x) = \sum_{m=1}^{\infty} m^\nu e^{-\frac{x}{m}} \tag{28}$$

immediately follows from (11) in the proper case.

Correspondingly to (16), we consider

$$\tilde{\mathcal{G}}_\nu(x) = \frac{1}{2\pi i} \int_{(\sigma_0)} \Gamma(-s)\,\zeta(s-\nu)\,x^s ds, \tag{29}$$

where σ_0 satisfies $-\frac{1}{2} < \sigma_0 < \mathrm{Re}\,\nu + 1 < 0$ or $-\frac{1}{2} < \sigma_0 < 0$ according to the proper and improper case.

As in the proof of Theorem 1, we have

$$\begin{aligned}\tilde{\mathcal{G}}_\nu(x) = \sum_{n=1}^{\infty} \frac{1}{(2\pi n)^{\nu+1}} &\left(e^{-\frac{1}{2}(\nu+1)\pi i} G_{0,2}^{2,0}\left(2\pi i n x \,\middle|\, \begin{matrix} - \\ 0 \ \ \nu+1 \end{matrix}\right) \right.\\ &\left. + e^{\frac{1}{2}(\nu+1)\pi i} G_{0,2}^{2,0}\left(-2\pi i n x \,\middle|\, \begin{matrix} - \\ 0 \ \ \nu+1 \end{matrix}\right)\right),\end{aligned}$$

whence by [3, p. 216, (4)], we have, as in the proof of Theorem 1,

$$\tilde{\mathcal{G}}_\nu(x) = \mathcal{G}_\nu(x). \tag{30}$$

This completes the proof.

3. General results. Katsurada's Theorem 1 is a special case of the following more general theorem, which in turn is a simple corollary of Theorem 4:

Theorem 3.

$$\begin{aligned}\frac{1}{\Gamma(1-b)} \sum_{n=0}^{\infty} \frac{(b)_n}{n!}\,\zeta(n-\nu)\,x^n &= x^{\nu+1} \sum_{k=1}^{\infty} \Big\{ e^{-2\pi i k x}\,\Psi(1-b, \nu+2; 2\pi i k x) \\ &\quad + e^{2\pi i k x}\,\Psi(1-b, \nu+2; -2\pi i k x) \Big\} \\ &\quad + x^{\nu+1} \frac{\Gamma(-\nu-1)}{\Gamma(-\nu-b)}.\end{aligned} \tag{31}$$

Remark 3. We note that the left-hand side member of Theorem 3 was already mentioned by Katsurada who, however, went on into another direction.

Theorem 4.

(32)

$$\frac{1}{\pi^s}\sum_{k=1}^{\infty}\frac{1}{k^{2s}}H_{2,2}^{1,1}\left(z\pi k^2 \,\middle|\, \begin{matrix}(1-a_1,2),(a_2,1)\\(s,1),(b,2)\end{matrix}\right)$$
$$+\operatorname{Res}\left(\frac{\Gamma(a_1+2w+2s-1)}{\Gamma\big(a_2-w-s+\frac{1}{2}\big)\Gamma(1-b+2w+2s-1)}\Gamma(w)Z(w)\,z^{s-\frac{1}{2}+w}, w=\tfrac{1}{2}\right)$$
$$=\frac{1}{\pi^{\frac{1}{2}-s}}\sum_{k=1}^{\infty}\frac{1}{k^{1-2s}}$$
$$H_{1,3}^{2,0}\left(\frac{\pi k^2}{z}\,\middle|\, \begin{matrix}(1-b,2)\\(\frac{1}{2}-s,1),(a_1,2),(1-a_2,1)\end{matrix}\right)$$
$$+\operatorname{Res}\left(\frac{\Gamma(a_1-2w+2s)}{\Gamma(a_2+w-s)\Gamma(1-b-2w+2s)}\Gamma(w)Z(w)\,z^{s-w}, w=\tfrac{1}{2}\right),$$

where $H_{p,q}^{m,n}$ *indicates the Fox H-function* (*cf.* [9,11])

Theorem 3 follows from Theorem 4 by specifying the parameters. Details of this and elucidaton of Thereom 2 together with other results will appear elsewhere.

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