Absolute zeta functions

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Abstract: Two new concepts of zeta functions for schemes over the field of one element are proposed. A localization formula and an explicit formula in the affine case are given. This allows for a computation for every scheme.

Key words: Zeta function; field of one element.

Introduction. In [2], the first named author has introduced a zeta function for $\mathbf{F}_1$-schemes generalizing an idea of Soulé's in [10]. Basically, this zeta function only detects free ranks of the involved groups, so it is insensitive to torsion. In the present paper we will introduce two new kinds of zeta functions for $\mathbf{F}_1$-schemes which are sensitive to torsion, yet still preserve the information on the ranks. The first new zeta function is of Weil type, where the finite fields are replaced by the basic finite monoids. The second is of Igusa type, as inspired by [7].

1. Soulé zeta function. Soulé [10], inspired by Manin [9], gave a definition of zeta functions over the field of one element $\mathbf{F}_1$. See also [8]. We describe the definition as follows: Let X be a scheme of finite type over $\mathbf{Z}$. For a prime number p one sets after Weil,

$$Z_X(p,T) \stackrel{\text{def}}{=} \exp\left(\sum_{n=1}^{\infty} \frac{T^n}{n} \#X(\mathbf{F}_{p^n})\right),$$

where $\mathbf{F}_{p^n}$ denotes the field of p^n elements. This is the local zeta function over p, and the global zeta function of X is given as

$$\zeta_{X|\mathbf{Z}}(s) \stackrel{\text{def}}{=} \prod_p Z_X(p,p^{-s})^{-1}.$$

Soulé considered in [10] the following condition: Suppose there exists a polynomial $N(x)$ with integer coefficients such that $\#X(\mathbf{F}_{p^n}) = N(p^n)$ for every prime p and every $n \in \mathbf{N}$. Then $Z_X(p,p^{-s})^{-1}$ is a rational function in p and p^{-s}. The vanishing order at $p = 1$ is $N(1)$. One may thus define

$$\zeta_{X|\mathbf{F}_1}(s) = \lim_{p\to 1} \frac{Z_X(p,p^{-s})^{-1}}{(p-1)^{N(1)}}.$$

One computes that if $N(x) = a_0 + a_1 x + \cdots + a_n x^n$, then

$$\zeta_{X|\mathbf{F}_1}(s) = s^{a_0}(s-1)^{a_1}\cdots(s-n)^{a_n}.$$

Based on ideas of [6], in the paper [1] there is given a definition of a scheme over $\mathbf{F}_1$ as well as an ascent functor $\cdot \otimes \mathbf{Z}$ from $\mathbf{F}_1$-schemes to $\mathbf{Z}$-schemes. For the convenience of the reader, we will briefly recall the definition. Recall a *monoid* is a set A with an associative composition and a unit element $1 \in A$, i.e., one has $1a = a1 = a$ for every $a \in A$. In this paper, all monoids will be commutative.

An *ideal* in the monoid A is a subset $\mathfrak{a} \subset A$ with $\mathfrak{a}A \subset A$. An ideal $\mathfrak{p}$ is a *prime ideal* if $S_{\mathfrak{p}} = A \smallsetminus \mathfrak{p}$ is a submonoid. Let $\operatorname{Spec} A$ denote the set of all prime ideals with the Krull topology [1]. For $\mathfrak{p} \in \operatorname{Spec} A$, let $A_{\mathfrak{p}} = S_p^{-1}A$ be the localization at $\mathfrak{p}$ and let $A_{\mathfrak{p}}^\times$ be its unit group. Then $A_{\mathfrak{p}}^\times$ is the quotient group of $S_{\mathfrak{p}} = A \smallsetminus \mathfrak{p}$.

On the topological space $\operatorname{Spec} A$ one has a canonical sheaf $\mathcal{O}_A$ of monoids with stalks being the localizations of A, so $\mathcal{O}_{A,\mathfrak{p}} = A_{\mathfrak{p}}$ for every $\mathfrak{p} \in \operatorname{Spec} A$. A *scheme over* $\mathbf{F}_1$ is a topological space X together with a sheaf $\mathcal{O}_X$ of monoids such that $(X, \mathcal{O}_X)$ is locally isomorphic to $(\operatorname{Spec} A, \mathcal{O}_A)$ for monoids A.

Let X be an $\mathbf{F}_1$-scheme of finite type. Note that this implies that X has finitely many points and that $\mathcal{O}_{X,p}^\times$ is a finitely generated group for every $p \in X$.

An affine $\mathbf{F}_1$-scheme is given by a commutative monoid and its lift to $\mathbf{Z}$ is given by the corresponding monoidal ring. This procedure extends to general schemes as it respects gluing. We say that a $\mathbf{Z}$-scheme is *defined over* $\mathbf{F}_1$, if it comes by ascent

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from a scheme over $\mathbf{F}_1$. The natural question arising is whether schemes defined over $\mathbf{F}_1$ satisfy Soulé's condition.

Simple examples show that this is not the case. However, schemes defined over $\mathbf{F}_1$ satisfy a slightly weaker condition which serves the purpose of defining $\mathbf{F}_1$-zeta functions as well, and which we give in the following theorem, proven in [2].

Theorem 1.1 (See [2])**.** *Let X be a $\mathbf{Z}$-scheme of finite type defined over $\mathbf{F}_1$. Then there exists a natural number e and a polynomial $N(x)$ with integer coefficients such that for every prime power q one has*

$$(q-1,e)=1 \Rightarrow \#X(\mathbf{F}_q)=N(q).$$

This condition determines the polynomial N uniquely (*independent of the choice of e*)*. We call it the* zeta-polynomial *of X.*

Using this polynomial $N(x)$, one defines the Soulé zeta function as above.

Proposition 1.2. *The Soulé zeta function satisfies the* localization formula

$$\zeta^S(s,X)=\prod_{p\in X}\zeta^S(s,\mathcal{O}_{X,p}^\times).$$

Proof. Let $X_{\mathbf{Z}}$ be the $\mathbf{Z}$-lift of X. For a prime power q let F_{q-1} be the monoid $(\mathbf{F}_q,\times)$. Then $|X_{\mathbf{Z}}(\mathbf{F}_q)|=|X(F_{q-1})|$ by [1], Theorem 1.1.

Generalizing the previous notation, for a natural number m let F_m be the monoid $\mu_m\cup\{0\}$, where μ_m denotes the cyclic group of m elements. The monoid operation on F_m is given by $0x=0$ for every $x\in\mathrm{F}_m$. The proposition follows from the next Lemma.

Lemma 1.3. *Let X be an $\mathbf{F}_1$-scheme. There is a canonical disjoint decomposition*

$$\mathrm{Hom}(\mathrm{Spec}\,F_m,X)=\coprod_{p\in X}\mathrm{Hom}(\mathcal{O}_{X,p}^\times,\mu_m).$$

Here on the right $\mathrm{Hom}(\mathcal{O}_{X,p}^\times,\mu_m)$ *corresponds to the set of all homomor-phisms $\phi:\mathrm{Spec}\,F_m\to X$ with $\phi(c)=p$, where c is the closed point of* $\mathrm{Spec}\,F_m$.

Proof. The set $\mathrm{Spec}\,F_m$ consists of two points, the generic point η and the closed point c. Let $\phi:\mathrm{Spec}\,F_m\to X$ be a homomorphism and let $U\subset X$ be an open affine subset containing $\phi(c)$. Then η is contained in the open set $\phi^{-1}(U)$, so that indeed, ϕ is a homomorphism from $\mathrm{Spec}\,F_m$ to $U=\mathrm{Spec}\,A$. In other words, ϕ is affine. It therefore suffices to prove the lemma in case of affine X. In this case ϕ is given by a monoid morphism $\varphi:A\to\mathrm{F}_m$. The set $\mathfrak{p}=\varphi^{-1}(0)$ is a prime ideal and φ induces a homomorphism $A\smallsetminus\mathfrak{p}\to\mu_m$. Thus one gets a disjoint decomposition

$$\mathrm{Hom}(A,\mathrm{F}_m)=\coprod_{\mathfrak{p}\in\mathrm{Spec}\,A}\mathrm{Hom}(A\smallsetminus\mathfrak{p},\mu_m).$$

The lemma and the proposition follow. $\square$

The factors of Soulé's zeta function can be calculated as

$$\zeta^S(T,\mathcal{O}_{X,p}^\times)=s^{a_0}(s-1)^{a_1}\cdots(s-r)^{a_r},$$

where $r=\mathrm{rank}(\mathcal{O}_{X,p}^\times)=\dim_{\mathbf{Q}}(\mathcal{O}_{X,p}^\times\otimes\mathbf{Q})$ is the rank of the finitely generated group $\mathcal{O}_{X,p}^\times$, and

$$a_j=(-1)^{r-j}\binom{r}{j}.$$

So $\zeta^S(T,A)$ only depends on the ranks of the local groups $\mathcal{O}_{X,p}^\times$ and ignores the finite parts. We will now introduce a zeta function of Weil type, which carries more information.

2. The absolute Weil zeta function. In the theory of schemes over $\mathbf{F}_1$, the monoids F_m, $m\in\mathbf{N}$ play the role of finite fields. Analogous to the above definition of the local Weil zeta function we define the *Weil zeta function* of a scheme X of finite type over $\mathbf{F}_1$ as formal power series in T by

$$\zeta^W(T,X)\overset{\mathrm{def}}{=}\exp\left(\sum_{m=1}^\infty\frac{|\mathrm{Hom}(\mathrm{Spec}\,F_m,X)|}{m}T^m\right).$$

In the case when m is equal to $q-1$ for a prime power q, the monoid F_m can be identified with the multiplicative monoid of the finite field $\mathbf{F}_q$. In that case there is a natural bijection

$$\mathrm{Hom}(\mathrm{Spec}_{\mathbf{F}_1}F_{q-1},X)\xrightarrow{\cong}\mathrm{Hom}(\mathrm{Spec}\,\mathbf{F}_q,X_{\mathbf{Z}}),$$

where $X_{\mathbf{Z}}$ denotes the $\mathbf{Z}$-ascent of X.

In the case when X is affine, i.e., $X=\mathrm{Spec}\,A$ for some monoid A, we will also write $\zeta^W(T,A)$ instead of $\zeta^W(T,\mathrm{Spec}\,A)$.

By Lemma 1.3, the zeta function ζ^W satisfies the same decomposition formula as the Soulé zeta function,

$$\zeta^W(T,X)=\prod_{p\in X}\zeta^W(T,\mathcal{O}_{X,p}^\times).$$

We will now compute $\zeta^W(T,A)$ for a finitely generated abelian group A. Any such group is isomorphic to $\mathbf{Z}^r\times C_{n_1}\times\cdots\times C_{n_k}$ for some $r\geq 0$ and $n_1,\ldots,n_k\in\mathbf{N}$. Here C_n denotes the cyclic group of n elements. Since for coprime numbers n, n' one has $C_{nn'} \cong C_n \times C_{n'}$, one can arrange the cyclic groups in a way that n_j divides n_{j+1} for every $j = 1, \ldots, k-1$. Assuming this, we can prove

Proposition 2.1. *For* $A = \mathbf{Z}^r \times C_{n_1} \times \cdots \times C_{n_k}$ *one has*

$$\zeta^W(T, A) = \begin{cases} \prod_{\underline{d}|\underline{n}} (1 - T^{|\underline{d}|})^{-\varphi(\underline{d}) d_1^{k-2} \cdots d_k^{-1}} & \text{for } r = 0, \\ \prod_{\underline{d}|\underline{n}} \exp\left(\frac{g_r(T^{|\underline{d}|})\varphi(\underline{d}) d_1^{r+k-1} \cdots d_k^{r-1}}{(1-T^{|\underline{d}|})^r}\right) & \text{for } r \geq 1, \end{cases}$$

where we have used the following notation. The products run over all tuples $\underline{d} = (d_1, \ldots, d_k) \in \mathbf{N}^k$ *such that* $d_1 | n_1$, $d_2 | \frac{n_2}{d_1}$, *and so on until* $d_k | \frac{n_k}{d_1 \cdots d_{k-1}}$. *These conditions are summarized in the notation* $\underline{d}|\underline{n}$. *Further,* $|\underline{d}| = d_1 \cdots d_k$, *and* $\varphi(\underline{d}) = \varphi(d_1) \cdots \varphi(d_k)$, *where* φ *is the Euler* φ*-function. We denote by* $g_r(T) \in \mathbf{Z}[T]$ *the Euler polynomials which are defined recursively by*

$$g_1(T) = T$$

and

$$g_{r+1}(T) = \sum_{k=1}^{r} \binom{r}{k-1} (T-1)^{r-k} g_k(T). \tag{1}$$

Proof. Let $A = \mathbf{Z}^r \times C_{n_1} \times \cdots \times C_{n_k}$ as above, then

$$|\mathrm{Hom}(A, \mu_m)| = m^r (m, n_1) \cdots (m, n_k),$$

where (m, n) denotes the greatest common divisor of m and n. Using the property of the φ-function, $\sum_{d|n} \varphi(d) = n$, we infer that the logarithm of $\zeta^W(T, A)$ equals

$$\begin{aligned} &\sum_{m=1}^{\infty} m^{r+k-1} \frac{(m, n_1)}{m} \cdots \frac{(m, n_s)}{m} T^m \\ &= \sum_{m=1}^{\infty} m^{r+k-1} \sum_{d_1|(m,n_1)} \frac{\varphi(d_1)}{m} \\ &\quad \times \frac{(m, n_2)}{m} \cdots \frac{(m, n_k)}{m} T^m \\ &\overset{m=d_1\nu}{=} \sum_{d_1|n_1} \varphi(d_1) \sum_{\nu=1}^{\infty} (d_1 \nu)^{r+k-2} \\ &\quad \times \frac{(\nu, \frac{n_2}{d_1})}{\nu} \cdots \frac{(\nu, \frac{n_k}{d_1})}{\nu} T^{d_1 \nu}. \end{aligned}$$

The sum over ν is now of the same form as the first sum. So we can iterate the argument to reach

$$\sum_{d_1|n_1} \varphi(d_1) d_1^{r+k-2} \sum_{d_2|\frac{n_2}{d_1}} \varphi(d_2) d_2^{r+k-3} \cdots \sum_{d_k|\frac{n_k}{d_1 \cdots d_{k-1}}} \varphi(d_k) d_k^{r+k-k-1} \sum_{\nu=1}^{\infty} \nu^{r-1} T^{d_1 \cdots d_k \nu}.$$

The argument is finished with the following lemma which yields

$$\exp\left(\sum_{\nu=1}^{\infty} \nu^{r-1} T^\nu\right) = \exp\left(\frac{g_r(T)}{(1-T)^r}\right).$$

□

Lemma 2.2. *For* $r = 1, 2, 3, \ldots$, *we have*

$$\sum_{\nu=1}^{\infty} \nu^{r-1} T^\nu = \frac{g_r(T)}{(1-T)^r},$$

where $g_r(T) \in \mathbf{Z}[T]$ *is defined by* (1).

Proof. Put

$$S_r = \sum_{\nu=1}^{\infty} \nu^{r-1} T^\nu.$$

Since

$$T S_{r+1} = \sum_{\nu=1}^{\infty} \nu^r T^{\nu+1} = \sum_{\nu=1}^{\infty} (\nu - 1)^r T^\nu,$$

we compute

$$\begin{aligned} &(1-T) S_{r+1} \\ &= \sum_{\nu=1}^{\infty} (\nu^r - (\nu-1)^r) T^\nu \\ &= \sum_{\nu=1}^{\infty} \sum_{k=1}^{r} \binom{r}{k} (-1)^{k+1} \nu^{r-k} T^\nu \\ &= \sum_{k=1}^{r} \binom{r}{k} (-1)^{k+1} S_{r-k+1} \\ &= \sum_{k=0}^{r-1} \binom{r}{k} (-1)^{r-k-1} S_{k+1} \\ &= \sum_{k=1}^{r} \binom{r}{k-1} (-1)^{r-k} S_k. \end{aligned} \tag{2}$$

If we put

$$S_r = \frac{f_r(T)}{(1-T)^r}$$

and substitute it to (2), we find that $f_r(T)$ is a polynomial in T with $f_r(T) = g_r(T)$, because $f_r(T)$ also satisfies (1). □

Example 2.3 (Euler polynomials)**.**

$$g_1(T) = T,$$
$$g_2(T) = T,$$
$$g_3(T) = T^2 + T,$$
$$g_4(T) = T^3 + 4T^2 + T.$$

3. The absolute Igusa zeta function. We also introduce another type of zeta function which has an additive localization formula,

$$\zeta^I(s, X) = \sum_{p \in X} \zeta^I(s, \mathcal{O}_{X,p}^{\times}).$$

It is defined for $s \in \mathbf{C}$ with $\mathrm{Re}(s) \gg 0$ as

$$\zeta^I(s, X) \stackrel{\mathrm{def}}{=} \sum_{m=1}^{\infty} \frac{|\mathrm{Hom}(\mathrm{Spec}\, F_m, X)|}{m^s}.$$

This is analogous to the global Igusa zeta function of a $\mathbf{Z}$-scheme X, which is defined by

$$Z^I(s, X) = \sum_{m=1}^{\infty} \frac{|\mathrm{Hom}(\mathrm{Spec}(\mathbf{Z}/m\mathbf{Z}), X)|}{m^s}.$$

It has an Euler product expression

$$Z^I(s, X) = \prod_{p:\ \mathrm{prime}} Z_p^I(s, X)$$

with the local Igusa zeta function

$$Z_p^I(s, X) = \sum_{k=0}^{\infty} \frac{|\mathrm{Hom}(\mathrm{Spec}(\mathbf{Z}/p^k\mathbf{Z}), X)|}{p^{ks}}.$$

We refer to [3–5] for details on $Z_p^I(s, X)$ containing the rationality in p^{-s}. The analytic nature of the global Igusa zeta function $Z^I(s, X)$ is not so well-known, and it has a natural boundary even for a rather simple X as is shown in [7].

Proposition 3.1. *Let* $A = \mathbf{Z}^r \times C_{n_1} \times \cdots \times C_{n_k}$ *be a finitely generated abelian group. Then its Igusa zeta function* $\zeta^I(s, A)$ *equals*

$$\zeta(s-r) \prod_{p|n} \Bigg(p^{v_p(n)(1+r-s)} + (1 - p^{r-s}) \times \sum_{j=0}^{k-1} p^{v_p(n_1 \cdots n_j)} \sum_{l=v_p(n_j)}^{v_p(n_{j+1})-1} p^{(k+r-j-s)l} \Bigg),$$

where $\zeta(s)$ *denotes the Riemann zeta function,* $n = n_1 \cdots n_k$*, and* v_p *is the* p*-adic valuation. We also put* $n_0 = 1$.

Proof. We compute

$$\begin{aligned}
\zeta^I(s, A) &= \sum_{m=1}^{\infty} (m, n_1) \cdots (m, n_k) m^{r-s} \\
&= \prod_p \sum_{l=0}^{\infty} (p^l, n_1) \cdots (p^l, n_k) p^{r-s} \\
&= \prod_p \Bigg(\sum_{l=v_p(n_k)}^{\infty} p^{v_p(n_1 \cdots n_k)} p^{l(r-s)} + \sum_{j=0}^{k-1} \sum_{l=v_p(n_j)}^{v_p(n_{j+1})-1} p^{v_p(n_1 \cdots n_j)} p^{l(k-j)} p^{l(r-s)} \Bigg) \\
&= \prod_p \Bigg(p^{v_p(n)} p^{v_p(n_k)(r-s)} \frac{1}{1 - p^{r-s}} + \sum_{j=0}^{k-1} p^{v_p(n_1 \cdots n_j)} \sum_{l=v_p(n_j)}^{v_p(n_{j+1})-1} p^{(k+r-j-s)l} \Bigg).
\end{aligned}$$

This implies the claim. □

Recall the *Hurwitz zeta function*, which for $\alpha \in \mathbf{C}$ with $\mathrm{Re}(\alpha) > 0$ is defined by

$$\zeta_{Hur}(s, \alpha) = \sum_{n=0}^{\infty} \frac{1}{(n+\alpha)^s}.$$

It extends to a meromorphic function with a simple pole at $s = 1$ of residue 1.

Proposition 3.2. *The Igusa zeta function of a finitely generated group* A *as above can also be expressed as*

$$\zeta^I(s, A) = n_k^{r-s} \sum_{l=1}^{n_k} (l, n_1) \cdots (l, n_k) \zeta_{Hur}(s - r, \tfrac{l}{n_k}).$$

As an application one gets the identity

$$\frac{1}{n_k} \sum_{l=1}^{n_k} (l, n_1) \cdots (l, n_k) = \prod_{p|n} \Bigg(1 + \left(1 - \frac{1}{p}\right) \sum_{j=0}^{k-1} p^{v_p(n_1 \cdots n_j)} \sum_{l=v_p(n_j)}^{v_p(n_{j+1})-1} p^{(k-j-1)l} \Bigg).$$

Proof. We compute

$$\begin{aligned}
\zeta^I(s, A) &= \sum_{m=1}^{\infty} (m, n_1) \cdots (m, n_k) m^{r-s} \\
&= \sum_{l=1}^{n_k} \sum_{\nu=0}^{\infty} (l, n_1) \cdots (l, n_k) (l + \nu n_k)^{r-s} \\
&= n_k^{r-s} \sum_{l=1}^{n_k} (l, n_1) \cdots (l, n_k) \times \zeta_{Hur}(s - r, \tfrac{l}{n_k}),
\end{aligned}$$

as claimed. The application comes about by comparing the residues of the two expressions at $s = r + 1$. □

Proposition 3.3. *The Igusa zeta functions of the affine line, the projective line, and* GL_n *are*

- $\zeta^I(s, \mathbf{A}^1) = \zeta(s) + \zeta(s-1)$,
- $\zeta^I(s, \mathbf{P}^1) = 2\zeta(s) + \zeta(s-1)$,
- $\zeta^I(s, \mathrm{GL}_n) = n!\zeta(s-n)$.

Proof. This follows from the localization formula and the known structure of $\mathbf{A}^1, \mathbf{P}^1, \mathrm{GL}_n$. $\square$

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