A note on normality of meromorphic functions

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(Communicated by Heisuke HIRONAKA, M.J.A., April 12, 2007)

Abstract: Let $\mathcal{F}$ be a family of all functions f meromorphic in a domain $D \subset \mathbf{C}$, for which, all zeros have multiplicity at least k, and $f(z) = 0 \Leftrightarrow f^{(k)}(z) = 1 \Rightarrow |f^{(k+1)}(z)| \leq h$, where $k \in \mathbf{N}$ and $h \in \mathbf{R}^+$ are given. Examples show that $\mathcal{F}$ is not normal in general (at least for $k = 1$ or $k = 2$). The example we give for $k = 1$ shows that a recent result of Y. Xu [5] is not correct. However, we prove that for $k \neq 2$, there exists a positive integer $K \in \mathbf{N}$ such that the subfamily $\mathcal{G} = \{f \in \mathcal{F} :$ all possible poles of f in D have multiplicity at least $K\}$ of $\mathcal{F}$ is normal. This generalizes our result in [1]. The case $k = 2$ is also considered.

Key words: Holomorphic functions, meromorphic functions, normal family.

1. Introduction and main results. Let $D \subset \mathbf{C}$ be a domain and $\mathcal{F}$ a family of meromorphic functions in D. $\mathcal{F}$ is said to be normal in D in the sense of Montel, if each sequence $\{f_n\} \subset \mathcal{F}$ contains a subsequence which converges spherically locally uniformly in D to a meromorphic function or ∞. See [3, 6].

The following result is due to X. C. Pang and L. Zalcman [4].

Theorem A. *Let $k \in \mathbf{N}$ and $h \in \mathbf{R}^+$, let $\mathcal{F}$ be a family of functions meromorphic in a domain $D \subset \mathbf{C}$ such that for any $f \in \mathcal{F}$, all zeros of f have multiplicity at least k, and $f(z) = 0 \Leftrightarrow f^{(k)}(z) = 1 \Rightarrow 0 < |f^{(k+1)}(z)| \leq h$. Then $\mathcal{F}$ is a normal family in D.*

In [4], for $k = 2$, the authors gave an example to show that in Theorem A, the condition $f^{(k+1)}(z)$ is non-zero at the 1-points of $f^{(k)}(z)$ can not be dropped even if $\mathcal{F}$ is a family of holomorphic functions. Recently, Y. Xu [5] said that for $k = 1$, this condition can be removed. We point out that Y. Xu's result is not correct. See the following example.

Example 1. For every $n \in \mathbf{N}$, let

$$f_n(z) = \frac{2(e^{nz} + 1)}{n(e^{nz} - 1)}.$$

Then, for any f_n, we have

$$f_n'(z) - 1 = -\frac{(e^{nz} + 1)^2}{(e^{nz} - 1)^2}, \; f_n''(z) - 1 = \frac{4ne^{nz}(e^{nz} + 1)}{(e^{nz} - 1)^3},$$

so that f_n satisfies $f_n(z) = 0 \Leftrightarrow f_n'(z) = 1 \Rightarrow f_n''(z) = 0$.

However, the family $\mathcal{F} = \{f_n\}$ is not normal in $\mathbf{C}$.

If the family $\mathcal{F}$ in Theorem A consists of holomorphic functions, then the condition $f^{(k+1)}(z)$ is non-zero at the 1-points of $f^{(k)}(z)$ can be dropped for $k \neq 2$. Indeed, we have proved in [1] the following two results.

Theorem B. *Let $k \in \mathbf{N}$ and $h \in \mathbf{R}^+$, let $\mathcal{F}$ be a family of functions holomorphic in a domain $D \subset \mathbf{C}$ such that for any $f \in \mathcal{F}$, all zeros of f have multiplicity at least k, and $f(z) = 0 \Rightarrow f^{(k)}(z) = 1 \Rightarrow |f^{(k+1)}(z)| \leq h$. For the case $k = 2$, suppose in addition that there exists an even positive integer $s \geq 4$ such that for any $f \in \mathcal{F}$, $f^{(k)}(z) = 1 \Rightarrow |f^{(s)}(z)| \leq h$. Then $\mathcal{F}$ is a normal family in D.*

Theorem C. *Let $k \in \mathbf{N}$ with $k \geq 2$ and $h \in \mathbf{R}^+$, let $\mathcal{F}$ be a family of functions holomorphic in a domain $D \subset \mathbf{C}$ such that for any $f \in \mathcal{F}$, $f(z) = 0 \Rightarrow f'(z) = 1 \Rightarrow |f^{(k)}(z)| \leq h$. Then $\mathcal{F}$ is a normal family in D.*

In this note, we prove that Theorem B and Theorem C are also valid if the family consists of meromorphic functions, all of whose poles have sufficiently large multiplicity.

Theorem 1. *Let $k \in \mathbf{N}$ and $h \in \mathbf{R}^+$, let $\mathcal{F}$ be a family of all functions f meromorphic in a domain $D \subset \mathbf{C}$, for which, all zeros have multiplicity at least k, and $f(z) = 0 \Rightarrow f^{(k)}(z) = 1 \Rightarrow |f^{(k+1)}(z)| \leq h$. For the case $k = 2$, suppose in addition that there exists an even positive integer $s \geq 4$ such that for any $f \in \mathcal{F}$, $f^{(k)}(z) = 1 \Rightarrow |f^{(s)}(z)| \leq h$. Then there exists an integer $K \in \mathbf{N}$ such that the subfamily $\mathcal{G}_K = \{f \in \mathcal{F} :$*

2000 Mathematics Subject Classiffcation. Primary 30D45.

all possible poles of f in D have multiplicity at least $K\}$ of $\mathcal{F}$ is normal in D.

In Theorem 1, when $k=2$, the additional condition is really necessary.

Example 2 [4]. For every $n \in \mathbf{N}$, let

$$f_n(z) = \frac{1}{2n^2}(e^{nz} + e^{-nz} - 2) = \frac{e^{-nz}}{2n^2}(e^{nz} - 1)^2.$$

Then for any positive integer $j \in \mathbf{N}$,

$$f_n^{(j)}(z) = \frac{1}{2} n^{j-2}(e^{nz} + (-1)^j e^{-nz}).$$

Thus one can see that all zeros of f_n have multiplicity at least 2, $f_n(z) = 0 \Rightarrow f_n''(z) = 1$ and $f_n''(z) = 1 \Rightarrow f^{(s)}(z) = 0$ for any odd positive integer s.

However, the family $\{f_n\}$ is not normal at $z = 0$.

Theorem 2. *Let $k \in \boldsymbol{N}$ with $k \geq 2$ and $h \in \boldsymbol{R}^+$, let $\mathcal{F}$ be a family of all functions f meromorphic in a domain $D \subset \boldsymbol{C}$, for which, $f(z) = 0 \Rightarrow f'(z) = 1 \Rightarrow |f^{(k)}(z)| \leq h$. Then there exists an integer $K \in \boldsymbol{N}$ such that the subfamily $\mathcal{G}_K = \{f \in \mathcal{F}$: all possible poles of f in D have multiplicity at least $K\}$ of $\mathcal{F}$ is normal in D.*

By the present examples, the integer K must be larger than 1. We conjecture that one may take $K = 2$.

2. Lemmas. We require some known results. The first two are the well-known Marty's theorem and Zalcman's Lemma respectively.

Lemma 1 (see [3,6]). *Let $\mathcal{F}$ be a family of functions meromorphic in D. Then $\mathcal{F}$ is normal in D if and only if for any compact subset E of D, there exists a positive number $M = M(E)$ such that for any $z \in E$ and any $f \in \mathcal{F}$,*

$$f^{\#}(z) = \frac{|f'(z)|}{1 + |f(z)|^2} \leq M.$$

Lemma 2 [4]. *Let $\mathcal{F}$ be a family of functions meromorphic in the unit disk $D = \{z : |z| < 1\}$, all of whose zeros have multiplicity at least k, and suppose that there exists $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$. Then if $\mathcal{F}$ is not normal, there exist, for each $0 \leq \alpha \leq k$,*

a) *a number $0 < r < 1$;*
b) *points z_n, $|z_n| < r$;*
c) *functions $f_n \in \mathcal{F}$; and*
d) *positive numbers $\rho_n \to 0$*

such that $\rho_n^{-\alpha} f_n(z_n + \rho_n\zeta) = g_n(\zeta) \to g(\zeta)$ spherically locally uniformly, where g is a nonconstant meromorphic function on $\boldsymbol{C}$, all of whose zeros have multiplicity at least k, such that $g^{\#}(\zeta) \leq g^{\#}(0) = kA + 1$.

Lemma 3 [2]. *Let f be an entire function. If there exists a positive number M such that $f^{\#}(z) \leq M$ for any $z \in \boldsymbol{C}$, then f is of order at most one.*

Lemma 4 [1]. *Let $k \in \boldsymbol{N}$, let f be a nonconstant entire function of order at most one. Suppose all zeros of f have multiplicity at least k, and $f(z) = 0 \Rightarrow f^{(k)}(z) = 1 \Rightarrow f^{(k+1)}(z) = 0$. For the case $k = 2$, suppose in addition that there exists an even positive integer $s \geq 4$ such that $f^{(k)}(z) = 1 \Rightarrow f^{(s)}(z) = 0$. Then f must be of the form $f(z) = \frac{1}{k!}(z - z_0)^k$, where z_0 is a constant.*

Lemma 5 [1]. *Let $k \in \boldsymbol{N}$ with $k \geq 2$, let f be a nonconstant entire function of order at most one. Suppose that $f(z) = 0 \Rightarrow f'(z) = 1 \Rightarrow f^{(k)}(z) = 0$. Then f must be of the form $f(z) = z - z_0$, where z_0 is a constant.*

Remark. In Lemma 4 (Lemma 5), the condition that f is of order at most one can be dropped, since it follows from the other conditions. Indeed, under the other conditions, by Theorem B (Theorem C), the corresponding family $\{f(z + \zeta)\}_{z \in \mathbf{C}}$ is normal at $\zeta = 0$, and then by Marty's theorem, the spherical derivative $f^{\#}$ of f is uniformly bounded on $\mathbf{C}$, and hence by Lemma 3, f is of order at most one.

3. Proofs of Theorem 1 and Theorem 2. Since the proofs of Theorem 1 and Theorem 2 are similar to each other, we only give the proof of Theorem 1.

Proof of Theorem 1. Suppose for any $K \in \mathbf{N}$, the family $\mathcal{G}_K$ is not normal at some point $z_K \in D$. Then by Zalcman's Lemma (Lemma 2), there exist points $z_n \to z_K$, positive numbers $\rho_n \to 0$ and functions $f_n \in \mathcal{G}_K$ such that

$$g_n(\zeta) = \rho_n^{-k} f_n(z_n + \rho_n\zeta) \to G_k(\zeta)$$

spherically locally uniformly, where G_K is a nonconstant meromorphic function on $\mathbf{C}$, all of whose zeros have multiplicity at least k and all of whose poles have multiplicity at least K, such that $G_K^{\#}(\zeta) \leq G_K^{\#}(0) = k + 1$.

Using the same argument in [1, P.334–336], we can see that

$$G_K(\zeta) = 0 \Rightarrow G_K^{(k)}(\zeta) = 1 \Rightarrow G_K^{(k+1)}(\zeta) = 0,$$

with additional property $G_K^{(k)}(\zeta) = 1 \Rightarrow G_K^{(s)}(\zeta) = 0$ for the case $k = 2$.

Now we consider the family $\{G_K\}_{K \in \mathbf{N}}$. Since $G^{\#}(\zeta) \leq k + 1$, by Marty's theorem, it is normal in the whole plane **C**. So there exists a subsequence of $\{G_K\}_{K\in\mathbf{N}}$, say itself without any loss of generality, such that $\{G_K\}_{K\in\mathbf{N}}$ converges spherically locally uniformly in **C** to a meromorphic function G or ∞.

By $G_K^{\#}(\zeta) \le G_K^{\#}(0) = k+1$, we see that $G_K \to G$ and $G^{\#}(\zeta) \le G^{\#}(0) = k+1$. Further we can see that G is a nonconstant entire function, all zeros of G have multiplicity at least k, and $G(\zeta) = 0 \Rightarrow G^{(k)}(\zeta) = 1 \Rightarrow G^{(k+1)}(\zeta) = 0$ with additional property $G^{(k)}(\zeta) = 1 \Rightarrow G^{(s)}(\zeta) = 0$ for the case $k = 2$. Thus by Lemma 4, we have $G(\zeta) = \frac{1}{k!}(\zeta - \zeta_0)^k$, where ζ_0 is a constant. Simple calculation shows that $G^{\#}(0) \le \frac{k}{2}+1$, which contradicts $G^{\#}(0) = k+1$.

The proof of Theorem 1 is completed.

Acknowledgements. This work was supported by the NNSF of China (Grant No. 10471065 and No 10671073) and by the Fred and Barbara Kort Sino-Israel Post Doctoral Fellowship Program at Bar-Ilan University.

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