A class of Banach spaces

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Abstract: Let G be a separable locally compact unimodular group of type I, $\widehat{G}$ be its dual, $\hat{p}$ is a measurable field of, not necessary bounded, operators on $\widehat{G}$ such that $\hat{p}(\pi)$ is self-adjoint, $\hat{p}(\pi) \geq I$ for $\mu-$almost all $\pi \in \widehat{G}$, and

$$A_{\hat{p}}(G) = \{f(x) := \int_{\widehat{G}} Tr[\hat{f}(\pi)\pi(x)^{-1}]d\mu(\pi), \hat{f} \in L_1(\widehat{G}), \|f\|_{\hat{p}} = \int_{\widehat{G}} Tr|\hat{p}(\pi)\hat{f}(\pi)|d\mu(\pi) < \infty\}.$$

We show that $A_{\hat{p}}(G)$ is a Banach space endowed with the norm $\|f\|_{\hat{p}}$, and we generalize this result to the matricial group $G = G_{nm}$, $m \geq n$, of a local field.

Key words: Banach spaces, Beurling-Domar weight, Fourier transform and cotransform on nonabelian groups, uncertainty principle.

Introduction. Domar [2] gave a natural generalization of the Beurling algebras to any locally compact Abelian group (LCA) G, where the weight is a measurable function $\hat{p}(\hat{x})$ on $\widehat{G}$, the dual group of G, bounded on every compact set and satisfying:

$$\forall \hat{x}, \hat{y} \in \widehat{G}, \qquad \hat{p}(\hat{x}) \geq 1, \qquad \hat{p}(\hat{x}+\hat{y}) \leq \hat{p}(\hat{x})\hat{p}(\hat{y}).$$

The associated Banach algebra is:

$$F_{\hat{p}}(G) = \{f(x) := \int_{\widehat{G}} \hat{f}(\hat{x})\overline{(x,\hat{x})}d\hat{x}, x \in G, \hat{f} \in L_1(\widehat{G}), \int_{\widehat{G}} |\hat{p}(\hat{x})\hat{f}(\hat{x})|d\hat{x} < \infty\},$$

endowed with the norm $\|f\|_{\hat{p}} = \int_{\widehat{G}} |\hat{p}(\hat{x})\hat{f}(\hat{x})|d\hat{x}$. In fact, the essential characterization given by Domar [2, p. 18] for this algebra is the following: $F_{\hat{p}}(G)$ is of type $F(G)$ (see [2] or [8, p. 15]) if and only if $\sum_1^\infty \frac{log[\hat{p}(n\hat{x}_0)]}{n^2} < \infty$, which is the case if and only if for every neighborhood V of the identity in G, there exists a function in $F_{\hat{p}}(G)$ which vanishes outside V (that is to say $F_{\hat{p}}(G)$ is of non-quasianalytic type when $G = \mathbf{R}$).

If G is not Abelian, $\widehat{G}$, the dual of G, is no more a group and the natural extension (from the point of view that the weight must be defined on $\widehat{G}$) of Domar's results to G is a very difficult problem. We generalize here the space $F_{\hat{p}}(G)$, as Banach space, to a separable locally compact unimodular type I group G and to some nonunimodular groups. Indeed, Let $\hat{p}$ be a measurable field of, not necessary bounded, operators on $\widehat{G}$ such that $\hat{p}(\pi)$ is self-adjoint, $\hat{p}(\pi) \geq I$ for $\mu-$almost all $\pi \in \widehat{G}$, and

$$A_{\hat{p}}(G) = \{f(x) := \int_{\widehat{G}} Tr[\hat{f}(\pi)\pi(x)^{-1}]d\mu(\pi), \hat{f} \in L_1(\widehat{G}), \|f\|_{\hat{p}} = \int_{\widehat{G}} Tr|\hat{p}(\pi)\hat{f}(\pi)|d\mu(\pi) < \infty\}.$$

We establish that $(A_{\hat{p}}(G), \|.\|_{\hat{p}})$ is a Banach space, and then we generalize this result to the matricial group $G = G_{nm}$, $m \geq n$, of a local field, which is not unimodular. Finally we can raise the following open problem: is $A_{\hat{p}}(G)$ a Banach algebra with respect to pointwise multiplication for some unbounded weight $\hat{p}$?

1. Separable locally compact unimodular type I groups. Let G be a separable locally compact unimodular group, then G is of type I if and only if G is postliminary by [1, th., p. 168], which is the case if and only if (by [1, p. 271]) for every irreducible unitary representation π of G, the norm adherence of $\pi(L_1(G))$ contains the space of compact operators on $\mathcal{H}_\pi$, the space of representation of π.

Henceforth G denotes a separable locally compact unimodular postliminary group (SLCUP). Let $A(G) := \{u = f * \tilde{g}, f, g \in L_2(G), \tilde{g} = \overline{g(x^{-1})}\}$, endowed with the norm $\|u\| = \inf\{\|f\|_2\|g\|_2, u = f * \tilde{g}\}$, be its Fourier algebra, $\widehat{G}$ be its dual, i.e., the set of (equivalence classes of) irreducible unitary represen-

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tations, and μ be the Plancherel measure on $\widehat{G}$ associated with the Haar measure of G [1, p. 328]. If $f \in L_1(G)$, $\hat{f}$ denotes the usual Fourier transform of f, $\hat{f}(\pi) = \int_G f(x)\pi(x)dx$, and if $f \in A(G)$, $\hat{f}$ denotes the only element of $L_1(\widehat{G})$ such that $f(x) = \int_{\widehat{G}} Tr[\hat{f}(\pi)\pi(x)^{-1}]d\mu(\pi)$ (which is possible according to [7, th. 3.1, p. 217]). Note that these two notations coincide when $f \in A(G) \cap L_1(G)$. The following result generalizes its Abelian analogue [6, th. 8, p. 377] and gives again another proof easier.

Proposition 1. *Let $f \in L_1(G)$, then $\hat{f} \in L_2(\widehat{G})$ if and only if $f \in L_2(G)$.*

Proof. In view of Plancherel theorem [7, th. 2.1, p. 213], we have the sufficiency. We obtain the necessity by applying Parseval theorem [7, th. 2.3, p. 214] and [7, cor. 2.4, p. 216]. □

Theorem 2. *Let $\hat{p}$ be a measurable field of, not necessary bounded, operators on $\widehat{G}$ such that $\hat{p}(\pi)$ is self-adjoint, $\hat{p}(\pi) \geq I$ for $\mu-$almost all $\pi \in \widehat{G}$, and*

$$A_{\hat{p}}(G) = \{f(x) := \int_{\widehat{G}} Tr[\hat{f}(\pi)\pi(x)^{-1}]d\mu(\pi),$$
$$\hat{f} \in L_1(\widehat{G}), \|f\|_{\hat{p}} = \int_{\widehat{G}} Tr|\hat{p}(\pi)\hat{f}(\pi)|d\mu(\pi) < \infty\}.$$

Then $A_{\hat{p}}(G)$ is a Banach space endowed with the norm $\|f\|_{\hat{p}}$.

Proof. According to [8, cor. 22, p. 41], for each $\pi \in \widehat{G}$ such that $\hat{p}(\pi) \geq I$, we have

$$Tr|\hat{f}(\pi)| \leq Tr|\hat{p}(\pi)\hat{f}(\pi)|, \tag{1}$$

from which follows that $\|f\|_{\hat{p}}$ is a norm on $A_{\hat{p}}(G)$. Establish that $(A_{\hat{p}}(G), \|.\|_{\hat{p}})$ is complete. Let f_n be a Cauchy sequence in $A_{\hat{p}}(G)$, then, by exceeding some rank n_0, we have

$$\|\hat{p}\hat{f}_n - \hat{p}\hat{f}_m\|_1 = \int_{\widehat{G}} Tr|\hat{p}(\pi)\hat{f}_n(\pi) - \hat{p}(\pi)\hat{f}_m(\pi)|d\mu(\pi)$$
$$= \|f_n - f_m\|_{\hat{p}} \leq \varepsilon,$$

which implies that $\hat{p}\hat{f}_n$ is a Cauchy sequence in $L_1(\widehat{G})$, thus there exists $\hat{g} \in L_1(\widehat{G})$ such that $\hat{p}\hat{f}_n \longrightarrow \hat{g}$ in $L_1(\widehat{G})$. It suffices to show that there exists $f \in A(G)$ such that $\hat{p}\hat{f} = \hat{g}$. In fact, from (1) follows that

$$\|\hat{f}_n - \hat{f}_m\|_1 := \int_{\widehat{G}} Tr|\hat{f}_n(\pi) - \hat{f}_m(\pi)|d\mu(\pi)$$
$$\leq \int_{\widehat{G}} Tr|\hat{p}\hat{f}_n(\pi) - \hat{p}\hat{f}_m(\pi)|d\mu(\pi) \leq \varepsilon.$$

Hence $\hat{f}_n$ is a Cauchy sequence in $L_1(\widehat{G})$. It converges to $F \in L_1(\widehat{G})$ and thus, in view of [7, th. 3.1, p. 217], there exists $f \in A(G)$ such that $\hat{f} = F$. Show that $\hat{p}\hat{f} = \hat{g}$. Indeed, since ($\|.\|_\infty$ denotes the uniform norm (operator norm) in $\mathcal{L}_\infty(\mathcal{H}_\pi)$ the space of bounded linear operators on $\mathcal{H}_\pi$)

$$\int_{\widehat{G}} \|\hat{f}_n(\pi) - \hat{f}(\pi)\|_\infty d_\mu(\pi)$$
$$\leq \int_{\widehat{G}} Tr|\hat{f}_n(\pi) - \hat{f}(\pi)|d\mu(\pi) \longrightarrow 0$$

and

$$\int_{\widehat{G}} \|\hat{p}\hat{f}_n(\pi) - \hat{g}(\pi)\|_\infty d\mu(\pi)$$
$$\leq \int_{\widehat{G}} Tr|\hat{p}\hat{f}_n(\pi) - \hat{g}(\pi)|d\mu(\pi) \longrightarrow 0,$$

then, according to Riesz theorem [4, p. 156], there exists a subsequence $\hat{f}_{nk}$ such that

$$\|\hat{f}_{nk}(\pi) - \hat{f}(\pi)\|_\infty \longrightarrow 0,$$
$$\text{and } \|\hat{p}\hat{f}_{nk}(\pi) - \hat{g}(\pi)\|_\infty \longrightarrow 0,$$

for $\mu-$almost all $\pi \in \widehat{G}$. It follows that

$$< \hat{f}_{nk}(\pi)y, \hat{p}(\pi)\hat{f}_{nk}(\pi)y > \longrightarrow < \hat{f}(\pi)y, \hat{g}(\pi)y >,$$

for all $y \in \mathcal{H}_\pi$ and $\mu-$almost all $\pi \in \widehat{G}$. Now $G([(\hat{p}(\pi)]^*)$, the graph of $[(\hat{p}(\pi)]^*$, is closed for $\mu-$almost all $\pi \in \widehat{G}$ (see for example [8, rq., p. 46], with $T = \hat{p}(\pi)$). Then $G(\hat{p}(\pi)) = G([(\hat{p}(\pi)]^*)$ is closed for $\mu-$almost all $\pi \in \widehat{G}$, thus $(\hat{f}(\pi)y, \hat{g}(\pi)y) \in G(\hat{p}(\pi))$ and $\hat{p}(\pi)\hat{f}(\pi)y = \hat{g}(\pi)y$ for all $y \in \mathcal{H}_\pi$ and for $\mu-$almost all $\pi \in \widehat{G}$. Consequently $\hat{p}\hat{f} = \hat{g}$. □

Problem 1 (open problem)**.** Is the Banach space $A_{\hat{p}}(G)$ a Banach algebra with respect to pointwise multiplication for some unbounded weight $\hat{p}$?

Remark. The dictionary which enables us to pass over from $F_{\hat{p}}(G)$ to $A_{\hat{p}}(G)$ is the following

$$F_{\hat{p}}(G) = \{f \in A(G), \int_{\widehat{G}} |\hat{p}(\hat{x})\hat{f}(\hat{x})|d\hat{x} < \infty\}$$

and

$$A_{\hat{p}}(G) = \{f \in A(G), \int_{\widehat{G}} Tr|\hat{p}(\pi)\hat{f}(\pi)|d\mu(\pi) < \infty\}.$$

2. The matricial group $G = G_{nm}$, $m \geq n$, of a local field. Let $\mathbf{K}$ be a local field, $n \leq m \in \mathbf{N}^*$. Let M_{nm} be the space of all $n \times m$-matrices with elements from $\mathbf{K}$, GL_n be the multiplicative group of all $n \times n$-invertible matrices with elements from $\mathbf{K}$, and $G = G_{nm}$ be the semi-direct product $M_{nm} \ltimes GL_n$, i.e., G_{nm} denotes the group of pairs

(b, a), where $b \in M_{nm}$ and $a \in GL_n$, with multiplication given by $(b, a)(b', a') = (b + ab', aa')$. Let $\mathcal{H}$ be the Hilbert space $L^2(GL_n, \frac{du}{|det(u)|^n})$, where $|.|$ is the module in $\mathbf{K}$. For all λ in M_{mn}, the formula

$$[\pi_\lambda(b, a)\xi](u) = \tau(Tr(b\lambda u))\xi(ua),$$

defines a unitary representation of G_{nm} in $\mathcal{H}$, where $(b, a) \in G$, $\xi \in \mathcal{H}$, $u \in GL_n$, and τ is a fixed additive unitary nontrivial character on $\mathbf{K}$. Letting $S = S_{mn}$ denote the canonical realization in M_{mn} (see [8, p. 56, 57], S is a well defined part of M_{mn}) which identifies with $\widehat{G}_{ess} = \{$equivalence classes of $\pi_\lambda, \lambda \in S\}$, the essential dual of $G = G_{nm}$, and which bears the Plancherel measure that we denote by $d\mathbf{s}(\lambda)$.

Now we shall introduce the notion of the regularized Fourier cotransformation on G, which helps as a guide to pass over from the unimodular case to the nonunimodular one, and translates, mainly vis-à-vis the Fourier inversion, the usual Fourier transformation on LCA and SLCUP groups. In fact, let $\mathcal{L}_1(\mathcal{H})$ be the space of nuclear operators on $\mathcal{H}$, $L^1(S, \mathcal{L}_1(\mathcal{H})) = \{F : S \to \mathcal{L}_1(\mathcal{H}), \int_S Tr|F(\lambda)|d\mathbf{s}(\lambda) < \infty\}$, and $\bar{\mathcal{F}}$ be the Fourier cotransformation, which an isometry of Banach spaces of $L^1(S, \mathcal{L}_1(\mathcal{H}))$ onto $A(G)$, defined by

$$\bar{\mathcal{F}}(F)(x) = \int_S Tr[\pi_\lambda(x)F(\lambda)]d\mathbf{s}(\lambda).$$

Then we define the regularized Fourier cotransform of a function $f \in A(G)$ by

$$\hat{f} := \bar{\mathcal{F}}^{-1}(\check{f}), \tag{2}$$

where $\check{f}(x) = f(x^{-1})$, and the following proposition justifies this notation. Recall that if G is a SLCUP group, then

$$\hat{f} \longrightarrow f(x) := \int_{\widehat{G}} Tr[\hat{f}(\pi)\pi(x)^{-1}]d_\mu(\pi)$$

is an isometry of Banach spaces of $L^1(\widehat{G})$ onto $A(G)$ by [7, th. 3.1, p. 217], and if $f \in A(G) \cap L^1(G)$, we have $\hat{f} = \mathcal{F}(f)$, the usual Fourier transform of f. If $G = G_{nm}$, definition (2) generalizes these notations:

Proposition 3. *The regularized Fourier cotransformation*

$$\hat{f} \longrightarrow f(x) := \int_S Tr[\hat{f}(\lambda)\pi_\lambda(x)^{-1}]d\boldsymbol{s}(\lambda)$$

is an isometry of Banach spaces of $L^1(S, \mathcal{L}_1(\mathcal{H}))$ *onto* $A(G)$. *If moreover* $f \in A(G) \cap L^1(G)$, *then* $\hat{f} = \mathcal{F}(f) \circ \delta_1$, *where* δ_1 *is the unbounded operator in* $\mathcal{H}$ *defined by* $\delta_1\xi(u) = |det(u)|^m\xi(u)$, *and* $\mathcal{F}(f)\lambda := \pi_\lambda(f)$ *if* $\lambda \in S$.

Proof. Since $\hat{f} := \bar{\mathcal{F}}^{-1}(\check{f})$, then $\bar{\mathcal{F}}(\hat{f}) = \check{f}$, and thus

$$f(x) = \check{f}(x^{-1}) = \bar{\mathcal{F}}(\hat{f})(x^{-1}) = \int_S Tr[\hat{f}(\lambda)\pi_\lambda(x)^{-1}]d\mathbf{s}(\lambda).$$

On the other hand, if $f \in A(G)$, then $\|f\| = \|\check{f}\|$ by [8, form. (1.1), p. 22]. It follows that $\hat{f} \longrightarrow f(x) := \int_S Tr[\hat{f}(\lambda)\pi_\lambda(x)^{-1}]d\mathbf{s}(\lambda)$ is an isometry of $L^1(S, \mathcal{L}_1(\mathcal{H}))$ onto $A(G)$. Suppose that $f \in A(G) \cap L^1(G)$, then in view of [8, th. 36, p. 58] we have

$$f(x) = \int_S Tr[\mathcal{F}f(\lambda) \circ \delta_1\pi_\lambda(x)^{-1}]d\mathbf{s}(\lambda).$$

Therefore $\mathcal{F}(f) \circ \delta_1 = \hat{f}$.

Note that the appearance of the unbounded operator δ_1 comes from the fact that G is not unimodular. □

Theorem 4. *Let* $\hat{p}$ *be a measurable function on* S *with values in* $\mathcal{L}(\mathcal{H})$, *the space of linear* (*not necessary bounded*) *operators in* $\mathcal{H}$, *such that* $\hat{p}(\lambda)$ *is self-adjoint*, $\hat{p}(\lambda) \geq I$ *for almost all* $\lambda \in S$, *and*

$$A_{\hat{p}}(G) = \{f(x) := \int_S Tr[\hat{f}(\lambda)\pi_\lambda(x^{-1})]d\boldsymbol{s}(\lambda), \quad \hat{f} \in L^1(S, \mathcal{L}_1(\mathcal{H})), \quad \int_S Tr|\hat{p}(\lambda)\hat{f}(\lambda)|d\boldsymbol{s}(\lambda) < \infty\}.$$

Then $A_{\hat{p}}(G)$ *is a Banach space under the norm* $\|f\|_{\hat{p}} = \int_S Tr|\hat{p}(\lambda)\hat{f}(\lambda)|d\boldsymbol{s}(\lambda)$.

Proof. The proof is analogous to the proof of Theorem 2. □

Examples (of weights). Let $x \in \mathbf{R}^*$, δ_x the unbounded operator in $\mathcal{H}$ defined by $\delta_x\xi(u) = |det(u)|^{mx}\xi(u)$, then the constant weight $\hat{p}$ defined by $\hat{p}(\lambda) := \delta_x + I$, for every $\lambda \in S$, satisfies the hypothesis of Theorem 4.

For recent results on the group G_{nm} when $m = n = 1$ see [9]. In that case the essential dual of G remounts to a single point denoted π and the space $L^1(S, \mathcal{L}_1(\mathcal{H}))$ is merely $\mathcal{L}_1(\mathcal{H})$.

As for the uncertainty principle for the matricial group of a local field, our results on the Hausdorff-Young theorem for G_{nm} and the inversion theorem for $L^p(G_{nm})$ enable us to give the following natural generalization of [9, th. 4 and cor. 5] to $G = G_{nm}$ ($n \leq m$):

Theorem 5. *Let* K *be a compact subset of* G, M *be a finite dimension subspace of* $\mathcal{H}$. *Then the space*

$$A_{K,M}(G) = \{f \in A(G),\ supp(f) \subseteq K,\ supp(\hat{f}) \subseteq M\},$$

where $supp(\hat{f}) \subseteq M$ *means that* $Im(\hat{f}(\lambda)) \subseteq M$ *for almost all* $\lambda \in S$, *is a Banach space of finite dimension.*

Corollary 6. *If* $\mathbf{K} = \mathbf{C}$ *or* $\mathbf{R}$, *then* $A_{K,M}(G) = 0$.

For recent results on (the weak and topological) Paley-Wiener property for group extensions and locally compact groups see [3, 5]. In our case $G = G_{nm}$, and by Corollary 6, if $\mathbf{K} = \mathbf{C}$ or $\mathbf{R}$, then the P.W property [9] is valid on G, in other words, a function $f \in A(G)$ with compact support is identically zero if and only if there exists a finite dimension subspace M of $\mathcal{H}$ such that $\text{supp}(\hat{f}) \subseteq M$.

If G is a LCA group, Theorem 5 can be read as the following: let K be a compact subset of G, $\widehat{K}_1$ be a compact subset of $\widehat{G}$, then the space

$$A_{K,\widehat{K}_1}(G) = \{f \in A(G),\ \text{supp}(f) \subseteq K,\ \text{supp}(\hat{f}) \subseteq \widehat{K}_1\}$$

is a Banach space of finite dimension. From which follows that the P.W property is valid on G (that is $A_{K,M}(G) = 0$ for all K and $\widehat{K}_1$ as above) if and only if G has no non-empty open compact subset. This yields to raise the following open problem: *what happens for Corollary 6 if* $\mathbf{K} \neq \mathbf{C}$ *and* $\neq \mathbf{R}$?

Note that if $\mathbf{K} \neq \mathbf{C}$ and $\neq \mathbf{R}$, then $G = G_{nm}$ does have non-empty open compact subsets.

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