Estimates of the proximate function of differential polynomials

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Abstract: We obtain a Clunie type theorem for a rather general form of functional equations involving differential polynomials. Our theorems can give a much sharper estimate on the error term of the proximity function of solutions of differential equations and functional equations than the upper bound obtained by Clunie, Doeringer, He-Xiao, Korhonen and etc. In particular, our theorem can also be applied to study various types of Painlevé differential equations.

Key words: Nevanlinna's value distribution theory, differential polynomial, Painlevé equations.

1. Introduction. Let f denote a function meromorphic on the complex plane. Nevanlinna theory of meromorphic functions has played an important role in the study of complex differential equations. One of the quantities that people always want to know in solving a complex differential equation is $m(r, f)$, the proximity function of its solution. In 1960's, J. Clunie [2] proved a lemma giving an estimate of a proximity functions, which has numerous applications to complex functional and differential equations. Recently, Shimomura (e.g. see: [9, 10]) and Steinmetz [13] use this kind of lemma in their studies in Painlevé differential equations. Since Clunie's work in 1960, there are several generalizations based on the Clunie's lemma. I. Laine called them as Clunie type lemmas in his book [7, Lemmas 2.4.1-2.4.5; pg.39-55]. In 2004, R. Korhonen [6] studied Clunie type lemma and had a sharper estimate of the error term in the Clunie's lemma. In 2006, Korhonen [6] corrected several errors in his theorems in [6] and improved an estimate of the error terms in two Clunie type lemmas. In this paper, we derive a better estimate on proximity functions than what were obtained by Clunie [2], Doeringer [3], He-Xiao [5] and Korhonen [6]. The general treatment of Clunie type lemmas is to utilize a logarithmic derivative lemma as in [2, 3, 5, 6]. Here we not only use a modified version of Gol'dberg-Grinshtein's logarithmic derivative lemma, but also treat every step with thorough analyses and technical calculations. The technique used in this paper can be used to deal with other kind of problems to get a better error term in the case of seeking sharper upper bound. Our main theorems enable one to derive, with easy, more precise estimates of the proximity functions $m(r, f)$ and $m(r, 1/(f-a))$, for any value $a \neq \infty$, for many well-known meromorphic functions such as $\exp z$, $\sin z$ and for meromorphic solutions f of various types of Painlevé differential equations. Moreover, we also show a sharper error term than that of Mohon'ko's result in [8].

2. Main results. Let n be a positive integer and

$$P(z, f) = \sum_{k=0}^{n} p_k(z) f^k(z)$$

a polynomial of f with meromorphic function coefficients p_k's. Let $\Lambda = \{(\lambda_0, \lambda_1, \cdots, \lambda_\mu) : \lambda_j$ is a non-negative integer and $0 \leq j \leq \mu < \infty\}$ be an index set with a finite cardinal number and let

$$A^*(z, f) = \sum_{\lambda \in \Lambda} a_\lambda(z) f^{\lambda_0} (f')^{\lambda_1} \cdots (f^{(\mu)})^{\lambda_\mu}$$

be a polynomial of f and its derivatives with meromorphic coefficients a_λ's. Clearly, $P(z, f)$ is a special form of $A^*(z, f)$. In the sequel, notation $A^*(z, f)$, an alphabet A with an asterisk, denotes a differential polynomial in general sense in f, while notation $A(z, f)$, an alphabet A without an asterisk, denotes a polynomial of f with meromorphic coefficients. Denote the length of $\lambda = (\lambda_0, \lambda_1, \cdots, \lambda_\mu) \in \Lambda$ and the total degree of $A^*(z, f)$ by $|\lambda| = \sum_{j=0}^{\mu} \lambda_j$ and $d(A^*) = \max_{\lambda \in \Lambda} |\lambda|$; and the weight length of λ and

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the weight degree of $A^*(z,f)$ by $w(\lambda)=\sum_{j=1}^{\mu} j\lambda_j$ and $w(A^*)=\max_{\lambda\in\Lambda} w(\lambda)$, respectively.

We begin with the Clunie's lemma in the form given by Laine [7].

Theorem A (Clunie). *Let f be a transcendental meromorphic solution of*

$$f^n A^*(z,f)=B^*(z,f),$$

where A^* and B^* have meromorphic coefficients a_λ and b_γ, respectively, with

$$m(r,a_\lambda)=o(T(r,f)) \ \text{ and } \ m(r,b_\gamma)=o(T(r,f)).$$

If $n\geq d(B^*)$, *then*

$$m(r,A^*(z,f))=o(T(r,f))$$

for all large r outside a set of a finite Lebesgue measure.

Theorem B (Korhonen). *Suppose f satisfies the equation in Theorem A with*

$$A^*(z,f)=\sum_{\lambda\in\Lambda} a_\lambda(z) f^{\lambda_0}(f')^{\lambda_1}\cdots(f^{(p)})^{\lambda_p} \quad \text{and}$$

$$B^*(z,f)=\sum_{\gamma\in\Delta} b_\gamma(z) f^{\gamma_0}(f')^{\gamma_1}\cdots(f^{(q)})^{\gamma_q}.$$

If $n\geq d(B^*)$, *Then, there exists r_0 such that*

$$m(r,A^*(z,f))\leq \left(\sum_{\lambda\in\Lambda} w(\lambda)+\sum_{\mu\in\Delta} w(\mu)\right)\log^+\frac{\rho T(\rho,f)}{r(\rho-r)} +\sum_\lambda m(r,a_\lambda)+\sum_\gamma m(r,b_\gamma)+O(1)$$

for all $r_0<r<\rho<\infty$.

Note the author in [6] *has an explicit expression of constant term in Theorem B.*

Theorem C (He-Xiao). *Let f be a transcendental meromorphic solution of*

$$P(z,f)A^*(z,f)=Q(z,f),$$

where all coefficients p_s, q_j and a_λ of P, Q and A^* satisfy, $m(r,p_s)=m(r,1/p_s)=m(r,q_s)=m(r,a_\lambda)=O(\log(rT(r,f)))$. If $d(P)\geq d(Q)$, *then*

$$m(r,A^*(z,f))=o(T(r,f))$$

for all large r outside a set of a finite Lebesgue measure.

Theorem D (Mohon'ko). *Let f be a transcendental meromorphic solution of the differential equation*

$$A^*(z,f)=0$$

with polynomial coefficients. If there is a constant c with $A^*(z,c)\not\equiv 0$, then

$$m(r,1/(f-c))=o(T(r,f))$$

as $r\to\infty$ outside of a possible exceptional set of finite Lebesgue measure.

The following is the main result in this paper.

Theorem 1. *Suppose that f is a meromorphic solution of the differential equation*

$$P(z,f)A^*(z,f)=B^*(z,f).$$

If $d(P)\geq d(B^*)$, then, there is a constant r_0 such that,

$$\begin{aligned} m(r,\ & A^*(z,f))\leq \\ & \max(w(A^*),\ w(B^*))\log^+\frac{\rho T(\rho,f)}{r(\rho-r)} \\ & +\sum_\lambda m(r,a_\lambda)+\sum_\gamma m(r,b_\gamma) \\ & +c\sum_{j=0}^{n-1}\frac{m(r,p_j)}{n-j}+\left(1+\sum_{j=1}^{n}\frac{c}{j}\right)m(r,1/p_n) \\ & +O(1), \end{aligned}$$

for any $\rho>r>r_0$, where $c=\sum_\lambda|\lambda|$.

Remark. The condition $d(P)\geq d(B^*)$ in the theorem is necessary. For example, $f=\sin z$ is a solution of the differential equation $f^2=1-(f')^2$. We can take

$$P(z,f)=A^*(z,f)=f \quad \text{and} \quad B^*(z,f)=1-(f')^2.$$

Thus, $d(P)<d(B^*)$ and the conclusion of the theorem is not true.

Let ψ and ϕ be increasing functions in $(0,\infty)$ with

$$\int_e^\infty \frac{dr}{r\psi(r)}<\infty \text{ and } \int_e^\infty \frac{dr}{\phi(r)}=\infty.$$

Applying a growth lemma (e.g. see. [1, Pg. 99]) to Theorem 1, we have

Corollary 1. *Under the assumptions of Theorem 1, we have*

$$\begin{aligned} m(r,\ & A^*(z,f))\leq \\ & \max(w(A^*),w(B^*))\log^+\frac{T(r,f)\psi(T(r,f))}{\phi(r)} \\ & +\sum_\lambda m(r,a_\lambda)+\sum_\gamma m(r,b_\gamma) \end{aligned}$$

$$+c\sum_{j=0}^{n-1}\frac{m(r,p_j)}{n-j}+\left(1+\sum_{j=1}^{n}\frac{c}{j}\right)m(r,1/p_n)$$
$$+O(1),$$

for all large r outside a set E with $\int_E dr/\phi(r)<\infty$.

By choosing a different ψ and ϕ, we can control the upper bound of $m(r,A^*(z,f))$ and the exceptional set E. For instance, when we take $\psi(r)=r$ and $\phi(r)=1$, our Corollary 1 gives a better estimate on the proximity function than those in Theorems A and C.

As a by-product of these ideas, we also can improve A. Mohon'ko and V. Mohon'ko's result in [8] as follows:

Theorem 2. *Let f be a transcendental meromorphic solution of the differential equation*

$$A^*(z,f)=0.$$

If there is a constant c with $A^*(z,c)\not\equiv 0$, then, there exists a positive constant r_0 such that

$$m\left(r,\frac{1}{f-c}\right)\le w(A^*)\log^+\frac{\rho T(\rho,f)}{r(\rho-r)}$$
$$+m\left(r,\frac{1}{A^*(z,c)}\right)+\sum_\lambda m(r,a_\lambda)+O(1),$$

for any $\rho>r>r_0$.

Corollary 2. *Under the assumptions of Theorem 2, there is a positive constant r_0 such that*

$$m\left(r,\frac{1}{f-c}\right)\le$$
$$w(A^*)\log^+\frac{T(r,f)\psi(T(r,f))}{\phi(r)}$$
$$+m\left(r,\frac{1}{A^*(z,c)}\right)+\sum_\lambda m(r,a_\lambda)+O(1),$$

for all large r outside a set E with $\int_E dr/\phi(r)<\infty$.

We give some applications of Theorems 1 and 2 as follows:

Example 1. Let $w(z)$ be a meromorphic solution of the fourth type Painlevé equation

$$ww''=\frac{1}{2}(w')^2+\frac{3}{2}w^4+4zw^3+2(z^2-\beta)w+\gamma.$$

Rewrite it as

$$\left(\frac{3}{2}w^2+4zw\right)w^2=ww''-\frac{1}{2}(w')^2-2(z^2-\beta)w-\gamma$$

with $P(z,w)=\dfrac{3}{2}w^2+4zw$, $A^*(z,w)=w^2$ and $B^*(z,w)=ww''-\frac{1}{2}(w')^2-2(z^2-\beta)w-\gamma$ in Theorem 1. Knowing that $d(P)=2=d(B^*)$, $w(A^*)=0$, $w(B^*)=2$ and $T(r,w)\le Cr^4$ in [9, 11, 13], we have, from Theorem 1, that, for all large r,

$$m(r,w)\le 5\log r+O(1).$$

Korhonen [6, (8.3)] proves $m(r,w)\le 15\log r+O(1)$. In many references, one can only get $m(r,w)\le O(\log r)$. If $\gamma\ne 0$, then, we obtain from Theorem 2 that for any complex number a, $m(r,1/(w-a))\le 9\log r+O(1)$. If $\gamma=0$, then, for any non-zero complex number a, $m(r,1/(w-a))\le 9\log r+O(1)$.

Example 2. Let $w(z)$ be a meromorphic solution of the second type Painlevé equation $w''=2w^3+zw+\alpha$. Rewrite it as $(2w)w^2=w''-zw-\alpha$ with $P(z,w)=2w$, $A^*(z,w)=w^2$ and $B^*(z,w)=w''-zw-\alpha$ in Theorem 1. Knowing that $T(r,w)\le Cr^3$ in [9, 11, 13], we have, from Theorem 1, that, $m(r,w^2)\le 4\log r+\log r+O(1)$. Therefore, $m(r,w)\le\frac{5}{2}\log r+O(1)$ for all large r. Korhonen in [6] proves $m(r,w)\le 5\log r+O(1)$ and one gets $m(r,w)=O(\log r)$ before. If $\alpha\ne 0$, then, we obtain from Theorem 2 that for any complex number a, $m(r,1/(w-a))\le 5\log r+O(1)$. If $\alpha=0$, then, for any non-zero complex number a, $m(r,1/(w-a))\le 5\log r+O(1)$.

Example 3. We also can use Theorems 1 and 2 to estimate the proximate functions for many classical meromorphic functions, such as, e^z, $\sin z$, $\cos z$, $\tan z$ and etc. For instance, let $f(z)=\sin z$. It satisfies the equations $f^2-1+(f')^2=0$ and $f''+f=0$. Recall that $T(r,f)=2r/\pi+O(1)$. For any complex number a, applying Theorem 2 to $f^2-1+(f')^2=0$ if $a\ne\pm 1$ and to $f''+f=0$ if $a=\pm 1$, we have $m(r,1/(\sin z-a))=O(1)$. This is much better than $m(r,1/(\sin z-a))=O(\log r)$ as stated in many references.

3. Proofs of results. We need following lemmas in our proofs.

Lemma 1. *Let f be a non-constant meromorphic function in the complex plane. Let s be a positive integer and α a positive real number with $0<\alpha s<1/2$. Then, there are two constants $r_0>1$ and $C=C(s,\alpha,r_0)$ such that, for all $r_0<r<\rho$,*

$$\int_0^{2\pi}\left|\frac{f^{(s)}}{f}(re^{i\theta})\right|^\alpha\frac{d\theta}{2\pi}\le C\left(\frac{\rho T(\rho,f)}{r(\rho-r)}\right)^{s\alpha}.$$

The lemma cited here is a simple version of the result of Z. Ye [14, Lemma 6]. The proof is based on a result of Gol'dberg-Grinshtein [4].

Lemma 2. *Let k be a positive integer and let f be a meromorphic function and*

$$g_j = f^{(j)}/f, //1 \le j \le k.$$

Assume that $\lambda_j (1 \le j \le k)$ are non-negative integers. Then, for any $\beta > 0$ with $0 < \beta \sum_{j=1}^{k} j\lambda_j < 1/2$, there are two positive constants r_0 and $C = C(k, \sum_{j=1}^{k} \lambda_j, \beta, r_0)$ such that, for all $r_0 < r < \rho$,

$$\int_0^{2\pi} \prod_{j=1}^{k} |g_j|^{\beta\lambda_j} \frac{d\theta}{2\pi} \le C \left(\frac{\rho T(\rho, f)}{r(\rho - r)} \right)^{\beta \sum_{j=1}^{k} j\lambda_j}.$$

Proof. The Holder inequality and Lemma 1 give

$$\int_0^{2\pi} \prod_{j=1}^{k} |g_j|^{\beta\lambda_j} \frac{d\theta}{2\pi} \le \prod_{j=1}^{k} \left(\int_0^{2\pi} |g_j|^{\beta\lambda_j k} \frac{d\theta}{2\pi} \right)^{1/k}$$
$$\le C \prod_{j=1}^{k} \left(\frac{\rho T(\rho, f)}{r(\rho - r)} \right)^{\beta j \lambda_j}.$$

It follows the lemma is proved.

Proof of Theorem 1. Let

$$u(z) = \max_{1 \le j \le n} \left(1, 2 \left| \frac{p_{n-j}}{p_n} \right|^{1/j} \right).$$

Thus,

$$(1) \qquad m(r, u) \le \sum_{j=0}^{n-1} \frac{m(r, p_j)}{n - j} + \left(\sum_{j=1}^{n} \frac{1}{j} \right) m(r, 1/p_n) + O(1).$$

Assume that, for any fixed $r > 0$, and $z = re^{i\theta}$,

$$E = E(r) = \{\theta \in [0, 2\pi) : \quad |f(z)| \le u(z)\}$$

and $F = F(r) = [0, 2\pi) \setminus E(r)$. Define $\chi_E(\theta) = 1$ when $\theta \in E$; and otherwise $\chi_E(\theta) = 0$.

Noting for any $\alpha > 0$ and $x_k \ge 0$, there is a positive constant $C(\alpha)$ such that $(\sum x_k)^\alpha \le C(\alpha) \sum x_k^\alpha$, we obtain, set $g_j = f^{(j)}/f$,

$$(2)\ |A^*(z, f)|^\alpha \le C(\alpha) \sum_{\lambda} \left(|a_\lambda f^{\lambda_0}|^\alpha \prod_{j=1}^{\mu} |f^{(j)}|^{\lambda_j \alpha} \right)$$
$$= C(\alpha) \sum_{\lambda} \left(|a_\lambda f^{|\lambda|}|^\alpha \prod_{j=1}^{\mu} |g_j|^{\lambda_j \alpha} \right)$$
$$\le C(\alpha) \left(\sum_{\lambda} |a_\lambda f^{|\lambda|}|^{2\alpha} \right)^{1/2} \left(\sum_{\lambda} \prod_{j=1}^{\mu} |g_j|^{\lambda_j 2\alpha} \right)^{1/2}.$$

So, for any $\theta \in E(r)$, we have

$$(3) \qquad |A^*(z, f)|^\alpha \le C(\alpha) \left(\sum_{\lambda} |a_\lambda u^{|\lambda|}|^{2\alpha} \right)^{1/2} \left(\sum_{\lambda} \prod_{j=1}^{\mu} |g_j|^{\lambda_j 2\alpha} \right)^{1/2} \chi_E(\theta).$$

For $\theta \in F(r)$, we get $|f(z)| > u(z) \ge 2|p_{n-j}/p_n|^{1/j}$ for $j = 1, \cdots, n$. Thus,

$$\frac{|p_{n-j}|}{|p_n|} \le \frac{|f|^j}{2^j} \quad \text{for} \quad j = 1, \cdots, n.$$

Therefore,

$$|P(z, f)| \ge |p_n||f|^n \left(1 - \sum_{j=1}^{n} \frac{|p_{n-j}|}{|f^j p_n|} \right) \ge \frac{|p_n||f|^n}{2^n}.$$

Set $h_k = f^{(k)}/f$. Similar to the computation of (2), and noting

$$|f(z)| > u(z) \ge 1, \text{ for } \theta \in F(r); \text{ and } d(P) \ge d(B^*),$$

we have, for any small $\alpha > 0$,

$$(4) \quad |A^*(z, f)|^\alpha = \left| \frac{B^*(z, f)}{P(z, f)} \right|^\alpha$$
$$\le \left(\frac{2^n}{|p_n||f|^n} \left| \sum_{\gamma} b_\gamma(z) f^{\gamma_0} (f')^{\gamma_1} \cdots (f^{(\nu)})^{\gamma_\nu} \right| \right)^\alpha$$
$$\le \left(\frac{2^n}{|p_n|} \sum_{\gamma} |b_\gamma| \prod_{k=1}^{\nu} |h_k|^{\gamma_k} \right)^\alpha$$
$$\le C(\alpha) \left(\frac{2^n}{|p_n|} \right)^\alpha \sum_{\gamma} \left(|b_\gamma| \prod_{k=1}^{\nu} |h_k|^{\gamma_k} \right)^\alpha$$
$$\le C(\alpha) \left(\frac{2^n}{|p_n|} \right)^\alpha \left(\sum_{\gamma} |b_\gamma|^{2\alpha} \right)^{1/2} \sum_{\gamma} \left(\prod_{k=1}^{\nu} |h_k|^{2\alpha\gamma_k} \right)^{1/2} \chi_F(\theta).$$

Combining (3) and (4) gives

$$(5) \qquad m(r, A^*) = \frac{1}{\alpha} \int_0^{2\pi} \log^+ |A^*(z, f)|^\alpha \frac{d\theta}{2\pi}$$

$$\leq \frac{1}{2\alpha}\int_0^{2\pi}\log^+\left(\sum_{\lambda}|a_\lambda u^{|\lambda|}|^{2\alpha}\right)\frac{d\theta}{2\pi}$$
$$+\frac{1}{2\alpha}\int_0^{2\pi}\log^+\left(\sum_{\gamma}|b_\gamma|^{2\alpha}\right)\frac{d\theta}{2\pi}$$
$$+\frac{1}{\alpha}\int_0^{2\pi}\log^+\left(\frac{2^n}{|p_n|}\right)^{\alpha}\frac{d\theta}{2\pi}+O(1)$$
$$+\frac{1}{2\alpha}\int_0^{2\pi}\log^+\left(\sum_{\lambda}\prod_{j=1}^{\mu}|g_j|^{\lambda_j 2\alpha}\right)\chi_E(\theta)\frac{d\theta}{2\pi}$$
$$+\frac{1}{2\alpha}\int_0^{2\pi}\log^+\sum_{\gamma}\left(\prod_{k=1}^{\nu}|h_k|^{2\alpha\gamma_k}\right)\chi_F(\theta)\frac{d\theta}{2\pi}$$
$$= I_1+I_2+I_3+I_4+I_5+O(1).$$

Therefore, we obtain

(6) $$I_1 \leq \sum_{\lambda} m(r,a_\lambda)+\left(\sum_{\lambda}|\lambda|\right)m(r,u) + O(1),$$

and,

(7) $$I_2+I_3 \leq \sum_{\gamma} m(r,b_\gamma)+m(r,1/p_n)+O(1).$$

Now to estimate I_4 and I_5 together. Indeed, set

$$V(r,\theta)=\left(\sum_{\lambda}\prod_{j=1}^{\mu}|g_j|^{\lambda_j 2\alpha}\right)\chi_E(\theta) + \left(\sum_{\gamma}\prod_{j=1}^{\nu}|h_j|^{\gamma_k 2\alpha}\right)\chi_F(\theta),$$

then, by applying the concavity of $\log^+$ and Lemma 2, we have

(8) $$I_4+I_5 = \frac{1}{2\alpha}\int_0^{2\pi}\log^+ V(r,\theta)\frac{d\theta}{2\pi}$$
$$\leq \frac{1}{2\alpha}\log^+\int_0^{2\pi}V(r,\theta)\frac{d\theta}{2\pi}$$
$$\leq \frac{1}{2\alpha}\log^+\left(\sum_{\lambda}\left(\frac{\rho T(\rho,f)}{r(\rho-r)}\right)^{2\alpha\sum j\lambda_j} + \sum_{\gamma}\left(\frac{\rho T(\rho,f)}{r(\rho-r)}\right)^{2\alpha\sum k\gamma_k}\right)+O(1)$$
$$\leq \max(w(A^*),w(B^*))\log^+\frac{\rho T(\rho,f)}{r(\rho-r)}+O(1).$$

It follows from (6), (1), (7), (8) and (5) that the theorem is proved.

Proof of Theorem 2. Without loss of the generality, we can assume that $A^*(z,0)\not\equiv 0$, otherwise we replace f by $f-c$. Thus, we may write

$$A^*(z,f)=a_0(z)+\sum_{|\lambda|\geq 1}a_\lambda\prod_{j=1}^{\mu}(f^{(j)})^{\lambda_j} = a_0+B^*(z,f),$$

where $a_0(z)=A^*(z,0)\not\equiv 0$. Set $E=E(r)=\{\theta\in(0,2\pi] : |f(re^{i\theta})|\leq 1\}$. Therefore, for any small $\alpha>0$, and noting $A^*(z,f)\equiv 0$, we have

$$m(r,\frac{1}{f}) = \int_E \log^+\left|\frac{1}{f}\right|\frac{d\theta}{2\pi}$$
$$=\frac{1}{\alpha}\int_E\log^+\left|\frac{B^*(z,f)}{f}\frac{1}{a_0}\right|^{\alpha}\frac{d\theta}{2\pi}$$
$$\leq \frac{1}{\alpha}\log^+\left(\sum_{|\lambda|\geq 1}\int_E\left(\prod_{j=1}^{\mu}\left|\frac{f^{(j)}}{f}\right|^{\lambda_j}\right)^{\alpha}\frac{d\theta}{2\pi}\right)$$
$$+m\left(r,\frac{1}{a_0}\right)+\sum_{|\lambda|\geq 1}m(r,a_\lambda)+O(1).$$

The rest of proof of the theorem follows from Lemma 2 as we have done in the proof of Theorem 1.

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