On meromorphic functions sharing two one-point sets and two two-point sets

By Manabu SHIROSAKI and Masaharu TAKETANI

Department of Mathematical Sciences, College of Engineering,
Osaka Prefecture University, Sakai 599-8531, Japan

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Abstract: We give a uniqueness theorem of two meromorphic functions sharing two one-point sets and two two-point sets CM.

Key words: Uniqueness theorem; shared values; Nevanlinna theory.

1. Introduction. For nonconstant meromorphic functions f and g on $\boldsymbol{C}$ and a finite set S in $\hat{\boldsymbol{C}} = \boldsymbol{C} \cup \{\infty\}$, we say that f and g share S CM (counting multiplicities) if $f^{-1}(S) = g^{-1}(S)$ and if for each $z_0 \in f^{-1}(S)$ two functions $f - f(z_0)$ and $g - g(z_0)$ have the same multiplicity of zero at z_0, where the notations $f - \infty$ and $g - \infty$ mean $1/f$ and $1/g$, respectively. In particular if S is a one-point set $\{a\}$, then we say also that f and g share a CM.

In [N], R. Nevanlinna showed

Theorem A1. *Let f and g be two distinct nonconstant meromorphic functions on $\boldsymbol{C}$ and $a_1, \cdots, a_4$ four distinct points in $\hat{\boldsymbol{C}}$. If f and g share $a_1, \cdots, a_4$ CM, then f is a Möbius transformation of g and there exists a permutation σ of $\{1,2,3,4\}$ such that $a_{\sigma(3)}, a_{\sigma(4)}$ are Picard exceptional values of f and g and the cross ratio $(a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}, a_{\sigma(4)}) = -1$.*

We get by this result a uniqueness theorem of meromorphic functions as a following corollary:

Corollary A2. *Let f and g be two nonconstant meromorphic functions on $\boldsymbol{C}$ sharing distinct four points a_1, a_2, a_3, a_4 CM. If any cross ratio of a_1, a_2, a_3, a_4 is not -1, then $f = g$.*

Also, in [T] Tohge considered two meromorphic functions sharing $1, -1, \infty$ and a two-point set containing none of them.

Theorem B. *Let f and g be two nonconstant meromorphic functions on $\boldsymbol{C}$ sharing $1, -1, \infty$ and a two-point set $S = \{a, b\}$ CM respectively, where $a, b \neq 1, -1, \infty$. If $a + b \neq 0$, $ab \neq 1$, $a + b \neq 2$, $a+b \neq -2$, $(a+1)(b+1) \neq 4$ and $(a-1)(b-1) \neq 4$, then $f = g$.*

By Tohge's result we can get a uniqueness theorem of meromorphic functions sharing three values and one two-point CM since given three points are mapped to $1, -1, \infty$, respcetively, by a suitable Möbius transformation.

In this paper we consider the uniqueness problem of meromorphic functions on $\boldsymbol{C}$ sharing two values and two two-point sets CM and it is enough to consider the case where meromorphic functions on $\boldsymbol{C}$ sharing $0, \infty$ CM by the same reason as above.

We prepare a terminology to state our result.

Definition 1.1. *Let $\mathcal{A} = \{S_1, \cdots, S_q\}$ be a finite collection of pairwise disjoint finite subsets of $\hat{\boldsymbol{C}}$ and let T be a Möbius transformation. We call a point z_0 a wandering point of T for $\mathcal{A}$ if z_0 and $T(z_0)$ do not belong to the same S_j $(j = 0, 1, \cdots, q)$, where $S_0 = \hat{\boldsymbol{C}} \setminus (\cup_{j=1}^{q} S_j)$.*

Theorem 1.2. *Let S_1 and S_2 be two disjoint two-point subsets in $\boldsymbol{C}$ not containg 0. Assume that f and g be two nonconstant meromorphic functions on $\boldsymbol{C}$ sharing $0, \infty, S_1, S_2$ CM. If for the collection $\{\{0\}, \{\infty\}, S_1, S_2\}$ each Möbius transformation except the indentity has at least three wandering points, then $f = g$.*

The six conditions about a and b in Theorem B imply that for the collection $\{\{1\}, \{-1\}, \{\infty\}, \{a, b\}\}$ each Möbius transformation except the identity has at least three wandering points.

Lemma 1.3. *Let $\mathcal{A} = \{S_1, \cdots, S_q\}$ be a finite collection of pairwise disjoint finite subsets of $\boldsymbol{C}$. Let f and g be two nonconstant meromorphic function on $\boldsymbol{C}$ with a relation $f = T(g)$, where T is a Möbius transformation not the identity. If f and g share each S_j CM $(j = 1, \cdots, q)$, then T has at most two wandering points for $\mathcal{A}$.*

Proof. If z_0 is not a Picard exceptional value of g, then z_0 is not a wandering point of T for $\mathcal{A}$. Hence, by the little Picard Theorem, we get the conclusion. □

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2. Representations of rank N and some lemmas. In this section we introduce the definition of representations of rank N. Let G be a torsion-free abelian multiplicative group, and consider a q-tuple $A = (a_1, a_2, \dots, a_q)$ of elements a_i in G.

Definition 2.1 *Let N be a positive integer. We call integers μ_j representations of rank N of a_j if*

$$\prod_{j=1}^{q} {a_j}^{\varepsilon_j} = \prod_{j=1}^{q} {a_j}^{\varepsilon'_j} \tag{2.1}$$

and

$$\sum_{j=1}^{q} \varepsilon_j \mu_j = \sum_{j=1}^{q} \varepsilon'_j \mu_j \tag{2.2}$$

are equivalent for any integers $\varepsilon_j, \varepsilon'_j$ with $\sum_{j=1}^{q} |\varepsilon_j| \le N$ and $\sum_{j=1}^{q} |\varepsilon'_j| \le N$. In particular we call representations of rank 1, simply, representations.

Remark. For the existence of representations of rank N, see [S]. However, according to the construction of them in [S], (2.1) always implies (2.2) for any integers $\varepsilon_j, \varepsilon'_j$. Hence, in Definition 2.1, it is significant that (2.2) implies (2.1) for any integers $\varepsilon_j, \varepsilon'_j$ with $\sum_{j=1}^{q} |\varepsilon_j| \le N$ and $\sum_{j=1}^{q} |\varepsilon'_j| \le N$.

We introduce the following Borel's Lemma, whose proof can be found, for example, on p.186 of [La].

Lemma 2.2. If entire functions $\alpha_0, \alpha_1, \dots, \alpha_n$ without zeros satisfy

$$\alpha_0 + \alpha_1 + \cdots + \alpha_n = 0,$$

then for each $j = 0, 1, \cdots, n$ there exists some $k \ne j$ such that α_j/α_k is constant.

Now we investigate the torsion-free abelian multiplicative group $G = \mathcal{E}/\mathcal{C}$, where $\mathcal{E}$ is the abelian group of entire functions without zeros and $\mathcal{C}$ is the subgroup of all non-zero constant functions.

Let $\alpha_1, \cdots, \alpha_q$ be elements in $\mathcal{E}$. We represent by $[\alpha_j]$ the element of $\mathcal{E}/\mathcal{C}$ with the representative α_j. Take representations μ_j of rank N of $[\alpha_j]$. For $\prod_{j=1}^{q} {\alpha_j}^{\varepsilon_j}$ we define its index by $\sum_{j=1}^{q} \varepsilon_j \mu_j$. The indices depend only on $\left[\prod_{j=1}^{q} {\alpha_j}^{\varepsilon_j}\right]$ under the condition $\sum_{j=1}^{q} |\varepsilon_j| \le N$.

We use the following Lemma in the proof of Theorem 1.2 which is an application of Lemma 2.2.

Lemma 2.3. *Assume that there is a relation $\Psi(\alpha_1, \cdots, \alpha_q) \equiv 0$ where $\Psi(X_1, \cdots, X_q) \in \boldsymbol{C}[X_1, \cdots, X_q]$ is a nonconstant polynomial of degree at most N of $X_1, \cdots, X_q$. Then each term $a{X_1}^{\varepsilon_1} \cdots {X_q}^{\varepsilon_q}$ of $\Psi(X_1, \cdots, X_q)$ has another term $b{X_1}^{\varepsilon'_1} \cdots {X_q}^{\varepsilon'_q}$ such that ${\alpha_1}^{\varepsilon_1} \cdots {\alpha_q}^{\varepsilon_q}$ and ${\alpha_1}^{\varepsilon'_1} \cdots {\alpha_q}^{\varepsilon'_q}$ have the same indices, where a and b are non-zero constants.*

Proof. By using Lemma 2.2 each term $a{X_1}^{\varepsilon_1} \cdots {X_q}^{\varepsilon_q}$ has another term $b{X_1}^{\varepsilon'_1} \cdots {X_q}^{\varepsilon'_q}$ such that $({\alpha_1}^{\varepsilon_1} \cdots {\alpha_q}^{\varepsilon_q})/({\alpha_1}^{\varepsilon'_1} \cdots {\alpha_q}^{\varepsilon'_q})$ is constant. This implies the conclusion of Lemma. □

3. Proof of Theorem 1.2. Let S_j be defined by the equation $P_j(z) := z^2 + a_j z + b_j = 0$ and ξ_j, η_j its elements. By assumption there exist entire functions without zeros $\alpha_0, \alpha_1, \alpha_2$ such that

$$f = \alpha_0 g \tag{3.1}$$

and

$$f^2 + a_j f + b_j = \alpha_j (g^2 + a_j g + b_j) \ (j = 1, 2). \tag{3.2}$$

Now we assume $f \ne g$.

Proposition 3.1. $f^{-1}(\xi_j) \ne g^{-1}(\eta_j)$ $(j = 1, 2)$.

Proof. Assume that $f^{-1}(\xi_1) = g^{-1}(\eta_1)$. Then there exist entire functions without zeros β and γ such that

$$f - \xi_1 = \beta(g - \eta_1), \ f - \eta_1 = \gamma(g - \xi_1).$$

Note that any of $\beta, \gamma, \beta/\gamma, \alpha_0/\beta, \alpha_0/\gamma$ is not constant since f is not any Möbius transformation of g by Lemma 1.3. By these two equations and (3.1) we get

$$(\xi_1 - \eta_1)(\beta\gamma - \alpha_0) - \eta_1 \beta + \xi_1 \gamma - \xi_1 \alpha_0 \gamma + \eta_1 \alpha_0 \beta = 0. \tag{3.3}$$

It follows from this, by Lemma 2.2 and the note above, that $\beta\gamma/\alpha_0$, $\alpha_0\beta/\gamma$ and $\alpha_0\gamma/\beta$ are constant. However, we can induce a contradiction $\beta^2 = \beta(\alpha_0\beta)/\alpha_0 = c_1\beta\gamma/\alpha_0 = c_1 c_2$, where c_1 and c_2 are constants.

The case of $f^{-1}(\xi_2) = g^{-1}(\eta_2)$ is the same. □

Proposition 3.2. *Any of α_j is not constant.*

Proof. If α_0 is constant $c \ne 0$, then for the Möbius transformation $T(z) = cz$ we have $f = T(g)$. By Lemma 1.3 and the assumption of Theorem 1.2 we have $c = 1$, which contradicts to $f \ne g$. Note that f is not any Möbius transformation of g.

Next assume α_1 is constant $c \ne 0$.

If $c = 1$, then f is a Möbius transformation of g. However it does not occur by Lemma 1.3. So $c \neq 1$ and it follows from (3.2) for $j = 1$ that f and g does not simultaneously take any finite value outside S_1. Therefore $f^{-1}(\eta_2) = g^{-1}(\xi_2)$, which is impossible by Proposition 3.1.

The case where α_2 is constant is the same. □

Proposition 3.3. *Any of α_j/α_0 $(j = 1, 2)$ is not constant.*

Proof. Assume that α_1/α_0 is constant c. Then

$$\frac{f^2 + a_1 f + b_1}{f} = c\frac{g^2 + a_1 g + b_1}{g}.$$

If $c = 1$, then $f = g$ or $fg = b_1$. In the latter case f is a Möbius transformation of g.

The case of $c \neq 1$ induces a contradiction $f^{-1}(\xi_2) \neq g^{-1}(\eta_2)$ to Proposition 3.1.

The case of α_2/α_0 is constant is the same. □

Proposition 3.4. *The entire function α_2/α_1 is not constant.*

Proof. Assume that α_2/α_1 is a constant c, then

$$\frac{f^2 + a_2 f + b_2}{f^2 + a_1 f + b_1} = c\frac{g^2 + a_2 g + b_2}{g^2 + a_1 g + b_1}. \tag{3.4}$$

If $c = 1$, then we have

$$(a_2 - a_1)fg + (b_2 - b_1)(f + g) + (a_1 b_2 - a_2 b_1) = 0$$

because of $f \neq g$, which implies f is a Möbius transformation of g.

Hence $c \neq 1$ and it follows from (3.4) that f and g does not simultaneously take any value outside $S_1 \cup S_2$ in $\hat{\boldsymbol{C}}$. In particular f and g are entire functions without zeros. By applying Borel's Lemma to (3.2) for $j = 1$, we can induce $f^3 = {\alpha_1}^3 g^3$. However, by this and (3.1), $(\alpha_1/\alpha_0)^3 = 1$, which contradicts to Proposition 3.3. □

Proposition 3.5. *Any of ${\alpha_0}^2/\alpha_j$ $(j = 1, 2)$ is not constant.*

Proof. Assume that ${\alpha_0}^2/\alpha_1$ is a constant c, then

$$\frac{f^2}{f^2 + a_1 f + b_1} = c\frac{g^2}{g^2 + a_1 g + b_1}. \tag{3.5}$$

If $c = 1$, then $a_1 fg + b_1(f + g) = 0$, which implies f is a Möbius transformation of g.

Hence $c \neq 1$ and it follows from (3.4) that f and g does not simultaneously take any value outside $S_1 \cup \{0\}$ in $\hat{\boldsymbol{C}}$. Hence $f^{-1}(\xi_2) = g^{-1}(\eta_2)$, which contradicts to Proposition 3.1.

The case where ${\alpha_0}^2/\alpha_1$ is constant is the same. □

By substituting (3.1) into (3.2) we have

$$({\alpha_0}^2 - \alpha_j)g^2 + a_j(\alpha - \alpha_j)g + b_j(1 - \alpha_j) = 0 \ (j = 1, 2).$$

Consider the resultant of these polynomials of g, then

$$\begin{aligned}
0 &= \begin{vmatrix} {\alpha_0}^2 - \alpha_1 & a_1(\alpha_0 - \alpha_1) & b_1(1 - \alpha_1) & 0 \\ 0 & {\alpha_0}^2 - \alpha_1 & a_1(\alpha_0 - \alpha_1) & b_1(1 - \alpha_1) \\ {\alpha_0}^2 - \alpha_1 & a_1(\alpha_0 - \alpha_1) & b_1(1 - \alpha_1) & 0 \\ 0 & {\alpha_0}^2 - \alpha_1 & a_1(\alpha_0 - \alpha_1) & b_1(1 - \alpha_1) \end{vmatrix} \\
&= -2b_1 b_2 \alpha_1 \alpha_2 {\alpha_0}^4 + b_1 {\alpha_1}^2 {\alpha_0}^4 + b_2 {\alpha_2}^2 {\alpha_0}^4 \\
&\quad + b_1\{a_2(a_1 - a_2) - 2(b_1 - b_2)\}\alpha_1 {\alpha_0}^4 \\
&\quad + b_2\{a_1(a_2 - a_1) - 2(b_2 - b_1)\}\alpha_2 {\alpha_0}^4 \\
&\quad + \{(a_1 - a_2)(a_1 b_2 - a_2 b_1) + (b_1 - b_2)^2\}{\alpha_0}^4 \\
&\quad + \{P_{32}(\alpha_1, \alpha_2) + P_{31}(\alpha_1, \alpha_2)\}{\alpha_0}^3 \\
&\quad + \{P_{23}(\alpha_1, \alpha_2) + P_{22}(\alpha_1, \alpha_2) + P_{21}(\alpha_1, \alpha_2)\}{\alpha_0}^2 \\
&\quad + \{P_{13}(\alpha_1, \alpha_2) + P_{12}(\alpha_1, \alpha_2)\}\alpha_0 \\
&\quad + \{(a_1 - a_2)(a_1 b_2 - a_2 b_1) + (b_1 - b_2)^2\}{\alpha_1}^2 {\alpha_2}^2 \\
&\quad + b_1\{a_2(a_1 - a_2) - 2(b_1 - b_2)\}\alpha_1 {\alpha_2}^2 \\
&\quad + b_2\{a_1(a_2 - a_1) - 2(b_2 - b_1)\}{\alpha_1}^2 \alpha_1 \\
&\quad + b_2 {\alpha_1}^2 + b_1 {\alpha_2}^2 - 2b_1 b_2 \alpha_1 \alpha_2.
\end{aligned}$$

Here $P_{kj}(X, Y)$ are homogeneous polynomials of degree j about X and Y, and in particular $P_{k3}(X, Y)$ have only two terms $X^2 Y$ and XY^2 with some coefficients.

Let μ_0, μ_1, μ_2 be representations of $[\alpha_0], [\alpha_1], [\alpha_2]$ with rank 6. Now we find the terms with the minimal or maximal indices in the expansion above. By assumption and Propositions above $\mu_j \neq 0$ $(j = 0, 1, 2)$, $\mu_0 \neq \mu_j$, $2\mu_0 \neq \mu_j$ $(j = 1, 2)$ and $\mu_1 \neq \mu_2$. We may assume (A)$\mu_0 < \mu_1 < \mu_2$ or (B)$\mu_1 < \mu_0 < \mu_2$.

In the case (A) it is enough to consider only four cases (i) $0 < \mu_0 < \mu_1 < \mu_2$; (ii) $\mu_0 < 0 < \mu_1 < \mu_2$; (iii) $\mu_0 < \mu_1 < 0 < \mu_2$; (iv) $\mu_0 < \mu_1 < \mu_2 < 0$.

In the case (i) if $\mu_0 > \frac{1}{2}\mu_1$, then only the term ${\alpha_1}^2$ has the minimal index $2\mu_1$; if $\mu_0 < \frac{1}{2}\mu_1$, then only the term ${\alpha_1}^4$ has the minimal index $4\mu_0$. In the case (ii) the coefficient of the term ${\alpha_0}^4$ is not zero since it is the resultant of $P_1(z)$ and $P_2(z)$, and only ${\alpha_0}^4$ has the minimal index $4\mu_0$. In the case (iii) only the term ${\alpha_2}^2$ has the maximal index $2\mu_2$. In the case (iv) only the term ${\alpha_0}^4 {\alpha_1}^2$ has the minimal index $4\mu_0 + 2\mu_1$. In all cases we get contradictions to Lemma 2.3.

In the case (B) it is enough to consider only two cases (i) $0 < \mu_1 < \mu_0 < \mu_2$; (ii) $\mu_1 < 0 < \mu_0 < \mu_2$. In each cases only the term ${\alpha_1}^2$, in the expansion, has the minimal index $2\mu_1$, which contradicts to Lemma 2.3. The proof is completed.

References

[La] S. Lang, *Introduction to complex hyperbolic spaces*, Springer, New York, 1987.

[Li] P. Li, Meromorphic functions sharing three values or sets CM, Kodai Math. J. **21** (1998), no. 2, 138–152.

[N] R. Nevanlinna, Einige Eindeutigkeitssätze in der Theorie der meromorphen Funktionen, Acta Math., **48** (1926), 367–391.

[S] M. Shirosaki, On meromorphic functions sharing three values and one set, Kodai Math. J., **29** (2006), 475–484.

[T] K. Tohge, Meromorphic functions covering certain finite sets at the same points, Kodai Math. J. **11** (1988), no. 2, 249–279.