On theta correspondences for Eisenstein series

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Abstract: There are three types of parabolic subgroups in $Sp(2, \mathbf{R})$. In this paper we show that the Eisenstein series with respect to the Siegel parabolic subgroup corresponds to the Eisenstein series with respect to the Jacobi parabolic subgroup by theta correspondences.

Key words: Siegel modular; theta correspondence; Eisenstein series.

As usual we put $g[x] = {}^t x g x$, $\Gamma_n = Sp(n, \mathbf{Z})$. We denote the Siegel upper half space of degree n by $\mathbf{H}_n$, that is,

$$(1) \qquad \mathbf{H}_n = \{Z = U + iV \in M_n(\mathbf{C}) \mid U, V \in M_n(\mathbf{R}), {}^t Z = Z, V > 0\}.$$

For $Z = U + iV \in \mathbf{H}_2$ with $U = \operatorname{Re} Z, V = \operatorname{Im} Z$ and for

$$F = \begin{pmatrix} & & & 4 & \\ & & & & 4 \\ & & 2 & & \\ 4 & & & & \\ & 4 & & & \end{pmatrix}, \quad H = \begin{pmatrix} 4 & & & & \\ & 4 & & & \\ & & 2 & & \\ & & & 4 & \\ & & & & 4 \end{pmatrix}$$

we define a theta function by

$$(2) \qquad \Theta(Z, F, g) = \sum_{\substack{X = \begin{pmatrix} X_1 \\ x \\ 4^{-1} X_2 \end{pmatrix}, \\ X_1, X_2 \in M_{2,2}(\mathbf{Z}), \\ x \in M_{1,2}(\mathbf{Z}), \\ \det X_2 \text{ odd}}} \exp(\pi i \operatorname{tr}(F[X]U + iH_g[X]V).$$

Here $H_g = H[\rho(g)]$ and for $g \in G = Sp(2, \mathbf{R})$, $\rho(g)$ is defined as follows:

$$\rho(g)\widehat{X} = \widehat{{}^t g X g}$$

denoting

$$\widehat{X} = \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} \text{ for } X = \begin{pmatrix} 0 & a & c/2 & -e \\ -a & 0 & -b & -c/2 \\ -c/2 & b & 0 & d \\ e & c/2 & -d & 0 \end{pmatrix}.$$

We note that $\rho(gh) = \rho(h)\rho(g)$ for $g, h \in G$. Put

$$\Gamma_0^{(n)}(N) = \left\{ \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n \,\middle|\, C \equiv 0 \bmod N \right\}$$

as usual. Then from [1] we obtain

$$\Theta(\gamma Z, F, g) = |\det(CZ + D)|^2 j(\gamma, Z)\Theta(Z, F, g),$$

for $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0(4) = \Gamma_0^{(2)}(4)$. Here $j(\gamma, Z) = \theta_2(\gamma Z)/\theta_2(Z)$ with $\theta_2(Z) = \sum_{x \in M_{2,1}(\mathbf{Z})} \exp(2\pi i Z[x])$. As for the transformation formula of $\Theta(Z, F, g)$ with respect to g, an easy direct calculation shows that $\Theta(Z, F, \gamma g k) = \Theta(Z, F, g)$ for $\gamma \in \Delta', k \in SO(4) \cap Sp(2, \mathbf{R})$, if we put

$$(3) \qquad \Delta = \left\{ \gamma \in \Gamma_2 \,\middle|\, \gamma \equiv \begin{pmatrix} * & * & * & 0 \\ 0 & * & 0 & 0 \\ * & * & * & 0 \\ * & * & * & * \end{pmatrix} \bmod 2 \right\},$$

$$(4) \qquad w = \begin{pmatrix} 1 & & & \\ & 2 & & \\ & & 1 & \\ & & & 1/2 \end{pmatrix}, \Delta' = w^{-1}\Delta w.$$

We use the following standard notations for $Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \in \mathbf{H}_2$ and for $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2, \mathbf{R})$:

$$Z^* = \tau, \gamma Z = (AZ + B)(CZ + D)^{-1}.$$

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Further let

$$\Gamma_\infty^{(n)} = \left\{ \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n \middle| C = 0 \right\},$$

$$P_J = \left\{ \begin{pmatrix} & & & * \\ & * & & * \\ & & & * \\ 0 & 0 & 0 & * \end{pmatrix} \in Sp(2, \mathbf{R}) \right\}.$$

If we put, for $z \in \mathbf{H}_1$, $Z \in \mathbf{H}_2$,

(5) $$\tilde{E}(z, s) = \sum_{\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty^{(1)} \backslash \Gamma_0^{(1)}(4)} j_1(\sigma, z)^{-1} |cz + d|^{-2s} \mathrm{Im}\, z^s,$$

where

$$j_1(\sigma, z) = \theta(\sigma z)/\theta(z)$$

with Riemann's theta function $\theta(z) = \sum_{x \in \mathbf{Z}} \exp(2\pi i x^2 z)$, and put

(6) $$\tilde{G}_B(Z, s, s_2) = \sum_{\gamma \in P_J \cap \Gamma_0(4) \backslash \Gamma_0(4)} \overline{\tilde{E}((\gamma Z)^*, \overline{s_2})} \Big/ \overline{j(\gamma, Z)} \det(\mathrm{Im}\, \gamma Z)^s (\mathrm{Im}(\gamma Z)^*)^{-s},$$

then we have

$$\tilde{G}_B(\gamma Z, s, s_2) = \overline{j(\gamma, Z)} \tilde{G}_B(Z, s, s_2)$$

for $\gamma \in \Gamma_0(4)$. As a double sum with respect to σ, γ, the series defining $\tilde{G}_B(Z, s, s_2)$ by (6), (5) converges absolutely in

(7) $$\mathcal{C}_0 = \{(s, s_2) \in \mathbf{C}^2 \mid 3/4 < \mathrm{Re}\, s_2, 1 + \mathrm{Re}\, s_2 < \mathrm{Re}\, s\}$$

(see [2, Satz2.8]). $\tilde{G}_B(Z, s, s_2)$ is the Eisenstein series with respect to the minimal parabolic subgroup of $\Gamma_0(4)$. The purpose of this paper is to compute the following integral:

(8) $$\tilde{I}(s, s_2, g) = \int_{\Gamma_0(4) \backslash \mathbf{H}_2} \tilde{G}_B(Z, s, s_2) \Theta(Z, F, g) (\det V)^{-3/2} dU\, dV$$

where we denote $Z = U + iV$ and $d\begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} = dx_1\, dx_2\, dx_3$. The convergence of this integral follows from that of the succeeding integrals (11), (12). We call an integral

(9) $$\int_{\Gamma_0(4) \backslash \mathbf{H}_2} F(Z) \Theta(Z, F, g) (\det V)^{-3/2} dU\, dV$$

the theta integral for $F(Z)$ in this paper. The theta integral (8) for $\tilde{G}_B$ gives a correspondence between Eisenstein series on $Sp(2, \mathbf{R})$ and those on its metaplectic cover. We note that theta integrals do not always converge, since $\Theta(Z, F, g)$ does not vanish at all boundary components. Unfolding the integral (8) as usual, we obtain the expression

(10) $$\int_{P_J \cap \Gamma_0(4) \backslash \mathbf{H}_2} \Theta(Z, F, g) \overline{\tilde{E}(Z^*, \overline{s_2})} \det(\mathrm{Im}\, Z)^{s-3/2} (\mathrm{Im}\, Z^*)^{-s} dU dV = \int_D \overline{\tilde{E}(\tau, \overline{s_2})} v^{-s} (vv' - y^2)^{s-3/2} \Theta(Z, F, g)\, du du' dv dv' dx dy$$

where $\tau = u + iv$, $z = x + iy$, $\tau' = u' + iv'$, $Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}$ and

$$D = \left\{ (u, u', v, v', x, y) \middle| \begin{array}{c} u + iv \in \Gamma_0^{(1)}(4) \backslash \mathbf{H}_1, \\ x + iy \in (\mathbf{Z} + \mathbf{Z}\tau) \backslash \mathbf{C}, \\ 0 \leqslant u' \leqslant 1, v' \leqslant \dfrac{y^2}{v} \end{array} \right\}.$$

Therefore we obtain

(11) $$\tilde{I}(s, s_2, g) = \int_{\Gamma_0^{(1)}(4) \backslash \mathbf{H}_1} f(\tau) \overline{\tilde{E}(\tau, \bar{s}_2)} v^{-2} du dv$$

with

(12) $$f(\tau) = \int_{D_2} \Theta(Z, F, g) t^{s-3/2} dx dy du' dt$$

where $\tau = u + iv$, $z = x + iy$, $\tau' = u' + iv'$, $v' = y^2/v + t$, $Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}$ and

$$D_2 = \left\{ (u', t, x, y) \middle| \begin{array}{l} x + iy \in (\mathbf{Z} + \mathbf{Z}\tau) \backslash \mathbf{C}, \\ 0 \leqslant u' \leqslant 1, 0 < t < \infty \end{array} \right\}.$$

Unfolding the integral in (11), we have

(13) $$\tilde{I}(s, s_2, g) = \int_{\Gamma_\infty^{(1)} \backslash \mathbf{H}_1} f(\tau) v^{s_2 - 3/2} du dv.$$

Since $\Theta(iV,F,g)=O(|V|^{-5/2}\exp(-c\,\mathrm{tr}\,V)$ for $V>0$ with a certain constant $c>0$, (11), (12) converge absolutely for $\mathrm{Re}\,s_2>2, \mathrm{Re}\,s>3$. To go further, it is more convenient to introduce another theta function $\tilde{\Theta}(Z,S,g)$. $\tilde{\Theta}(Z,S,g)$ is defined as follows:

$$(14)\qquad \tilde{\Theta}(Z,S,g)=\sum_{\substack{X=\begin{pmatrix}X_1\\x\\X_2\end{pmatrix},\\ X_1,X_2\in M_{2,2}(\mathbf{Z}),\\ x\in M_{1,2}(\mathbf{Z}),\\ \det X_2\text{ odd}}}\exp(\pi i\,\mathrm{tr}(S[X]U+iK^g[X]V)),$$

where

$$S=\begin{pmatrix} & & & 1 & \\ & & & & 1\\ & & 2 & & \\ 1 & & & & \\ & 1 & & & \end{pmatrix},\quad K=\begin{pmatrix}1 & & & & \\ & 1 & & & \\ & & 2 & & \\ & & & 1 & \\ & & & & 1\end{pmatrix},$$

$K^g=K[\rho'(g)]$ and ρ' is defined by $\rho'(g)\widetilde{X}={}^t\widetilde{gXg}$ denoting

$$\widetilde{X}=\begin{pmatrix}a\\b\\c\\d\\e\end{pmatrix}\text{ for }X=\begin{pmatrix}0 & a & c & -e\\ -a & 0 & -b & -c\\ -c & b & 0 & d\\ e & c & -d & 0\end{pmatrix}.$$

It is easy to see

$$(15)\qquad \begin{aligned}\rho'(g)&=W_1^{-1}\rho(g)W_1,\\ H[\rho(g)][W_2]&=K[\rho'(wg)]\end{aligned}$$

for $g\in Sp(2,\mathbf{R})$, w in (14) and for

$$W_1=\begin{pmatrix}1 & & & & \\ & 1 & & & \\ & & 2 & & \\ & & & 1 & \\ & & & & 1\end{pmatrix},\quad W_2=\begin{pmatrix}1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1/4 & \\ & & & & 1/4\end{pmatrix}.$$

Therefore we get

$$(16)\qquad \Theta(Z,F,g)=\tilde{\Theta}(Z,S,wg).$$

Changing the variables x,y for $x,y/v$, the integrations with respect to u,u',x in (12), (13) imply that we have only to consider X in the definition (14) of $\tilde{\Theta}(Z,S,g)$, such that X is of the form $(\tilde{A},\tilde{B})$ and $S[(\tilde{A},\tilde{B})]=0$ when we replace $\Theta(Z,F,g)$ with $\tilde{\Theta}(Z,S,g)$ in (12). If $X=\begin{pmatrix}X_1\\x\\X_2\end{pmatrix}=(\tilde{A},\tilde{B})$ satisfies the condition in the summation of (14) and $S[(\tilde{A},\tilde{B})]=0$, A,B must be of the following form:

$$(17)\qquad \begin{aligned}A=A_h&=\begin{pmatrix}0 & l[(m,n)]J\\ J(l[(m,n)]) & jJ\end{pmatrix}[h],\\ B=B_h&=\begin{pmatrix}0 & 0\\ 0 & kJ\end{pmatrix}[h],\end{aligned}$$

where $k,j,l,m,n\in\mathbf{Z}$; m,l,k odd; $k,m>0$; $(m,n)=1$; $J=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ and $h\in\Delta_0\backslash\Delta$ with $\Delta_0=\left\{\sigma\,\middle|\,\sigma=\begin{pmatrix}A & B\\ 0 & D\end{pmatrix}\in\Delta\right\}$. Therefore we have

$$(18)\qquad I(s,s_2,g)=\int_{D_3}\sum_{\substack{h\in\Delta_0\backslash\Delta;\,k,m>0;\\ k,j,l,m,n\in\mathbf{Z};\\ m,l,k\text{ odd};\,(m,n)=1}}\exp(-\pi\,\mathrm{tr}(K^g[\widetilde{A_h},\widetilde{B_h}]V)t^{s-3/2}dtv^{s_2-1/2}dydv,$$

where A_h,B_h are defined in (17), $V=\begin{pmatrix}v & vy\\ vy & t+vy^2\end{pmatrix}$ and

$$D_3=\{(v,t,y)\,|\,0<v,0\leqslant y\leqslant 1,0<t<\infty\}.$$

Put

$$(19)\qquad \begin{aligned}T&=\begin{pmatrix}\sqrt{v} & 0\\ y\sqrt{v} & \sqrt{t}\end{pmatrix},\quad B_1=\begin{pmatrix}0 & 0\\ 0 & kJ\end{pmatrix},\\ A_1&=\begin{pmatrix}0 & l[(m,n)]J\\ J(l[(m,n)]) & jJ\end{pmatrix},\end{aligned}$$

and

$$(20)\qquad A_1'=\sqrt{v}A_1+y\sqrt{v}B_1,\ B_1'=\sqrt{t}B_1$$

then

$$(21)\qquad \begin{aligned}&\mathrm{tr}(K^g[(\widetilde{A_h},\widetilde{B_h})]V)\\ &=\mathrm{tr}(K^{hg}[(\widetilde{A_1},\widetilde{B_1})T])=\mathrm{tr}(K^{hg}[(\widetilde{A_1'},\widetilde{B_1'})]).\end{aligned}$$

We note that for any alternating matrix A of degree 4

(22) $S[\tilde{A}] = 1/2\,\mathrm{tr}\left(A\begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}{}^tA\begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}\right),$

(23) $K[\tilde{A}] = 1/2\,\mathrm{tr}(A^tA)$

where $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

For $Z = X + iY \in \mathbf{H}_2$, we hereafter put

(24) $g = g_Z = \begin{pmatrix} E & X \\ 0 & E \end{pmatrix}\begin{pmatrix} \sqrt{Y} & 0 \\ 0 & \sqrt{Y}^{-1} \end{pmatrix}.$

From (23) we can replace hg with $g_{Z'}$ where $Z' = hZ = hgi = hg_Zi = X' + iY'$ in the further expressions. Since

(25) $B_1'[g_{Z'}] = \sqrt{t}(\det Y')^{-1/2}\begin{pmatrix} 0 & 0 \\ 0 & kJ \end{pmatrix},$

(26) $A_1'[g_{Z'}] = (\det Y')^{-1/2}$
$$\begin{pmatrix} 0 & l[(m,n)\sqrt{Y'}]J \\ J(l[(m,n)\sqrt{Y'}]) & y'J \end{pmatrix},$$

where $y' = X'l[(m,n)] + Jl[(m,n)]X'J^{-1} + ky + j$, we have

(27) $\mathrm{tr}(K^g[(\widetilde{A_h}, \widetilde{B_h})]V) = (\det Y')^{-1}$
$$(vl^2(Y'[{}^t(m,n)])^2 + vy'^2 + k^2t)$$

by (17), (21), (23). In (18) interchanging the order of integration and summation, extending the interval of the integration with respect to y to $(-\infty,\infty)$ as usual, and changing y to y', we obtain

(28)
$$\tilde{I}(s,s_2,g) = \sum_{\substack{h\in\Delta_0\backslash\Delta;\,k,m>0;\\ k,l,m,n\in\mathbf{Z};\\ m,l,k\ \mathrm{odd};\,(m,n)=1}} \int_0^\infty\int_0^\infty\int_{-\infty}^\infty \exp(-\pi(\det Y')^{-1} (vl^2(Y'[{}^t(m,n)])^2 + vy'^2 + k^2t)) t^{s-3/2}dt v^{s_2-1/2}dv dy'$$
$$= \sum_{\substack{h\in\Delta_0\backslash\Delta;\,k,m>0;\\ k,l,m,n\in\mathbf{Z};\\ l,m,k\ \mathrm{odd},\,(m,n)=1}} (\det Y')^{1/2}\int_0^\infty\int_0^\infty \exp(-\pi(\det Y')^{-1} (vl^2(Y'[{}^t(m,n)])^2 + k^2t)) t^{s-3/2}dt v^{s_2-1}dv$$
$$= 2\pi^{-s-s_2+1/2}\zeta_2(2s-1)\zeta_2(2s_2) \Gamma(s-1/2)\Gamma(s_2) \sum_{h\in\Delta_0\backslash\Delta} (\det Y')^{s+s_2} \sum_{\substack{m,n\in\mathbf{Z},\\ m>0,\,(m,2n)=1}} (Y'[{}^t(m,n)])^{-2s_2}$$

with $\zeta_2(s) = (1-2^{-s})\zeta(s)$. We are ready to state our theorem. For a function $f(x+iy)$ on $\mathbf{H}_1$ and $Y = |Y|^{1/2}\begin{pmatrix} (x^2+y^2)/y & x/y \\ x/y & 1/y \end{pmatrix}$ we denote $f(Y) = f(x+iy)$. For an Eisenstein series $G(z,s) = \sum_{(2n,m)=1,m>0} \frac{\mathrm{Im}\,z^s}{|2nz+m|^{2s}}$, put $h(z,s) = 2^sG(-1/(2z),s)$ and

(29)
$$G_B(Z,s,s_2) = \sum_{\gamma\in\Delta_0\backslash\Delta} \det(\mathrm{Im}\,\gamma Z)^s h(\mathrm{Im}(\gamma Z), 2s_2).$$

Then we obtain

Theorem 1. *For* $\mathrm{Re}\,s_2 > 2, \mathrm{Re}\,s > 1 + \mathrm{Re}\,s_2$,
$$\int_{\Gamma_0(4)\backslash\mathbf{H}_2} \tilde{G}_B(Z,s,s_2)\Theta(Z,F,w^{-1}g_W) (\det V)^{-3/2}dU\,dV = 2\pi^{-s-s_2+1/2}\Gamma(s-1/2)\Gamma(s_2) \zeta_2(2s-1)\zeta_2(2s_2)G_B(W,s,s_2)$$
with $Z = U + iV$ *and* $W \in \mathbf{H}_2$.

The right-hand side of the equality in Theorem 1 has the meromorphic continuation by [2]. [5] contains many results concerning analytic properties of $\tilde{G}_B(Z,s,s_2)$. However, we do not use them in this paper because, in order to prove our next theorem, we need only absolute convergences of Eisenstein series with respect to three types of paraboric subgroups (see [2–4]) in addition to Fourier expansions of Eisenstein series of one complex variable. To rewrite $\tilde{G}_B(Z,s,s_2)$ in a different form, put

(30)
$$E(z,s) = \sum_{\sigma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma_\infty^{(1)}\backslash\Gamma_0^{(1)}(1)} |cz+d|^{-2s}\mathrm{Im}\,z^s.$$

Then, using the notation just above Theorem 1, we have

(31) $$\tilde{G}_B(Z,s,s_2)=\sum_{\gamma\in\Gamma_\infty^{(2)}\backslash\Gamma_0(4)}E(\operatorname{Im}\gamma Z,s-s_2)\,\overline{j(\gamma,Z)}^{-1}\det(\operatorname{Im}\gamma Z)^{s+s_2}.$$

This rewriting is justified by the abusolute convergence, stated above (7), of $\tilde{G}_B(Z,s,s_2)$ and hereafter we consider that (31) is the definition of $\tilde{G}(Z,s,s_2)$.

Let us investigate the behavior of $(s_2-s+1)\tilde{G}_B(Z,s,s_2)$ when s_2-s+1 is a negative real and approaches to 0. Since the constant term of the Fourier expansion of $E(z,s)$ is

(32) $$y^s+\frac{\sqrt{\pi}\Gamma(s-1/2)\zeta(2s-1)}{\Gamma(s)\zeta(2s)}y^{1-s},$$

for any constant c and $1\le s\le c$ there exists a positive constant C_1 such that

(33) $$(s-1)E(z,s) < C_1((s-1)y^c+(s-1)\zeta(2s-1))$$

uniformly on

$$\mathcal{D}=\{z=x+iy\in\mathbf{H}_1 \mid |x|<1/2, |z|>1\}.$$

Thus, for $\sigma\in\Gamma_1$,

(34) $$\begin{aligned}(s-1)E(z,s) &< C_1((s-1)\operatorname{Im}(\sigma z)^c + (s-1)\zeta(2s-1))\\ &< C_1((s-1)E(z,c)+(s-1)\zeta(2s-1))\end{aligned}$$

uniformly in $z\in\sigma^{-1}\mathcal{D}$. Therefore, for $z\in\mathbf{H}_1$,

(35) $$(s-1)E(z,s) < C_1((s-1)E(z,c) + (s-1)\zeta(2s-1)).$$

Assuming $1\le s-s_2\le c$ and substituting $s-s_2$ for s in (35), the abusolute convergence of a double sum $\tilde{G}_B(Z,s,s-1-c)$, that of Eisenstein series with respect to the Sigel parabolic subgroup (see [3] for instance) and the inequality (35) show that $(s_2-s+1)\tilde{G}_B(Z,s,s_2)$, defined by (31) as a sum with respec to γ, converges uniformly in $1\le s-s_2\le c$. Changing the order of the limiting and the summation and using the Fourier expansion of $E(z,s)$, we obtain, for $\operatorname{Im}s>4$,

(36) $$\lim_{s_2-s\to-1-0}(s_2-s+1)\tilde{G}_B(Z,s,s_2) = -3\pi^{-1}\tilde{G}_S(Z,s),$$

where

(37) $$\tilde{G}_S(Z,s)=\sum_{\gamma\in\Gamma_\infty^{(2)}\backslash\Gamma_0(4)}\overline{j(\gamma,Z)}^{-1}\det(\operatorname{Im}\gamma Z)^{s-1/2}$$

is the Eisenstein series with respect to the Siegel parabolic subgroup. The same thing holds for $G_B(Z,s,s_2)$. Denote the fundamental domain of $\Gamma_0^{(1)}(2)$ by $\mathcal{D}_2$. Then, for $1\le s\le c$, $h(z,s)=2^sG(-1/(2z),s)$ defined just before Theorem 1 has an estimate

(38) $$|(s-1)h(z,s)| < C_2(y^s+y^+(\operatorname{Im}(-1/z))^s+1)$$

where C_2 is a constant independent of s and $z\in\mathcal{D}_2$. Thus, in the same way as before, we have

(39) $$|(s-1)h(z,s)| < C_2(h(z,c)+h(-1/(2z),c+1).$$

Therefore, since the constant term of the Fourier expansion of $h(z,s)=2^sG(-1/(2z),s)$ is

(40) $$y^{1-s}\frac{\sqrt{\pi}\Gamma(s-1/2)\zeta_2(2s-1)}{\Gamma(s)\zeta_2(2s)},$$

we have

(41) $$\lim_{s_2-s\to-1-0}(s_2-s+1)G_B(Z,s,s_2) = -3\pi^{-1}G_J(Z,s)$$

where

(42) $$G_J(Z,s)=\sum_{\gamma\in\Delta\cap P_J\backslash\Delta}\det(\operatorname{Im}\gamma Z)^{2s-1}(\operatorname{Im}(\gamma Z)^*)^{-2s+1}$$

is the Eisenstein series with respect to the Jacobi parabolic subgroup. The theta integral, defined in (9), for $\tilde{G}_S(Z,s)$ converges for $\operatorname{Im}s>4$ in view of the estimate stated just after (13). This combined with (35) and the convergence of the theta integral for $\tilde{G}_B(Z,s,s-1-c)$ permit changing the order of the limiting and the integration. Therefore we obtain

Theorem 2. *For* $\operatorname{Im}s>4$

$$\begin{aligned}&\int_{\Gamma_0(4)\backslash\mathbf{H}_2}\tilde{G}_S(Z,s)\Theta(Z,F,w^{-1}g_W)(\det V)^{-3/2}dU\,dV\\ &= 2\pi^{-2s+3/2}\Gamma(s-1/2)\Gamma(s-1)\zeta_2(2s-1)\zeta_2(2s-2)G_J(W,s)\end{aligned}$$

with $Z=U+iV$ *and* $W\in\mathbf{H}_2$.

We note that the theta integrl in Theorem 2 can directly be calculated without limiting process. We can also investigate the behavior of $\tilde{G}_B(Z, s, s_2)$, $G_B(Z, s, s_2)$ when s_2 approaches to $1/2$. But, in this cace, the theta integral does not converge and some modifications of the statement are needed. So, we do not go further in this topic.

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