Analytic torsions for hyperbolic manifolds with cusps

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Abstract: In this paper, we announce a result on the relation of the analytic torsion with the Laurent expansion of the Ruelle zeta function at $s = 0$ for odd dimensional noncompact hyperbolic manifolds with cusps.

Key words: Analytic torsion; Ruelle zeta function; hyperbolic manifold with cusps.

1. Introduction. In his seminal paper [1], Fried derived a formula relating the analytic torsion to the Laurent expansion of the Ruelle zeta function at $s = 0$ for compact hyperbolic manifold of odd dimension. The corresponding formula for the eta invariant and the value of the odd type Selberg zeta function at $s = 0$ had been proved by Millson in [6]. Recently in [7] the formula of Millson has been generalized to the case of noncompact hyperbolic manifolds with cusps. Here the eta invariant is defined by certain regularized trace of odd heat operator, which is essentially the same as the b-trace of Melrose introduced in [5]. We also applied the result for the weighted unipotent orbital integral in [4] to compute the contribution from cusps. Hence it is a natural question whether a generalization of the formula of Fried for analytic torsion could be obtained employing these methods. In this paper, we announce such a generalization of the formula of Fried, the relationship of the analytic torsion with the Laurent expansion of the Ruelle zeta function at $s = 0$ for noncompact hyperbolic manifolds with cusps. First we follow the idea of Melrose in [5] to define the analytic torsion for this noncompact case, which is explained in Section 3. Recently in [2,3] it is also shown that the Ruelle zeta function has the meromorphic extension over $\mathbf{C}$ for odd dimensional hyperbolic manifolds with cusps. This is briefly reviewed in Proposition 4.1. The detailed proofs of results announced in this paper will be given in [8].

2. Laplacians over hyperbolic manifolds with cusps. Let us recall that a $(2n+1)$-dimensional noncompact hyperbolic manifold with cusps is given by

$$X_\Gamma = \Gamma \backslash \mathrm{SO}_0(2n+1, 1)/\mathrm{SO}(2n+1)$$

where Γ is a cofinite discrete subgroup of $G = \mathrm{SO}_0(2n+1,1)$ and $K = \mathrm{SO}(2n+1)$ is a maximal compact subgroup of $\mathrm{SO}_0(2n+1,1)$. Throughout this paper, we assume that the group generated by the eigenvalues of Γ contains no root of unity. Its consequences are that Γ is torsion free and

$$\Gamma \cap P = \Gamma \cap N \qquad (1)$$

for a Γ-cuspidal minimal parabolic subgroup P and a Langlands decomposition $P = MAN$ where $M = \mathrm{SO}(2n) \subset K = \mathrm{SO}(2n+1)$.

Let (ρ, V_ρ) be a finite-dimensional unitary representation of $\pi_1(X_\Gamma) = \Gamma$. The vector bundle E^k_ρ over X_Γ of k-forms twisted by ρ is given by

$$E^k_\rho = V_\rho \times_\rho G \times_{\tau_k} V_{\tau_k}$$

where τ_k denotes the fundamental representation of $K = \mathrm{SO}(2n+1)$ acting on $V_{\tau_k} = \wedge^k \mathbf{R}^{2n+1} \otimes \mathbf{C}$. Then the Laplacian acting on $C^\infty_0(X_\Gamma, E^k_\rho)$ has the unique self adjoint extension to $L^2(X_\Gamma, E^k_\rho)$ denoted by Δ_k. In general, the operator Δ_k on $L^2(X_\Gamma, E^k_\rho)$ has the discrete spectrum $\sigma_p(\Delta_k)$ as well as the continuous spectrum $[(n-k)^2, \infty)$. The continuous spectrum of Δ_k is mainly controlled by the scattering operators $C^k_\rho(\sigma_k, s)$ and $C^k_\rho(\sigma_{k-1}, s)$ for purely imaginary numbers $s = i\lambda \in \mathbf{C}$. Here σ_k denotes the fundamental representation of $M = \mathrm{SO}(2n)$ acting on $\wedge^k \mathbf{R}^{2n} \otimes \mathbf{C}$ for $k = 0, 1, \ldots, (n-1)$ and $\sigma_n = \sigma_+ \oplus \sigma_-$ with the half spin representations σ_+, σ_- acting on $\wedge^n \mathbf{R}^{2n} \otimes \mathbf{C}$. These scattering operators have the matrix forms of size $d_c(\rho)$ where

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$$d_c(\rho) = \sum_{j=1}^{\kappa} d_j(\rho).$$

Here κ denotes the number of cusps and $d_j(\rho)$ denotes the dimension of the maximal subspace of V_ρ over which $\rho|_{P_j\cap\Gamma}$ acts trivially for $P_j \in \mathfrak{P}$ where $\mathfrak{P} := \{P_1, \dots, P_\kappa\}$ denotes the set of representatives of Γ-conjugacy classes of Γ-cuspidal minimal parabolic subgroups corresponding to the cusps of X_Γ. The scattering operator $C_\rho^n(\sigma_n, s)$ has the size $2\ d_c(\rho)$ since $\sigma_\pm$ is un-ramified.

3. Analytic torsions for hyperbolic manifolds with cusps. Now let us recall that the heat operator $e^{-t\Delta_k}$ is not of trace class for noncompact hyperbolic manifold with cusps, so that we could not take the usual trace of it. To overcome this, we follow the idea of Melrose in [5] as follows. First let us observe that each cusp corresponds to a Γ-cuspidal parabolic subgroup $P = MAN$ and each cuspidal end is modelled on $A \cdot \Gamma_N \backslash N$ where $\Gamma_N := \Gamma \cap P = \Gamma \cap N$ by (1). The standard Haar measure on G (for instance given in [9]) induces naturally a metric over X_Γ, which has the form $du^2 + e^{-2u} dn^2$ over a cuspidal end where dn^2 is the induced metric over $\Gamma_N \backslash N$. For sufficiently large $a \gg 0$, we put X_Γ^a to be the complement in X_Γ of the cuspidal ends whose u-coordinates are larger than a. Now, by the Maass-Selberg relation, we could remove the diverging term of the expansion of

$$\int_{X_\Gamma^a} \operatorname{tr} e^{-t\Delta_k}(x,x)\ dx \qquad \text{as} \quad a \to \infty$$

and define the regularized trace $\mathrm{Tr_r}(\cdot)$ of $e^{-t\Delta_k}$ to be the remaining finite part of it. Then we have

$$\begin{aligned}\mathrm{Tr_r}\big(e^{-t\Delta_k}\big) = & \sum_{\lambda_j \in \sigma_p(\Delta_k)} e^{-t\lambda_j} \\ & + \sum_{\ell = k, k-1} \Bigg(\frac{d(\sigma_\ell)}{4} e^{-t d_\ell^2}\, \mathrm{Tr}\Big(C_\rho^k(\sigma_\ell, 0)\Big) \\ & - \frac{d(\sigma_\ell)}{4\pi} \int_{-\infty}^{\infty} e^{-t(\lambda^2 + d_\ell^2)} \mathrm{Tr}\big(C^k(\ell,\lambda)\big)\, d\lambda \Bigg)\end{aligned}$$

where $d_\ell = (n-\ell)$, $d(\sigma_\ell) = \dim(V_{\sigma_\ell})$ and

$$C^k(\ell,\lambda) = C_\rho^k(\sigma_\ell, s)^{-1} \frac{d}{ds} C_\rho^k(\sigma_\ell, s)\Big|_{s=i\lambda}.$$

Actually this trace is the same as the geometric side of the Selberg trace formula applied to the test function given by the lifted heat kernel of Δ_k to G.

Now we define the spectral zeta function of Δ_k by

$$\zeta_{\Delta_k}(s) := \frac{1}{\Gamma(s)} \left(\int_0^1 + \int_1^\infty \right) t^{s-1} \mathrm{Tr_r}\big(e^{-t\Delta_k} - P_k\big)\, dt$$

where P_k denotes the orthogonal projection onto $\ker_{L^2}(\Delta_k)$. Here the small, large time integrals $\int_0^1$, $\int_1^\infty$ are defined for $\Re(s) \gg 0$ and $\Re(s) \ll 0$ respectively. The first result in this paper is

Theorem 3.1. *For* $0 \le k \le (2n+1)$, *the spectral zeta function* $\zeta_{\Delta_k}(s)$ *has the meromorphic extension over* $\mathbf{C}$ *and is regular at* $s = 0$.

The proof of Theorem 3.1 is an application of the Selberg trace formula in [10] with complete computation of the weighted unipotent orbital integral applied to the test function given by the lifted heat kernel of Δ_k to G. The detail of proof will be given in [8].

By Theorem 3.1, we can define the regularized determinant of Δ_k by

$$\det{}_\zeta \Delta_k := \exp\left(-\frac{d}{ds}\Big|_{s=0} \zeta_{\Delta_k}(s) \right)$$

and the analytic torsion $T(X_\Gamma, \rho)$ by

$$\begin{aligned} T(X_\Gamma,\rho) := & \frac{\det_\zeta \Delta_1}{\big(\det_\zeta \Delta_2\big)^2} \cdot \frac{\big(\det_\zeta \Delta_3\big)^3}{\big(\det_\zeta \Delta_4\big)^4} \cdots \\ & \cdots \frac{\big(\det_\zeta \Delta_{2n-1}\big)^{2n-1}}{\big(\det_\zeta \Delta_{2n}\big)^{2n}} \cdot \big(\det_\zeta \Delta_{2n+1}\big)^{2n+1}. \end{aligned}$$

Note that our definition of analytic torsion is reduced to the square of the one given in [1] when X_Γ is compact.

4. Expansion of Ruelle zeta function at $s = 0$. Let us recall that the Ruelle zeta function $R_\rho(s)$ is defined by

$$R_\rho(s) := \prod_\gamma \det\big(\mathrm{Id} - \rho(\gamma) e^{-s\, l_\gamma}\big)^{-1}$$

for $\Re(s) > 2n$. Here γ runs over the Γ-conjugacy classes of the primitive hyperbolic elements in Γ, the determinant denoted by det is taken over the representation space V_ρ of ρ, and l_γ denotes the length of the prime geodesic determined by γ. Note that the above definition of the Ruelle zeta function is the inverse of the one in [1]. In [2,3], the following fundamental properties of $R_\rho(s)$ are proved,

Proposition 4.1.

(a) *The Ruelle zeta function* $R_\rho(s)$ *defined a priori* *for $\Re(s) > 2n$ has the meromorphic extension over* $\mathbf{C}$.

(b) *Let N_0 denote the order of the singularity of $R_\rho(s)$ at $s = 0$ such that $\lim_{s\to 0} s^{N_0} R_\rho(s)$ is a nonzero finite value. Then the integer N_0 is given by*

$$2\sum_{k=0}^{n}(-1)^k(n+1-k)\beta_k + \sum_{k=0}^{n-1}(-1)^{k+1}d(\sigma_k)b_k + d_c(\rho)\sum_{k=1}^{n}(-1)^k\, k\, d(\sigma_k)$$

where $\beta_k := \dim\ker_{L^2}(\Delta_k)$ and b_k is the order of singularity of $\det C^k_\rho(\sigma_k, s)$ *at $s = n - k$.*

By Proposition 4.1, we can see that the behavior of the Ruelle zeta function $R_\rho(s)$ at $s = 0$ is related to the spectral data of the Laplacians Δ_k's. Hence it is a natural question whether the nonzero constant $\lim_{s\to 0} s^{N_0} R_\rho(s)$ may have a relationship with certain spectral data. It turned out that this is given by the analytic torsion (up to a constant) for compact case, which is the formula of Fried in [1]. The second result in this paper states that the essentially same formula holds for hyperbolic manifolds with cusps when we use the analytic torsion defined in Section 3. To state this, we need to introduce some notation. Let us recall that $\det C^k_\rho(\sigma_k, s)$ is a meromorphic function over $\mathbf{C}$ and $C^k_\rho(\sigma_k, s)$ satisfies the following functional equation

$$C^k_\rho(\sigma_k, s)C^k_\rho(\sigma_k, -s) = \mathrm{Id}.$$

Hence the order of the singularity of $\det C^k_\rho(\sigma_k, s)$ at $s = -(n-k)$ is $-b_k$. Now we put

$$\begin{aligned} S_\rho(k) &= \lim_{s\to -(n-k)} s^{-b_k}\det C^k_\rho(\sigma_k, s) \\ &= (-1)^{b_k}\lim_{s\to (n-k)}\Big(s^{b_k}\det C^k_\rho(\sigma_k, s)\Big)^{-1}. \end{aligned}$$

Theorem 4.2. *The following equality holds up to sign,*

$$\lim_{s\to 0}\big(s^{N_0}R_\rho(s)\big)^{-1} = C_1\cdot C_2^{d_c(\rho)}\cdot C_3\cdot T(X_\Gamma, \rho).$$

Here

$$C_1 := \prod_{k=0}^{n-1}\Big(-4(n-k)^2\Big)^{(-1)^k\alpha_k}$$

$$C_2 := \prod_{k=0}^{n-1} 2^{(-1)^{k+1}d(n,k)}\cdot(n-k)^{(-1)^k(d(n,k)+d(\sigma_k))}$$

where

$$\alpha_k := \beta_k - \beta_{k-1} + \beta_{k-2} - \ldots \pm \beta_0,$$

$$d(n,k) := \binom{2n}{k} - \binom{2n-1}{k}$$

and

$$C_3 := \prod_{k=0}^{n-1} S_\rho(k)^{(-1)^{k+1}d(\sigma_k)}.$$

When X_Γ is compact, the equality in Theorem 4.2 is reduced to the formula of Fried in [1]. Actually we can see that the same formula holds under a more general condition that $d_c(\rho) = 0$. In fact, if $d_c(\rho) = 0$, then $C_2^{d_c(\rho)} = C_3 = 1$ and N_0 is given only by β_k's. Moreover the sign ambiguity in Theorem 4.2 disappear since this comes from the scattering operators. The proof of Theorem 4.2 is mainly a complete analysis of the geometric side of the Selberg trace formula in [10], in particular, of the weighted unipotent orbital integral. The detail of proof will be given in [8].

References

[1] D. Fried, Analytic torsion and closed geodesics on hyperbolic manifolds, Invent. Math. **84** (1986), no. 3, 523–540.

[2] Y. Gon and J. Park, Ruelle zeta function for odd dimensional hyperbolic manifolds with cusps. (Preprint).

[3] Y. Gon and J. Park, The zeta functions of Ruelle and Selberg for hyperbolic manifolds with cusps. (in preparation).

[4] W. Hoffmann, The Fourier transforms of weighted orbital integrals on semisimple groups of real rank one, J. Reiner Angew. Math. **489** (1997), 53–97.

[5] R. B. Melrose, *The Atiyah-Patodi-Singer index theorem*, Research Notes in Mathematics, 4. A K Peters, Ltd., Wellesley, MA, 1993.

[6] J. Millson, Closed geodesics and the η-invariant, Ann. of Math.(2) **108** (1978), no. 1, 1–39.

[7] J. Park, Eta invariants and regularized determinants for odd dimensional hyperbolic manifolds with cusps, Amer. J. Math. **127** (2005), no. 3, 493–534.

[8] J. Park, Analytic torsion and closed geodesics for hyperbolic manifolds with cusps. (in preparation).

[9] M. Wakayama, Zeta functions of Selberg's type associated with homogeneous vector bundles, Hiroshima Math. J. **15** (1985), no. 2, 235–295.

[10] G. Warner, Selberg's trace formula for nonuniform lattices: the R-rank one case, in *Studies in algebra and number theory*, 1–142, Academic Press, New York.