Steiner ratio for hyperbolic surfaces

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Abstract: We prove that the Steiner ratio for hyperbolic surfaces is 1/2.

Key words: Steiner ratio; Steiner tree; Riemannian geometry; geodesic; hyperbolic geometry.

1. Introduction. Let M be a complete Riemannian manifold without boundary. Let P be a finite set of points in M. A shortest network interconnecting P is called a *Steiner minimum tree* which is denoted as SMT(P). An SMT(P) may have vertices which are not in P. Such vertices are called *Steiner points*. A spanning tree on P is a tree with vertex set P. A shortest spanning tree on P is called a *minimum spanning tree* on P which is denoted as MST(P). Let $L(T)$ be the total length of edges in a tree T. The *Steiner ratio* is given by

$$\rho = \rho(M) = \inf_P \frac{L(\mathrm{SMT}(P))}{L(\mathrm{MST}(P))}.$$

Du and Hwang ([1]) have proved that $\rho = \sqrt{3}/2$ if M is the Euclidean plane. This was the affirmative answer of a famous conjecture of Gilbert and Pollak ([2]). Rubinstein and Weng ([4]) have proved that $\rho = \sqrt{3}/2$ if M is a 2-dimensional sphere of constant curvature. Ivanov, Tuzhilin and Cieslik ([3]) have estimated some Steiner ratios for manifolds. In particular, they proved that $\rho \leq 3/4$ if M is a simply connected complete surface of constant curvature -1 without boundary, and that $\rho < \sqrt{3}/2$ if M is a surface of constant curvature -1.

In the present note we prove the following theorem.

Theorem 1. *The Steiner ratio $\rho(M)$ is $1/2$ if M is a simply connected complete surface of negative constant curvature without boundary.*

A simply connected complete Riemannian manifold of negative constant curvature without boundary is called a *hyperbolic space*.

2. The lower bound of Steiner ratio. Let M be a complete surface without boundary and P a set of n points in M. Then, SMT(P) satisfies the following properties.

(1) All terminal points of SMT(P) are points in P.
(2) Any two edges meet at an angle of at least 120°.
(3) Every Steiner point has degree exactly three.
(4) There are at most $(n-2)$ Steiner points in SMT(P).

We say that a tree T is a *Steiner tree* if T satisfies (1) to (3). A Steiner tree T is by definition *full* if T has exactly $n-2$ Steiner points. Any Steiner tree can be decomposed into an edge-disjoint union of full Steiner trees.

Any tree T will become a polygonal region with boundary if its edges are replaced with ϵ-belts as its widen edges. Two terminal points are *adjacent* in T if they are consecutive on the boundary.

The following lemma is stated in [1]. However, we give a proof here because the idea will be important in proving Theorem 1.

Lemma 2. *Let M be a complete surface without boundary. Then,*

$$\rho(M) \geq \frac{1}{2}.$$

Proof. Let $P = \{p_1, \ldots, p_n\}$ be a set of points in M where p_i and p_{i+1} are adjacent in SMT(P) and $p_{n+1} = p_1$. We may assume that SMT(P) is full. Let $S(p_i, p_{i+1})$ be the minimal subtree from p_i to p_{i+1} of SMT(P). Then, we have

$$d(p_i, p_{i+1}) \leq L(S(p_i, p_{i+1}))$$

for $i = 1, \ldots, n$ where $d(\cdot, \cdot)$ is the distance induced from the Riemannian metric of M. We set $L = \sum_{i=1}^{n} d(p_i, p_{i+1})$. Since

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$$\sum_{i=1}^{n} L(S(p_i,p_{i+1})) = 2L(\mathrm{SMT}(P)),$$

we see that $L \le 2L(\mathrm{SMT}(P))$. Since

$$\frac{L}{n} \le \max\{d(p_i,p_{i+1}) \mid i=1,\ldots,n\},$$

we have

$$L(\mathrm{MST}(P)) \le L - \max\{d(p_i,p_{i+1}) \mid i=1,\ldots,n\} \le \frac{n-1}{n}L.$$

Therefore, we have the inequality

$$\frac{L(\mathrm{SMT}(P))}{L(\mathrm{MST}(P))} \ge \frac{n}{(n-1)L}\frac{L}{2} > \frac{1}{2}.$$

This completes the proof of Lemma 2. □

3. Proof of Theorem 1. Let H be the Poincaré disk, namely, $H = \{(x,y) \mid x^2+y^2 < 1\}$ with Riemannian metric

$$ds^2 = \frac{4\left(dx^2+dy^2\right)}{c(1-x^2-y^2)^2}$$

for a positive c. Any complete simply connected surface M of negative constant curvature $-c$ without boundary is isometric to H. The geodesic lines are circles which meet at the right angle to the boundary ∂H. Let $T(p,q)$ be the unique geodesic segment connecting points p and q in H. The most important property to be used in our proof is that any sequence of segments $T(p_k,q_k)$ converges to a geodesic line T connecting p_∞ and q_∞ if the sequences of points p_k and q_k converge to points p_∞ and q_∞ in ∂H, respectively.

Let n be an integer greater than 2. Let O be the origin in H and $\gamma_i\colon [0,\infty) \longrightarrow H$ unit speed geodesic rays for $i=1,\ldots,n$ such that $\gamma_i(0)=O$, $\angle(\dot\gamma_i(0),\dot\gamma_{i+1}(0)) = 2\pi/n$, where $\dot\gamma_i(0)$ is the tangent vector of γ_i at $t=0$ and $\angle(\dot\gamma_i(0),\dot\gamma_{i+1}(0))$ is the angle of $\dot\gamma_i(0)$ with $\dot\gamma_{i+1}(0)$ and $\gamma_{n+1}=\gamma_1$. Let $P(s) = \{\gamma_i(s) \mid i=1,\ldots,n\}$ for a positive s. Let $S(\gamma_i(s),\gamma_{i+1}(s))$ be the minimal subtree from $\gamma_i(s)$ to $\gamma_{i+1}(s)$ of $\mathrm{SMT}(P(s))$. We prove the following lemma.

Lemma 3. *For all $i=1,\ldots,n$, we have*

$$\lim_{s\to\infty} \frac{L(S(\gamma_i(s),\gamma_{i+1}(s)))}{d(\gamma_i(s),\gamma_{i+1}(s))} = 1.$$

Proof. Let ${}^s\alpha_i\colon [-d_i(s),d_i(s)] \longrightarrow H$ be the geodesic segment from $\gamma_i(s)$ to $\gamma_{i+1}(s)$ where $d_i(s) = d(\gamma_i(s),\gamma_{i+1}(s))/2$. Then, ${}^s\alpha_i$ converges to the geodesic line $\alpha_i\colon (-\infty,\infty) \longrightarrow H$ connecting $\gamma_i(\infty) = \alpha_i(-\infty)$ and $\gamma_{i+1}(\infty) = \alpha_i(\infty)$. The set of Steiner points for $P(s)$ is nonempty because of the inequality $\angle(-{}^s\dot\alpha_{i-1}(d_{i-1}(s)), {}^s\dot\alpha_i(-d_i(s))) < 120°$ for sufficiently large s. The geodesic polygon $K_i(s) = S(\gamma_i(s),\gamma_{i+1}(s)) \cup {}^s\alpha_i([-d_i(s),d_i(s)])$ surrounds a convex domain. Let $b_j(s)$, $j=0,\ldots,m$, be the vertices of $S(\gamma_i(s),\gamma_{i+1}(s))$ such that they are in this order on it, $b_0(s) = \gamma_i(s)$ and $b_m(s) = \gamma_{i+1}(s)$. Then, $m \le n-2$. The inner angles of $K_i(s)$ at $b_j(s)$ $(j=1,\ldots,m-1)$ are 120°. Let $\beta_k\colon [0,\infty) \longrightarrow H$, $k=1,2,3$, be three geodesic rays such that $\beta_k(0) = b_1(s)$, $\beta_1([0,\infty)) \supset T(b_0(s),b_1(s)), \beta_2([0,\infty)) \supset T(b_1(s),b_2(s))$ and $\angle(\dot\beta_3(0),\dot\beta_1(0)) = \angle(\dot\beta_3(0),\dot\beta_2(0)) = 120°$. The geodesic rays $\beta_k([0,\infty))$ $(k=1,2,3)$ divide H into three parts. The geodesic polygons $K_i(s)$ and $K_{i-1}(s)$ are contained in one of them, respectively.

We first claim that $b_1(s)$ is bounded as $s \longrightarrow \infty$. Suppose $b_1(s) \longrightarrow \gamma_i(\infty)$ as $s \longrightarrow \infty$. Since $d(\alpha_{i-1}(t),\alpha_i(\mathbf{R})) \longrightarrow 0$ as $t \longrightarrow \infty$, it follows that at least one of $\beta_k([0,\infty))$ $(k=2,3)$ intersects either $\alpha_i(\mathbf{R})$ or $\alpha_{i-1}(\mathbf{R})$, contradicting the construction of the geodesic polygons $K_i(s)$ and $K_{i-1}(s)$ for sufficiently large s. In the same way we can prove that $b_{m-1}(s)$ is bounded as $s \longrightarrow \infty$. Combining these facts, we see that the set $\{b_1(s),\ldots,b_{m-1}(s)\}$ is bounded as $s \longrightarrow \infty$.

Let $f_j(s)$, $j=1,\ldots,m-1$, be the foot of $b_j(s)$ on ${}^s\alpha_i([-d_i(s),d_i(s)])$, namely, $f_j(s)$ is the unique point in ${}^s\alpha_i([-d_i(s),d_i(s)])$ with $d(b_j(s),f_j(s)) = d(b_j(s),{}^s\alpha_i([-d_i(s),d_i(s)]))$. Then, we have

$$d(\gamma_i(s),\gamma_{i+1}(s)) \le L(S(\gamma_i(s),\gamma_{i+1}(s))) \le d(\gamma_i(s),\gamma_{i+1}(s)) + 2\sum_{j=1}^{m-1} d(b_j(s),f_j(s)).$$

Since $b_j(s)$ and $f_j(s)$ are bounded as $s \longrightarrow \infty$, we see that

$$\frac{L(S(\gamma_i(s),\gamma_{i+1}(s)))}{d(\gamma_i(s),\gamma_{i+1}(s))} \longrightarrow 1 \quad \text{as} \quad s \longrightarrow \infty.$$

This completes the proof of Lemma 3. □

Proof of Theorem 1. As was seen in Lemma 2, the Steiner ratio is greater than or equal to 1/2. We will show that

$$\frac{L(\mathrm{SMT}(P(s)))}{L(\mathrm{MST}(P(s)))} \longrightarrow \frac{n}{2(n-1)} \quad \text{as} \quad s \longrightarrow \infty.$$

This fact implies that $\rho(H) = 1/2$. By the choice of $P(s)$ we have

$$L(\mathrm{MST}(P(s))) = (n-1)d(\gamma_1(s), \gamma_2(s)).$$

Hence, we have

$$\begin{aligned}
&\frac{L(\mathrm{SMT}(P(s)))}{L(\mathrm{MST}(P(s)))}\\
&= \frac{1}{2}\frac{\sum_{i=1}^{n} L(S(\gamma_i(s), \gamma_{i+1}(s)))}{(n-1)d(\gamma_1(s), \gamma_2(s))}\\
&= \frac{1}{2}\frac{n}{n-1}\frac{\sum_{i=1}^{n} L(S(\gamma_i(s), \gamma_{i+1}(s)))}{nd(\gamma_1(s), \gamma_2(s))}\\
&= \frac{1}{2}\frac{n}{n-1}\frac{\sum_{i=1}^{n} L(S(\gamma_i(s), \gamma_{i+1}(s)))}{\sum_{i=1}^{n} d(\gamma_i(s), \gamma_{i+1}(s))}.
\end{aligned}$$

Therefore, it follows Lemma 3 that

$$\lim_{s\to\infty} \frac{L(\mathrm{SMT}(P(s)))}{L(\mathrm{MST}(P(s)))} = \frac{n}{2(n-1)}.$$

This completes the proof of Theorem 1. □

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