Collision or non-collision problem for interacting Brownian particles

By Kiyokazu INUKAI

Graduate School of Information Science, Nagoya University, Chikusa-ku, Nagoya 464-8601, Japan

(Communicated by Heisuke HIRONAKA, M. J. A., April 12, 2006)

Abstract: The purpose of this paper is to study the collision or non-collision problem for interacting Brownian particles in the framework of theory of Dirichlet forms. The result is closely related to a question on existence and uniqueness of strong solutions for stochastic differential equations with singular drifts.

Key words: Interacting Brownian particles; diffusion; collision; Dirichlet form; capacity.

1. Introduction and main results.

One-dimensional interacting Brownian particles are stochastic dynamics of N-particles moving in $\mathbf{R}$ with effects of mutual interactions and external forces. Mathematically, we consider a diffusion process $\{X(t) = (X_1(t), \dots, X_N(t)),\ t \geqq 0\}$ given as the solution for the stochastic differential equation (SDE) of the following form: for a permutation invariant function $F\colon \mathbf{R}^N \to \mathbf{R}$ and for an N-dimensional Brownian motion $\{(B_1(t), \dots, B_N(t))\}$,

$$dX_i(t) = dB_i(t) + \frac{\partial F}{\partial x_i}(X_1(t), \dots, X_N(t))\, dt, \tag{1.1}$$

$i = 1, \dots, N$.

In many important examples (cf. [7, 8]), the potential F is given in the form

$$F(x_1, \dots, x_N) = \sum_{1 \leqq i < j \leqq N} \Phi(x_i, x_j) + \sum_{i=1}^{N} \Psi(x_i) \tag{1.2}$$

for a symmetric function $\Phi\colon \mathbf{R}^2 \to \mathbf{R}$ and $\Psi\colon \mathbf{R} \to \mathbf{R}$. The functions Φ and Ψ are called an interacting and a self potential, respectively.

In particular, when the potentials Φ and Ψ are given by

$$\Phi(\xi, \eta) = \gamma \log|\xi - \eta|, \quad \Psi(\xi) = -\frac{1}{2}\beta\xi^2,$$

for positive constants γ and β, the corresponding diffusion process is called Dyson's model (in finite dimension), which Dyson [3] studied in connection with the statistics of the eigenvalues of some random matrices.

Throughout this paper, taking into account Dyson's model, the eigenvalue processes of Wishart processes discussed below and so on, we assume that the potential F is given by (1.2) and that $\Phi(\xi, \eta)$ diverges as ξ tends to η for all $\eta \in \mathbf{R}$.

The purpose is to study the (non)collision problem for the solutions of the SDE (1.1) and (1.2) in the framework of theory of Dirichlet forms. We say that the interacting Brownian particles collide, if $\{X(t)\}$ hits the diagonal set D,

$$D = \{x \in \mathbf{R}^N\,;\, x_i = x_j \quad \text{for some} \quad i \neq j\}$$

in finite time with positive probability. As is mentioned later, this problem is closely related to that on the existence and uniqueness of a strong solution for (1.1).

For Dyson's model, Rogers-Shi [7] have proven that the particles do not collide if $\gamma \geqq 1/2$, and Cépa-Lépingle [2] have proven that a strong solution exist uniquely by using some results on multi-valued SDE's. These results may be obtained from our results and we will also show that the particles collide if $\gamma < 1/2$.

On the other hand, the eigenvalue processes of Wishart processes may also be treated in our framework. Let $\{B(t)\}$ be an $N \times N$ Brownian matrix, that is, a matrix-valued stochastic process whose elements consist of independent 1-dimensional Brownian motions. A Wishart process $\{S_t\}$ of dimension $n > N - 1$ may be obtained as a solution for the SDE

$$dS(t) = \sqrt{S(t)}\, dB(t) + dB(t)^T \sqrt{S(t)} + nI_N\, dt, \tag{1.3}$$

where I_N is the N-dimensional identity matrix and we denote by A^T the transpose of a matrix A. In particular, when n is a positive integer, letting $\{\beta(t)\}$ be an $n \times N$ Brownian matrix, $\{S(t)\}$ may be realized as $S(t) = \beta(t)^T \beta(t)$.

2000 Mathematics Subject Classification. Primary 60J60.

Bru [1] has shown that, if $n > N-1$, (1.3) has a unique solution and that the eigenvalue process $\{(X_1(t), \dots, X_N(t))\}$ of $\{S(t)\}$ satisfies an SDE. We present the SDE satisfied by $Y_i(t) = \sqrt{X_i(t)}$: for $i = 1, \dots, N$,

(1.4)
$$dY_i(t) = dW_i(t) + \frac{1}{2Y_i(t)}\left\{(n-1) + \sum_{j=1, j\neq i}^{N} \frac{(Y_i(t))^2 + (Y_j(t))^2}{(Y_i(t))^2 - (Y_j(t))^2}\right\} dt,$$

where $\{(W_1(t), \dots, W_N(t))\}$ is an N-dimensional Brownian motion. She has also proven that the particles (eigenvalues) do not collide.

Now we formulate our results. We follow the notions and the notations in [4].

Let ρ be a locally bounded, non-negative function on $\mathbf{R}^N$. Throughout this paper we assume:
(A-1) $\rho(x)$ is permutation invariant.
We denote by $L^2(\mathbf{R}^N; \rho)$ the space of real-valued functions on $\mathbf{R}^N$ which are square integrable with respect to $\rho(x)\,dx$.

We consider a symmetric bilinear form $\mathcal{E}$ on $L^2(\mathbf{R}^N; \rho)$ defined by

$$\mathcal{E}(f, g) = \frac{1}{2}\sum_{i=1}^{N} \int_{\mathbf{R}^N} \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_i}\rho(x)\,dx, \tag{1.5}$$

for $f, g \in C_0^\infty(\mathbf{R}^N)$, and assume the following:
(A-2) the symmetric form $\mathcal{E}$ is closable.

We denote its closure by $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, using the same symbol. It is known (cf. Osada [6]) that, the lower semicontinuity of the density ρ is sufficient for (A-2).

$(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a regular Dirichlet space relative to $L^2(\mathbf{R}^N; \rho)$ and we denote by $(X(t), P_x)$ the diffusion process starting from $x \in \mathbf{R}^N \setminus D$ associated with $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. For simplicity, we also assume the following:
(A-3) the diffusion process is conservative.

Let σ_D denote the hitting time to D defined by

$$\sigma_D = \inf\{t > 0\,;\, X(t) \in D\}. \tag{1.6}$$

If ρ is of C^1-class and positive outside D and if we set

$$b_i(x) = \frac{1}{2}\rho(x)^{-1}\frac{\partial \rho(x)}{\partial x_i},$$

then $\{X(t)\}$ may be realized as a solution for the following SDE, based on an N-dimensional Brownian motion $\{(B_1(t), \dots, B_N(t))\}$,

(1.7)
$$\begin{cases} dX_i(t) = dB_i(t) + b_i(X(t))\,dt, \quad i = 1, \dots, N, \\ X_0 = x \in \mathbf{R}^N \setminus D, \end{cases}$$

for $t < \sigma_D$. Its generator is given by,

$$L = \frac{1}{2}\sum_{i=1}^{N} \rho(x)^{-1}\frac{\partial}{\partial x_i}\left(\rho(x)\frac{\partial}{\partial x_i}\right). \tag{1.8}$$

Since, taking the two examples into account, we do not assume the smoothness of the drift coefficient b at D, the existence and uniqueness of the global strong solution for (1.7) is nontrivial. If ρ is smooth outside the set D and if one proves that the particles do not hit D in finite time, then one obtains the unique existence of a strong solution for (1.7). This observation is one of the motivations of our study about the collision or non-collision problem for this diffusion process.

A sufficient condition for a diffusion process not to hit a set with measure zero is found in [4] (cf. Theorem 6.3.3 p.291), where a transformation of drift is used. We cannot apply this result since we do not assume the global smoothness of ρ. We will prove the results by some estimates for the capacity.

Now we present our results.

Theorem 1.1. *In addition to* (A-1), (A-2), (A-3), *we assume that, for each* $a \in D$ *with* $a_i = a_j$ $(i \neq j)$, *there exist a positive constant* λ *and a non-negative continuous function* $h(\eta) = h_{\lambda,a}(\eta)$ *on* $(0, \lambda)$ *such that*

$$\rho(x) \leqq h(|x_i - x_j|)$$

for all $x \in U(a, \lambda) \equiv \{z\,;\, |z - a| < \lambda\}$ *and*

$$\int_{0+} h(\eta)^{-1}\,d\eta = \infty.$$

Then we have $P_x(\sigma_D < \infty) = 0$ *for* q.e.x $\in \mathbf{R}^N \setminus D$.

Remark 1.1. If ρ is smooth outside D, we can take a version of P_x which is continuous outside D, and obtain the result $P_x(\sigma_D < \infty) = 0$ for all $x \in \mathbf{R}^N \setminus D$.

As is mentioned above, we obtain the following from Theorem 1.1.

Corollary 1.2. *If the function* $\rho(x)$ *is of* C^1-*class on* $\mathbf{R}^N \setminus D$ *and satisfies the assumptions of Theorem* 1.1, *then the SDE* (1.7) *has a unique strong solution.*

We next consider the collision case.

Theorem 1.3. *In addition to* (A-1), (A-2), (A-3), *we assume that* $\rho(x)$ *is locally uniformly posi-*

tive on $\mathbf{R}^N \setminus D$ *and that there exist* $a \in D$ *with* $a_i = a_j$ $(i \neq j)$, $\lambda > 0$ *and a non-negative continuous function* $h(\eta) = h_{\lambda,a}(\eta)$ *on* $(0, \lambda)$ *such that*

$$\rho(x) \geqq h(|x_i - x_j|)$$

for all $x \in U(a, \lambda)$ *and*

$$\int_{0+} h(\eta)^{-1}\, d\eta < \infty. \tag{1.9}$$

Then we have $P_x(\sigma_D < \infty) > 0$ *for* q.e.x $\in \mathbf{R}^N \setminus D$.

We apply these theorems to the two examples above.

Example 1.1 (Dyson's model). We set

$$\rho(x) = \exp\left(-\beta \sum_{i=1}^{N} x_i^2\right) \cdot \prod_{i<j}^{N} |x_i - x_j|^{2\gamma}.$$

Then we obtain the Dyson's model mentioned above. Hence if $\gamma \geqq 1/2$, we have non-collision for the solution of SDE (1.1). In particular, when $N = 2$ and $\beta = 0$, non-collision property may be obtained from Feller's test for explosion [5]. In fact, setting $Y(t) = X_1(t) - X_2(t)$, we easily see that $\{Y(t)\}$ satisfy the SDE,

$$dY(t) = \sqrt{2}\, dW(t) + \frac{2\gamma}{Y(t)}\, dt, \tag{1.10}$$

where $\{W(t)\,;\, 0 \leqq t < \infty\}$ is a 1-dimensional Brownian motion. Hence, letting $\{R(t)\}$ be a $(2\gamma + 1)$-dimensional Bessel process, we see that $\{Y(t)\}$ is identical in law with $\{R_{2t}\}$. Hence, if $\gamma \geqq 1/2$, $\{Y(t)\}$ never hits zero a.s. In the case $\gamma < 1/2$, we obtain $P(\sigma_D < \infty) > 0$ by Theorem 1.3, namely, we have collision.

Example 1.2 (Eigenvalue processes of Wishart process). Setting

$$\rho(x) = \left(\prod_{i=1}^{N} x_i^{n-N}\right)\left(\prod_{i<j}^{N} |x_i + x_j|\right)\left(\prod_{i<j}^{N} |x_i - x_j|\right),$$

we obtain the generator (1.8). Hence we have non-collision for the solution of the SDE (1.4).

2. Proof of Theorem 1.1. First of all, we recall the notion of capacity. Let $\mathcal{O}$ denote the collection of all open sets in $\mathbf{R}^N$. For $A \in \mathcal{O}$, the capacity $\mathrm{Cap}(A)$ is defined by

$$\mathrm{Cap}(A) = \inf\{\mathcal{E}(f, f) + (f, f)_{L^2}\,;\, f \in \mathcal{L}_A\},$$
$$\mathcal{L}_A = \{f \in \mathcal{D}(\mathcal{E})\,;\, f \geqq 1,\ \rho\text{-a.e. on } A\}.$$

For a general subset $B \subset \mathbf{R}^N$, we define $\mathrm{Cap}(B) = \inf_{B \subset A \in \mathcal{O}} \mathrm{Cap}(A)$.

By the general theory of Dirichlet forms [4], Theorem 1.1 follows if we show

$$\mathrm{Cap}(D) = 0. \tag{2.1}$$

For this purpose, let $D_{i,j}(k, l)$ denote the following subset of $\mathbf{R}^N$:

$$D_{i,j}(k, l) = \{x \in \mathbf{R}^N\,;\, x_i = x_j,\ k \leqq x_i < k + 1,\ |x_\nu| \leqq l,\ \nu \neq i, j\}.$$

Then, by the sub-additivity of capacity, if we show

$$\mathrm{Cap}(D_{i,j}(k, l)) = 0 \tag{2.2}$$

for every $i, j = 1, \ldots, N$, $l \in \mathbf{N}$ and $k \in \mathbf{Z}$, we obtain (2.1). For details of the general theory of Dirichlet forms and the corresponding diffusion processes, we refer to [4].

To prove (2.2), we need to introduce another set $D_{i,j}^\delta(k, l)$, $\delta > 0$, given by

$$D_{i,j}^\delta(k,l) = \{x \in \mathbf{R}^N\,;\, |x_i - x_j| < \sqrt{2}\,\delta,\ k \leqq x_i < k + 1,\ |x_\nu| \leqq l,\ \nu \neq i, j\}$$

and show the following proposition. Since

$$\lim_{\lambda \downarrow 0} \lim_{\delta \downarrow 0} F(\delta, \lambda) = 0,$$

by assumption, we obtain Theorem 1.1 from the proposition.

Proposition 2.1. *Let* λ *be a constant given in Theorem* 1.1. *Then, there exists an absolute constant* C *such that*

$$\mathrm{Cap}(D_{i,j}^\delta(k, l)) \leqq C \cdot F(\delta, \lambda)$$

holds for any $\delta < \lambda$ *and for any* $i, j = 1, \ldots, N$, $l \in \mathbf{N}$, $k \in \mathbf{Z}$, *where*

$$F(\delta, \lambda) = \left(\int_\delta^\lambda h(\eta)^{-1}\, d\eta\right)^{-1} + \int_0^\lambda h(\eta)\, d\eta.$$

Proof. For notational simplicity, we only consider the case $i = 1$ and $j = 2$. Moreover, without loss of generality, we assume $x_1 > x_2$.

We change the variables by $x_1 - x_2 = \sqrt{2}\,\xi_1$, $x_1 + x_2 = \sqrt{2}\,\xi_2$, $x_j = \xi_j$, $j \geqq 3$. Letting $\widetilde{D}_{1,2}^\delta(l)$ be a subset of $\mathbf{R}^{N-1}$ given by

$$\widetilde{D}_{1,2}^\delta(l) = \{(\xi_2, \xi_3, \ldots, \xi_N) \in \mathbf{R}^{N-1}\,;\ 0 \leqq \xi_2 \leqq \sqrt{2},\ |\xi_\nu| \leqq l,\ \nu \geqq 3\}$$

and ψ be a non-negative C^∞ function on $\mathbf{R}^{N-1}$ with compact support such that $\psi(\xi_2, \xi_3, \ldots, \xi_N) = 1$ on $\widetilde{D}_{1,2}^\delta(l)$, we set

$$\psi_k(\xi_2, \xi_3, \ldots, \xi_N) = \psi(\xi_2 - \sqrt{2}\, k, \xi_3, \ldots, \xi_N).$$

Moreover, letting h be the function given in Theorem 1.1 and setting $g(\xi_1) = h(\sqrt{2}\,\xi_1)$, we consider the function ϕ_δ on $[0, \infty)$ such that $\phi_\delta(\xi_1) = 1$ for $0 \leqq \xi_1 \leqq \delta$, $\phi_\delta(\xi_1) = 0$ for $\xi_1 \geqq \lambda' \equiv \lambda/\sqrt{2}$ and

$$\phi_\delta(\xi_1) = \frac{\int_{\xi_1}^{\lambda'} g(\eta)^{-1} d\eta}{\int_{\delta}^{\lambda'} g(\eta)^{-1} d\eta} \quad \text{for} \quad \delta \leqq \xi_1 \leqq \lambda'.$$

Now let us consider a test function f given by

$$f(x_1, x_2, \ldots, x_N) = \phi_\delta(\xi_1)\psi_k(\xi_2, \xi_3, \ldots, \xi_N).$$

Then we have

$$\begin{aligned}
&\mathcal{E}(f,f) + (f,f)_{L^2} \\
&= \frac{1}{2}\int_0^{\lambda'}\int_{\mathbf{R}^{N-1}} \left\{ \left(\frac{\partial \phi_\delta}{\partial \xi_1}\right)^2 \psi_k^2 + \phi_\delta^2 \sum_{j=2}^{N} \left(\frac{\partial \psi_k}{\partial \xi_j}\right)^2 \right\} \\
&\qquad\qquad \times \rho(x)\, d\xi_1\, d\xi_2 \cdots d\xi_N \\
&\quad + \int_0^{\lambda'}\int_{\mathbf{R}^{N-1}} \phi_\delta^2 \psi_k^2 \rho(x)\, d\xi_1\, d\xi_2 \cdots d\xi_N.
\end{aligned}$$

Since $\rho(x) \leqq g(\xi_1)$ if $0 < \xi_1 < \lambda'$ by assumption, we have,

$$\begin{aligned}
&\mathcal{E}(f,f) + (f,f)_{L^2} \\
&\leqq \frac{1}{2}\int_0^{\lambda'} \left(\frac{\partial \phi_\delta}{\partial \xi_1}\right)^2 g(\xi_1)\, d\xi_1 \int_{\mathbf{R}^{N-1}} \psi_k^2\, d\xi_2 \cdots d\xi_N \\
&\quad + \frac{1}{2}\int_0^{\lambda'} \phi_\delta^2 g(\xi_1)\, d\xi_1 \int_{\mathbf{R}^{N-1}} \sum_{j=2}^{N} \left(\frac{\partial \psi_k}{\partial \xi_j}\right)^2 d\xi_2 \cdots d\xi_N \\
&\quad + \frac{1}{2}\int_0^{\lambda'} \phi_\delta^2 g(\xi_1)\, d\xi_1 \int_{\mathbf{R}^{N-1}} \psi_k^2\, d\xi_2 \cdots d\xi_N.
\end{aligned}$$

Hence there exists a constant K, which depends only on the function ψ and its first derivatives, such that

$$\begin{aligned}
&\mathcal{E}(f,f) + (f,f)_{L^2} \\
&\leqq K \left(\int_0^{\lambda'} \left(\frac{\partial \phi_\delta}{\partial \xi_1}\right)^2 g(\xi_1)\, d\xi_1 + \int_0^{\lambda'} \phi_\delta^2 g(\xi_1)\, d\xi_1 \right).
\end{aligned}$$

Moreover, by the definition of the function $\phi_\delta(\xi_1)$, we obtain

$$\begin{aligned}
&\mathcal{E}(f,f) + (f,f)_{L^2} \\
&\leqq K \left\{ \left(\int_\delta^{\lambda'} g(\eta)^{-1}\, d\eta \right)^{-1} + \int_0^{\delta} g(\eta)\, d\eta \right. \\
&\qquad \left. + \left(\int_\delta^{\lambda'} g(\eta)^{-1}\, d\eta \right)^{-2} I \right\},
\end{aligned}$$

where

$$I = \int_\delta^{\lambda'} \left(\int_{\xi_1}^{\lambda'} g(\eta)^{-1} d\eta \right)^2 g(\xi_1)\, d\xi_1.$$

By integration by parts, we obtain

$$\begin{aligned}
I &\leqq -\left(\int_\delta^{\lambda'} g(\eta)^{-1}\, d\eta \right)^2 \int_0^{\delta} g(\eta)\, d\eta \\
&\quad + 2\left(\int_\delta^{\lambda'} g(\eta)^{-1} d\eta \right)^2 \int_0^{\lambda'} g(\eta)\, d\eta
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{E}(f,f) + (f,f)_{L^2} \\
&\leqq K \left\{ \left(\int_\delta^{\lambda'} g(\eta)^{-1}\, d\eta \right)^{-1} + 2\int_0^{\lambda'} g(\eta)\, d\eta \right\},
\end{aligned}$$

which implies the assertion of the proposition. □

3. Proof of Theorem 1.3. We give a proof, assuming $i = 1$ and $j = 2$. For $a \in D$ with $a_1 = a_2$, we let $V_a(\lambda, \mu)$, $\lambda > 0$, $\mu > 0$, be the set defined by

$$\begin{aligned}
V_a(\lambda,\mu) = \{x \in \mathbf{R}^N\,;\, &|x_2 - x_1| < \lambda, \\
&|x_1 + x_2 - 2a_1| < \mu, \\
&|x_3 - a_3| < \mu, \ldots, |x_N - a_N| < \mu\}
\end{aligned}$$

and consider

$$\mathcal{E}^1(f,f) = \frac{1}{2}\sum_{i=1}^{N} \int_{V_a(\lambda,\mu)} \left(\frac{\partial f}{\partial x_i}\right)^2 \rho(x)\, dx,$$

$$\mathcal{E}^2(f,f) = \frac{1}{4}\int_{V_a(\lambda,\mu)} \left(\frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2}\right)^2 h(|x_1 - x_2|)\, dx$$

for $f \in C_0^\infty(\mathbf{R}^N)$. We denote by $(\mathcal{E}^1, \mathcal{D}(\mathcal{E}^1))$ and $(\mathcal{E}^2, \mathcal{D}(\mathcal{E}^2))$ the corresponding Dirichlet spaces relative to $L^2(\mathbf{R}^N; \rho)$.

Lemma 3.1. *Let* $\mathrm{Cap}, \mathrm{Cap}^1, \mathrm{Cap}^2$ *be the capacities associated with the regular Dirichlet spaces* $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, $(\mathcal{E}^1, \mathcal{D}(\mathcal{E}^1))$, $(\mathcal{E}^2, \mathcal{D}(\mathcal{E}^2))$, *respectively. Then we have*

$$\mathrm{Cap}(D) \geqq \mathrm{Cap}^1(D) \geqq \mathrm{Cap}^2(D),$$

where $D = \{x \in \mathbf{R}^N\,;\, x_i = x_j$ *for some* $i \neq j\}$.

Proof. Since $\mathcal{E}(f,f) \geqq \mathcal{E}^1(f,f)$ for any $f \in \mathcal{D}(\mathcal{E})$ and $\mathcal{D}(\mathcal{E}) \subseteq \mathcal{D}(\mathcal{E}^1)$, we have $\mathrm{Cap}(D) \geqq \mathrm{Cap}^1(D)$.

For the other inequality, we note the trivial inequality

$$\sum_{i=1}^{N} \left(\frac{\partial f}{\partial x_i}\right)^2 \geqq \frac{1}{2}\left(\frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2}\right)^2.$$

Then, by assumption, we have $\mathcal{E}^1(f,f) \geqq \mathcal{E}^2(f,f)$ for $f \in C_0^\infty(\mathbf{R}^N)$, which implies the assertion. □

From the lemma, we obtain Theorem 1.3 if we show $\mathrm{Cap}^2(D) > 0$. For the purpose, we use the same change of variables as in the proof of Theorem1.1. Then we have

$$\mathcal{E}^2(f,f) = \frac{1}{2}\int_{V_a(\lambda,\mu)} \left(\frac{\partial f}{\partial \xi_1}\right)^2 h\big(\sqrt{2}\,\xi_1\big)\, d\xi_1\, d\xi_2 \cdots d\xi_N.$$

Now we consider one more regular Dirichlet space $(\mathcal{E}^3, \mathcal{D}(\mathcal{E}^3))$ relative to $L^2([0,\infty); h(\sqrt{2}\,\cdot))$ defined from

$$\mathcal{E}^3(\phi,\phi) = \frac{1}{2}\int_0^{\lambda/\sqrt{2}} \left(\frac{\partial \phi(\xi)}{\partial \xi}\right)^2 h\big(\sqrt{2}\,\xi\big)\, d\xi,$$

for $\phi \in C_0^\infty([0,\infty))$ and denote by Cap^3 the corresponding capacity. Then it is easy to see that $\mathrm{Cap}^2(D) > 0$ if and only if $\mathrm{Cap}^3(\{0\}) > 0$. The assumption (1.9) means that, for the diffusion process corresponding to $\mathcal{E}^3$, 0 is a regular boundary by Feller's test. Hence we obtain $\mathrm{Cap}^3(\{0\}) > 0$ and have proven Theorem 1.3.

Acknowledgements. I would like to express my profound gratitude to Professors Hirofumi Osada and Hiroyuki Matsumoto for their encouragement and several fruitful comments during the study. I would also like to thank the anonymous referee for pointing out some mistakes in the first version of the paper.

References

[1] M.-F. Bru, Diffusions of perturbed principal component analysis, J. Multivariate Anal. **29** (1989), no. 1, 127–136.

[2] E. Cépa and D. Lépingle, Diffusing particles with electrostatic repulsion, Probab. Theory Related Fields **107** (1997), no. 4, 429–449.

[3] F. J. Dyson, A Brownian-motion model for the eigenvalues of a random matrix, J. Mathematical Phys. **3** (1962), 1191–1198.

[4] M. Fukushima, Y. Ōshima, and M. Takeda, *Dirichlet forms and symmetric Markov processes*, de Gruyter, Berlin, 1994.

[5] K. Itô and H. P. McKean, Jr., *Diffusion processes and their sample paths*, Springer, Berlin, 1974.

[6] H. Osada, Dirichlet form approach to infinite-dimensional Wiener processes with singular interactions, Comm. Math. Phys. **176** (1996), no. 1, 117–131.

[7] L. C. G. Rogers and Z. Shi, Interacting Brownian particles and the Wigner law, Probab. Theory Related Fields **95** (1993), no. 4, 555–570.

[8] H. Spohn, Interacting Brownian particles: a study of Dyson's model, in *Hydrodynamic behavior and interacting particle systems*, ed. by G. C. Papanicolaou (*Minneapolis*, *Minn.*, 1986), 151–179, IMA Vol. Math. Appl., 9, Springer, New York, 1987.